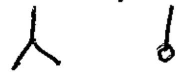
Talk for CT08 at Calais — June 2008

Title: Quantum categories (with Day, Pastro, Chikhladze)

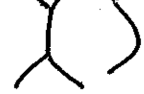
$\mathcal{V}$ monoidal category

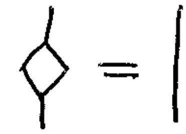
1) Monoid in $\mathcal{V}$ $\quad A\otimes A \xrightarrow{\mu} A \xleftarrow{\eta} I$

2) Comonoid in $\mathcal{V}$ = monoid in $\mathcal{V}^{op}$ $\quad A\otimes A \xleftarrow{\delta} A \xrightarrow{\varepsilon} I$



3) Frobenius monoid = monoid + comonoid satisfying

(which consequently = )

4) Separable monoid = Frobenius + 

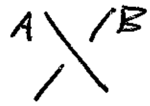
5) Example $\mathcal{V} = \underset{\sim}{\mathrm{Set}}$: comonoid = set; monoid = unital semigroup; Frobenius monoid = separable monoid = singleton set

6) Example $\mathcal{V} = \underset{\sim}{\mathrm{Vect}}_k$: monoid = k-algebra; comonoid = k-coalgebra; Frobenius monoid = Frobenius k-algebra = finite dimensional k-algebra A with $A \xrightarrow{\varepsilon} k$ such that $A\otimes A \xrightarrow{\mu} A \xrightarrow{\varepsilon} k$ is non-degenerate.

7) Example $\mathcal{V} = \underset{\sim}{\mathrm{Vect}}_k$, G finite group, $A = kG$ the group algebra over k

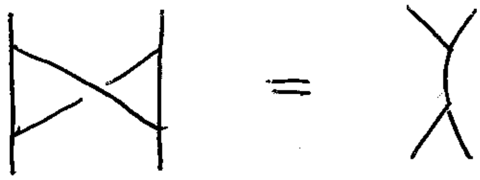
A becomes Frobenius with $\delta : A \to A \otimes A$ defined by $\delta(x) = \sum_y y \otimes y^{-1}x$ and $\varepsilon(x) = \begin{cases} 1 & x=1 \\ 0 & x \neq 1 \end{cases}$.
Then $\mu\,\delta(x) = (\#G)x$. If $\#G$ is invertible in k then A becomes separable with
$\delta(x) = \frac{1}{\#G} \sum_y y \otimes y^{-1}x$.

$\mathcal{V}$ braided monoidal category



$$A \otimes B \xrightarrow{c_{A,B}} B \otimes A$$

8) <u>Bimonoid</u> = monoid + comonoid satisfying



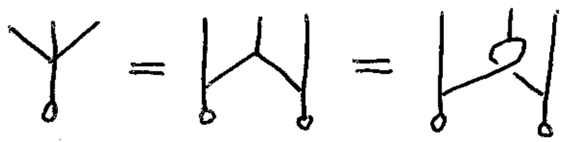
(b)

(u)

This concept is self-dual.

9) <u>Example</u> $\mathcal{V} = \mathrm{Vect}_k$, G any group, $A = kG$ the group algebra over k becomes a bimonoid with $\delta(x) = x \otimes x$, $\varepsilon(x) = 1 \in k$.

10) <u>Weak bimonoid</u> = monoid + comonoid satisfying (b) +



(v)

(w)

Every bimonoid is a weak bimonoid.

11) <u>Example</u> $\mathcal{V} = \underline{\text{Vect}}_k$, $\mathcal{C}$ category with finitely many objects, $A = k(\text{ar}\,\mathcal{C})$ the vector space with basis the set of arrows of $\mathcal{C}$. An associative multiplication is defined on A by

$$f \cdot g = \begin{cases} g \circ f & \text{when this composition is defined} \\ 0 & \text{otherwise} \end{cases}$$

for arrows f, g in $\mathcal{C}$. A unit is $1 = \sum_a 1_a$, the sum of the identity arrows of the objects a of $\mathcal{C}$. A comonoid structure is defined as usual by

$$\delta(f) = f \otimes f \quad \text{and} \quad \varepsilon(f) = 1 \in k.$$

Axiom (b) holds but none of (a) does. This A is a weak bimonoid: both (v) and (w) hold.

