Frobenius Ob's in Cartesian Bicat's/Duals Invert

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A bicategory **B** is cartesian if

1) $\mathrm{Map}\mathbf{B}$ has $\times$ and 1, products (as a bicategory)
2) Each $\mathbf{B}(X, A)$ has $\wedge$ and $\top$, products
3) The lax functors

$$\mathbf{B}\times\mathbf{B} \xrightarrow{\otimes} \mathbf{B} \xleftarrow{I} \mathbf{1}$$

$$(R\otimes S := p^* R p \wedge r^* S r \qquad \begin{array}{ccccc} X & \xleftarrow{p_{X,Y}} & X\times Y & \xrightarrow{r_{X,Y}} & Y \\ R\downarrow & \xleftarrow{\pi} & \downarrow R\otimes S & \xrightarrow{\rho} & \downarrow S \\ A & \xrightarrow[p^*_{A,B}]{} & A\times B & \xleftarrow[r^*_{A,B}]{} & B \end{array}$$

$$I \text{ is the unique monad on } \top{:}1 \to 1)$$

are pseudofunctors.

- In general, $\times$ and 1 are not products for **B**.
- A cartesian bicategory *is* a symmetric monoidal bicategory.

For any A in $\mathbf{B}$ we have the equality

$$
\begin{array}{ccc}
A & \xrightarrow{\quad d \quad} & A \times A \\
 & & \downarrow{\scriptstyle d \times 1} \\
{\scriptstyle d}\downarrow & \xrightarrow{=} & (A \times A) \times A \\
 & & \downarrow{\scriptstyle a} \\
A \times A & \xrightarrow[1 \times d]{} & A \times (A \times A)
\end{array}
$$

Mates $\delta_0 : d.d^* \to 1_A \otimes d^*.a.d \otimes 1_A \quad \delta_1 : d.d^* \to d^* \otimes 1_A.a^*.1_A \otimes d$
A is Frobenius if either δ_i, equivalently the other, is invertible.
THEOREM For an arrow $R : X \to A$ in a cartesian bicategory, with X and A Frobenius, the following are equivalent:

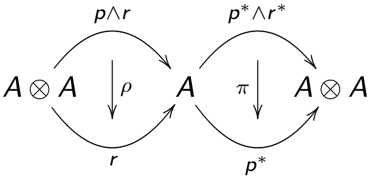
1) R is a map
2) R is a comonoid homomorphism
3) $R \dashv R^\circ$ (the mate of R wrt $X \dashv X = X^\circ$ and $A \dashv A = A^\circ$). $\square$

THEOREM If A is a Frobenius object in a cartesian bicategory **B**, then, for all X in **B**, the hom-category $\mathrm{Map}\mathbf{B}(X, A)$ is a groupoid.

LEMMA With reference to the 2-cell $\delta_1 : dd^* \to (d^* \otimes 1_A)(1_A \otimes d)$,

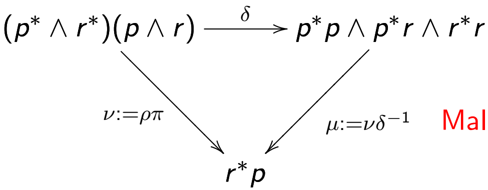
$$dd^* \xrightarrow{\simeq} (p^* \wedge r^*)(p \wedge r) \quad \text{and} \quad (d^* \otimes A)(A \otimes d) \xrightarrow{\simeq} p^*p \wedge p^*r \wedge r^*r$$

and these canonical isomorphisms identify δ_1 with $\langle \pi\pi, \pi\rho, \rho\rho \rangle$. Here the components are horizontal composites of the local product projection 2-cells. For example, $\pi\rho$ is



We will write

$$\delta = \langle \pi\pi, \pi\rho, \rho\rho \rangle : (p^* \wedge r^*)(p \wedge r) \to p^*p \wedge p^*r \wedge r^*r : A \otimes A \to A \otimes A$$



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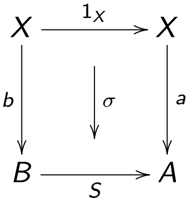
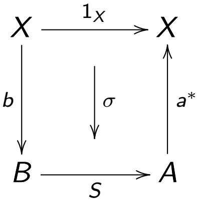
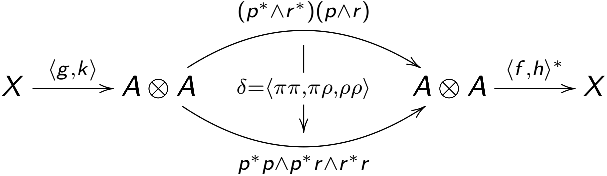
HOM NOTATION





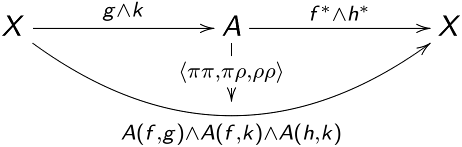
We can write $\sigma \in S(a, b) = a^*Sb$. If $S = 1_A{:}A \to A$, and $\alpha{:}f \to g$ in $\mathrm{Map}\mathbf{B}(X, A)$ we can write $\alpha \in A(f, g) = f^*1_Ag = f^*g$.