12) <u>Construction</u> There is an autonomous monoidal bicategory $\underline{\text{Comod}(\mathcal{V})}$ (assuming $V \otimes -$ preserves coreflexive equalizers). The objects are comonoids C in $\mathcal{V}$. For C, D comonoids, put

$$\text{Comod}(\mathcal{V})(C, D) = \mathcal{V}^{C \otimes - \otimes D},$$

the category of Eilenberg-Moore coalgebras for the comonad $C \otimes - \otimes D$ on $\mathcal{V}$: morphisms of $\text{Comod}(\mathcal{V})$ are therefore left C-, right D-bicomodules $M : C \nrightarrow D$. Composition of comodules $C \overset{M}{\nrightarrow} D \overset{N}{\nrightarrow} E$ is given by the equalizer

$$N \circ M = M \underset{D}{\otimes} N \xrightarrow{\ i\ } M \otimes N \underset{1 \otimes \delta_l}{\overset{\delta_r \otimes 1}{\rightrightarrows}} M \otimes D \otimes N.$$

The tensor product of Comod($\mathcal{V}$) is that of comonoids; it uses the braiding in the coaction:

$$C\otimes D \xrightarrow{\delta\otimes\delta} C\otimes C\otimes D\otimes D \xrightarrow{1\otimes c_{C,D}\otimes 1} C\otimes D\otimes C\otimes D.$$

The dual C^{o} of C is C with coaction  $C \xrightarrow{\delta} C\otimes C \xrightarrow{c_{C,C}} C\otimes C$;
unit $I \overset{\eta}{\nrightarrow} C^{o}\otimes C$, counit $C\otimes C^{o} \overset{e}{\nrightarrow} I$ are both C with appropriate coactions. The biadjunction $C \dashv_{b} C^{o}$ generates a pseudomonoid structure on $C^{o}\otimes C$:

$$C^{o}\otimes C\otimes C^{o}\otimes C \overset{P=1\otimes e\otimes 1}{\nrightarrow} C^{o}\otimes C \overset{J=\eta}{\nleftarrow} I.$$

13) <u>Quantum category</u> (C,A) <u>in</u> $\mathcal{V}$ = comonoid C + monoidal comonad $A : C^{o}\otimes C \nrightarrow C^{o}\otimes C$ in Comod($\mathcal{V}$)

$$\begin{array}{ccc} C^{o}\otimes C & \overset{A}{\nrightarrow} & C^{o}\otimes C \\ & \Downarrow\delta \quad \overset{\varepsilon}{\Rightarrow} & \\ {\scriptstyle A}\searrow & C^{o}\otimes C & \nearrow{\scriptstyle A},\ \curvearrowright 1 \end{array}$$

$$\begin{array}{ccc} C^{o}\otimes C\otimes C^{o}\otimes C & \overset{A\otimes A}{\nrightarrow} & C^{o}\otimes C\otimes C^{o}\otimes C \\ P \downarrow & \Downarrow\mu & \downarrow P \\ C^{o}\otimes C & \overset{A}{\nrightarrow} & C^{o}\otimes C \\ {\scriptstyle J}\nwarrow & \Nwarrow\eta \ \ I & \nearrow{\scriptstyle J} \end{array}$$

The comonad Eilenberg-Moore construction is

$$\begin{array}{ccc} & A & \\ {\scriptstyle \varepsilon_{*}}\swarrow & \overset{\delta}{\Leftarrow} & \searrow{\scriptstyle \varepsilon_{*}} \\ C^{o}\otimes C & \underset{A}{\nrightarrow} & C^{o}\otimes C \end{array}$$

(universal A-coalgebra)

so that (by Moerdijk & McCrudden) yields a pseudomonoid structure $P : A\otimes A \nrightarrow A$, $J : I \nrightarrow A$ for which ε_{*} is strong monoidal.

$$\begin{array}{ccc} & \text{Comod}(\mathcal{V})(I,A) \text{ (right } A\text{-comodules)} & \\ {\scriptstyle \varepsilon_{*}\circ -}\swarrow & \overset{\delta}{\Leftarrow} & \searrow{\scriptstyle \varepsilon_{*}\circ -} \\ \text{Comod}(\mathcal{V})(I, C^{o}\otimes C) & \longrightarrow & \text{Comod}(\mathcal{V})(I, C^{o}\otimes C) \\ \wr\! \| & & \wr\! \| \\ \text{Comod}(\mathcal{V})(C,C) & \underset{G}{\longrightarrow} & \text{Comod}(\mathcal{V})(C,C) \end{array}$$

(C-bicomodules) monoidal comonad in <u>CAT</u>

The comonoid morphism $\varepsilon : A \to C^{o} \otimes C$ determines two comonoid morphisms

$$s = (A \xrightarrow{\varepsilon} C^{o} \otimes C \xrightarrow{1 \otimes \varepsilon} C^{o}), \quad t = (A \xrightarrow{\varepsilon} C^{o} \otimes C \xrightarrow{\varepsilon \otimes 1} C)$$

called <u>source</u> and <u>target</u>; of course, μ is <u>composition</u> and η provides <u>identities</u>.

14) <u>Example</u> $\mathcal{V} = \underset{\sim}{\mathrm{Set}}$, $\mathrm{Comod}(\mathcal{V}) \simeq \mathrm{Span}(\underset{\sim}{\mathrm{Set}})$: a quantum category in $\underset{\sim}{\mathrm{Set}}$ is a category in $\underset{\sim}{\mathrm{Set}}$ in the usual sense.

$$P : \quad C^4 \xleftarrow{1 \times \Delta \times 1} C^3 \xrightarrow{1 \times ! \times 1} C^2 \qquad J : \quad 1 \xleftarrow{!} C \xrightarrow{\Delta} C^2$$

$$A : \quad C \times C \xleftarrow{(s,t)} A \xrightarrow{(s,t)} C \times C$$

$$A \circ A = \{(\alpha, \beta) \in A \times A \mid a \underset{\beta}{\overset{\alpha}{\rightrightarrows}} b\} \xleftarrow{\delta = \Delta} A \xrightarrow{\varepsilon = (s,t)} C \times C$$

as spans from $C \times C$ to itself

$$P \circ (A \times A) = \{(\alpha, \beta) \in A \times A \mid a \xrightarrow{\alpha} b \xrightarrow{\beta} c\} \xrightarrow{\mu} C \times A = A \circ P$$

$$(\alpha, \beta) \longmapsto (b, \beta \circ \alpha)$$

$$C \xrightarrow{\eta} \{\xi \in A \mid a \circlearrowleft^{\xi}\} = A \circ J, \quad a \longmapsto a \circlearrowleft^{1_a}$$

G is a monoidal comonad on $\mathrm{Set}/C \times C$

$$G(X \xrightarrow{(u,v)} C \times C) = \{(x, \alpha) \in X \times A \mid u(x) \xrightarrow{\alpha} v(x)\}$$

$$G(X \xrightarrow{(u,v)} C \times C) \underset{C}{\times} G(Y \xrightarrow{(h,k)} C \times C) = \{(x, \alpha, y, \beta) \mid u(x) \xrightarrow{\alpha} v(x) = h(y) \xrightarrow{\beta} k(y)\}$$

$$G\left(\begin{matrix} X \underset{C}{\times} Y \\ \downarrow \\ C \times C \end{matrix}\right) = \{(x, y, \gamma) \mid v(x) = h(y), u(x) \xrightarrow{\gamma} k(y)\}$$

$$\mu(x, \alpha, y, \beta) = (x, y, \beta \circ \alpha)$$

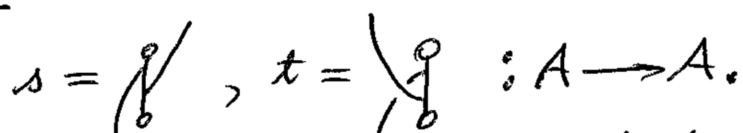
$$\eta : C \longrightarrow G\left(\begin{matrix} C \\ \downarrow (1,1) \\ C \times C \end{matrix}\right) = \{(c, \alpha) \mid c \xrightarrow{\alpha} c\}, \quad \eta(a) = (a, 1_a)$$

15) <u>Example</u> $\mathcal{V} = \mathrm{Vect}_k^{op}$: a quantum category here is a Takeuchi bialgebroid (he called them $\times_R$-bialgebras).
[Mitsuhira Takeuchi, J. Math. Soc. Japan 29 (1977) 459–492]

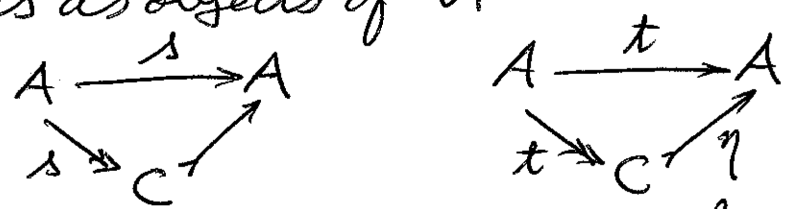
16) <u>Example</u> A quantum category with $C = I$ is a quasibimonoid A in $\mathcal{V}$ (i.e. pseudomonoid in the 2-category of comonoids in $\mathcal{V}$).