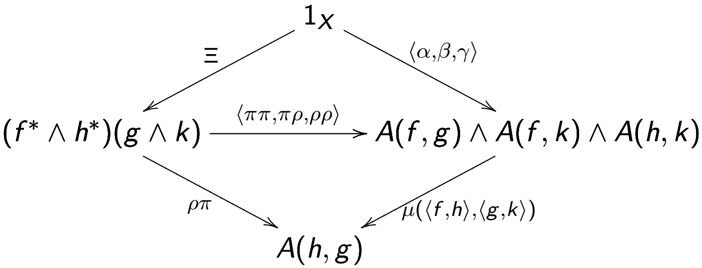
LEMMA For objects f, h, g, k of $\mathrm{Map}\mathbf{B}(X, A)$, the whisker composite



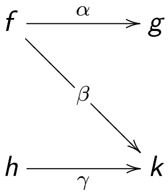
$$(p^*\wedge r^*)(p\wedge r)(\langle f, h\rangle, \langle g, k\rangle) \xrightarrow{\delta(\langle f,h\rangle,\langle g,k\rangle)} (p^*p\wedge p^*r\wedge r^*r)(\langle f, h\rangle, \langle g, k\rangle)$$

is





$\langle \alpha, \beta, \gamma \rangle \in_{1_X} A(f,g) \wedge A(f,k) \wedge A(h,k)$ an "S" configuration



$\Xi \in_{1_X} (f^* \wedge h^*)(g \wedge k) = (p^* \wedge r^*)(p \wedge r)(\langle f, h \rangle, \langle g, k \rangle)$?

Those obtained by pasting a 1_X-element of $(p^* \wedge r^*)(\langle f, h\rangle, x)$ to a 1_X-element of $(p \wedge r)(x, \langle g, k\rangle)$, for some $x\colon T \to A$, !

$$
\begin{array}{ccccc}
T & \xrightarrow{\;1_T\;} & T & \xrightarrow{\;1_T\;} & T \\
{\scriptstyle \langle g,k\rangle}\Big\downarrow & \Downarrow & \Big\downarrow{\scriptstyle x} & \Downarrow & \Big\downarrow{\scriptstyle \langle f,h\rangle} \\
A \otimes A & \xrightarrow[p \wedge r]{} & A & \xrightarrow[p^* \wedge r^*]{} & A \otimes A
\end{array}
$$

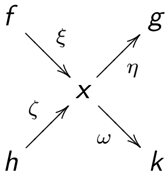
$$(p^* \wedge r^*)(\langle f, h\rangle, x) = \langle f, h\rangle^*(p^* \wedge r^*)x \cong (f^* \wedge h^*)x \cong f^*x \wedge h^*x$$

$$= A(f, x) \wedge A(h, x)$$

$$(p \wedge r)(x, \langle g, k\rangle) = x^*(p \wedge r)\langle g, k\rangle \cong x^*(g \wedge k) \cong x^*g \wedge x^*k$$

$$= A(x, g) \wedge A(x, k)$$

Equivalence classes of configurations in $\mathrm{Map}\mathbf{B}(X, A)$ of the form



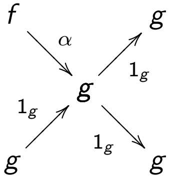
LEMMA For a 1_X-element Ξ arising from an "X" configuration as above, $\delta\Xi = \langle \eta\xi, \omega\xi, \omega\zeta \rangle$ and $\nu\Xi = \eta\zeta$. $\square$

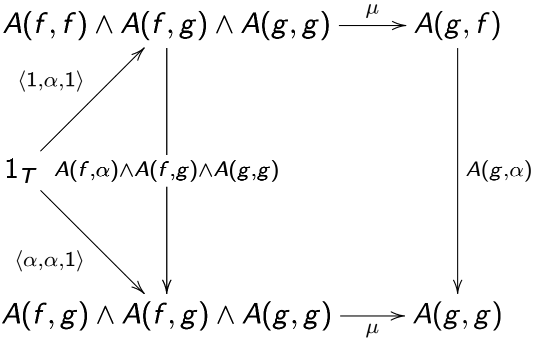
Given a 2-cell $\alpha{:}f \to g$ we have the "S" $\langle 1_f, \alpha, 1_g \rangle$. $\alpha^\dagger := \mu\langle 1, \alpha, 1 \rangle$