17) <u>Theorem</u> [Böhm–Szlachányi, Nill, Schauenburg, Pastro–St]
<u>Weak bimonoids in</u> $\mathcal{V}$ <u>are precisely quantum categories</u> (C, A) <u>in</u> $\mathcal{V}$ <u>for which</u> C <u>is separable</u>.

<u>Sketch</u> Let A be a weak bimonoid in $\mathcal{V}$. Put

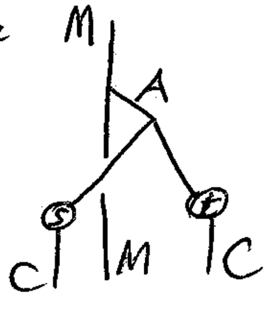


These turn out to be idempotents with isomorphic images as objects of $\mathcal{V}$.



If we make C a comonoid in $\mathcal{V}$ such that the second triangle consists of comonoid morphisms then $s : A \longrightarrow C^{o}$ is also a comonoid morphism. Also, C becomes a monoid such that $t : A \longrightarrow C$ becomes a monoid morphism. One checks that C is Frobenius and $\mu\,\delta = 1_C$.

Each right A-comodule M becomes a comodule $M : C \rightarrow\!\!\!\!+ C$ via



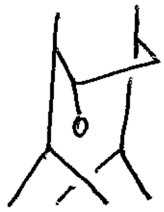
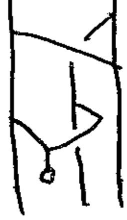
and, for right A-comodules M and N, we obtain $M \otimes_C N$ by splitting the idempotent

$m =$ 

to obtain $M \otimes_C N$ as a right A-comodule via

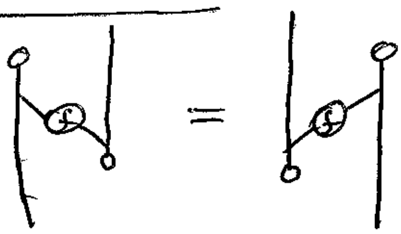
$\gamma =$ 

In particular, we have $P = A \otimes_C A$ with

$\delta_\ell =$  $: P \to A \otimes A \otimes P$ and $\delta_r =$  $: P \to P \otimes A$

□

18) Proposition *If R and S are Frobenius monoids in* $\mathcal{V}$ *and* $f : R \to S$ *is a monoid and comonoid morphism then* f *has inverse*



Corollary *If A and B are weak bimonoids in* $\mathcal{V}$ *and* $f : A \to B$ *is a monoid and comonoid morphism then* f *induces an invertible comonoid morphism on the "object of objects" of the quantum categories arising from A and B.*

19) <u>Quantum functor</u> $(f, \varphi) : (\mathcal{C}, A) \longrightarrow (\mathcal{D}, B)$ consists of a comonoid morphism $f : C \longrightarrow D$ and a monoidal 2-cell

$$\begin{array}{ccc} D^{o}\otimes D & \xrightarrow{\;f_*^{o}\otimes f^*\;} & C^{o}\otimes C \\ B\downarrow & \Downarrow \tilde{\varphi} & \downarrow A \\ D^{o}\otimes D & \xrightarrow[\;f_*^{o}\otimes f^*\;]{} & C^{o}\otimes C \end{array}$$

whose mate φ, with $f^{*o}\otimes f_*$, forms a comonad morphism

$$\begin{array}{ccc} C^{o}\otimes C & \xrightarrow{\;f^{*o}\otimes f_*\;} & D^{o}\otimes D \\ A\downarrow & \overset{\varphi}{\Longrightarrow} & \downarrow B \\ C^{o}\otimes C & \xrightarrow[\;f^{*o}\otimes f_*\;]{} & D^{o}\otimes D . \end{array}$$

20) <u>Frobenius pseudomonoid</u> = pseudomonoid E equipped with a morphism $\ell : E \longrightarrow I$ such that

$$E\otimes E \xrightarrow{\;p\;} E \xrightarrow{\;\ell\;} I$$

is a counit for $E \dashv_{\ell} E$. This is a general definition within any monoidal bicategory $\mathcal{M}$.

21) <u>Example</u> $\mathcal{M} = \mathrm{Mod}\ (= \mathrm{Prof} = \mathrm{Dist})$ Recall that a monoidal category $\mathcal{C}$ is <u>$*$-autonomous</u> (Barr) when it is equipped with an equivalence of categories $S : \mathcal{C}^{op} \longrightarrow \mathcal{C}$ and a natural isomorphism

$$\mathcal{C}(B, SA) \cong \mathcal{C}(I, S(A\otimes B)).$$

It follows that

$$\mathscr{C}(A\otimes B, SC) \cong \mathscr{C}(I, S(C\otimes A\otimes B))$$
$$\cong \mathscr{C}(B, S(C\otimes A))$$
$$\cong \mathscr{C}(C\otimes A, S^{-1}B) \text{ and}$$

$$\mathscr{C}(B, SA) \cong \mathscr{C}(A\otimes B, L)$$

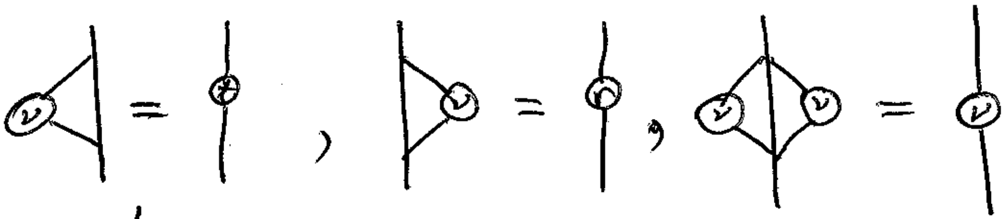
where $L = S^{-1}I$ is called a <u>dualizing object</u>. Then $\mathscr{C}^{op}$ becomes a Frobenius pseudomonoid in Mod where $p = P : \mathscr{C}^{op} \times \mathscr{C}^{op} \nrightarrow \mathscr{C}^{op}$ is defined by $P(C; A, B) = \mathscr{C}(A\otimes B, C)$ and $l = L^* : \mathscr{C}^{op} \nrightarrow 1$ (since $\mathscr{C}^{op} \times \mathscr{C}^{op} \xrightarrow{P} \mathscr{C}^{op} \xrightarrow{L^*} 1$ corresponds to the equivalence $S_* : \mathscr{C}^{op} \nrightarrow \mathscr{C}$ under the biduality $\mathscr{C} \dashv_b \mathscr{C}^{op}$).

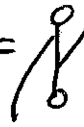
22) <u>Example</u> $\mathcal{M} = \mathrm{Comod}(\mathcal{V})$; C comonoid in $\mathcal{V}$ with a comonoid isomorphism $v : C \to C^{oo}$. Then the pseudomonoid $C^o \otimes C$ in $\mathrm{Comod}(\mathcal{V})$ is Frobenius when it is equipped with

$$l = \left(C^o\otimes C \xrightarrow{1\otimes v} C^o \otimes C^{oo} \xrightarrow{e_{C^o}} I\right).$$

23) <u>Quantum groupoid</u> in $\mathcal{V}$ = quantum category (C, A) equipped with a comonoid isomorphism $v : C \to C^{oo}$ and a Frobenius structure $l : A \nrightarrow I$ on the pseudomonoid A "compatible with that on $C^o \otimes C$".

24) <u>Weak Hopf monoids in $\mathcal{V}$</u> = weak bimonoid A in $\mathcal{V}$ equipped with an "antipode" morphism $\nu : A \to A$ satisfying



where $r =$  (which is generally different from $s =$ ).

It can be shown that $\nu : A^{\circ} \to A$ is a comonoid morphism.

25) <u>Example</u> Each weak Hopf monoid with invertible antipode determines a quantum groupoid with $\upsilon : C^{\circ\circ} \to C$ induced by $\nu\nu : A^{\circ\circ} \to A$ and $\nu : A^{\circ} \to A$ yielding $A \dashv_{\ell} A$.