LEMMA $\alpha^\dagger = \alpha^{-1}$

PROOF

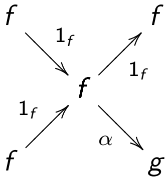
$\Xi :=$





$$\alpha\alpha^{\dagger} = \alpha\mu\langle 1, \alpha, 1\rangle = \mu\langle \alpha, \alpha, 1\rangle = \mu\delta\Xi = \nu\Xi = 1_g$$

For $\alpha^{\dagger}\alpha$ use



□

THEOREM [N. Saavedra Rivano, SLNM 265, 1972] For $\tau : F \Rightarrow G : \mathcal{V} \to \mathcal{W}$, if F, G strong monoidal, τ monoidal, V part of a dual situation in $\mathcal{V}$ then τV is invertible. □

$V^+ \dashv V$ adjunction wrt $\otimes$, F strong $\Longrightarrow FV^+ \dashv FV$

$$(\tau_V : FV \to GV)^{-1} = (\tau_{V^+} : FV^+ \to GV^+)^-$$

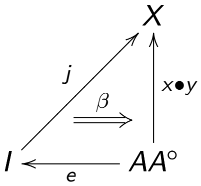
$(-)^-$ denotes mate with respect to $FV^+ \dashv FV$ and $GV^+ \dashv GV$

THEOREM [B.J Day and C. Pastro, 2008] For $\tau : F \Rightarrow G : \mathcal{V} \to \mathcal{W}$, if F, G Frobenius, τ monoidal and comonoidal, V part of a dual situation in $\mathcal{V}$ then τV is invertible. □

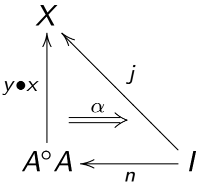
NOW WORK IN A MONOIDAL BICATEGORY $(\mathcal{M}, \otimes, I, \ldots)$

$(X, p, j, \alpha, \lambda, \rho)$, $(A \otimes B \xrightarrow{x \otimes y} X \otimes X \xrightarrow{p} X) = (AB \xrightarrow{x \bullet y} X)$

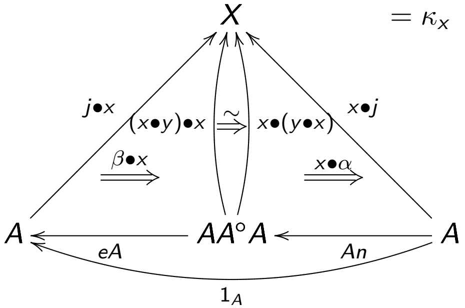
For A with right bidual and $(X, p, j, \alpha, \lambda, \rho)$ monoidal object in $\mathcal{M}$ an exact pairing for arrows $x{:}A \rightarrow X$ and $y{:}A^\circ \rightarrow X$ in $\mathcal{M}$ consists of the further data

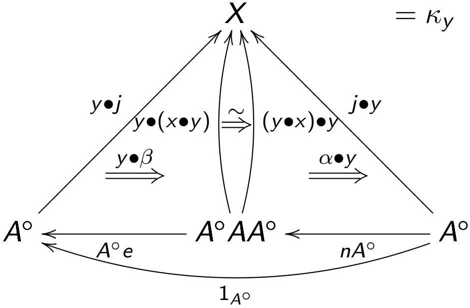


and



subject to the equations





S. Lack: $x{:}A \rightarrowtriangle X$ has a right bidual in $\mathcal{M}/\!\!/X$

THEOREM If A has a right bidual, X and Y are monoidal objects, $(x^+, x; \alpha, \beta): A \twoheadrightarrow X$ is an exact pairing, and $\tau: f \twoheadrightarrow g: X \twoheadrightarrow Y$ is a both monoidal and comonoidal 2-cell between Frobenius arrows then $\tau x: fx \twoheadrightarrow gx$ is invertible with

$$(\tau x)^{-1} = (\tau x^+)^-$$

and $\tau x^+: fx^+ \twoheadrightarrow gx^+$ is invertible with

$$(\tau x^+)^{-1} = (\tau x)^+.$$

COROLLARY If A has left dualization as a monoidal object of $\mathcal{M}$ ($1_A^+ \dashv 1_A$ exact pairing), X is a monoidal object, and $\tau: f \twoheadrightarrow g: A \twoheadrightarrow X$ is a monoidal and comonoidal 2-cell between Frobenius arrows then τ is invertible with

$$\tau^{-1} = (\tau 1_A^+)^-$$

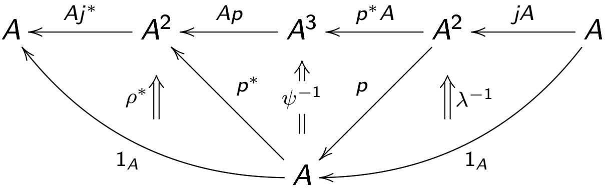
In any monoidal bicategory $\mathcal{M}$, monoidal object $(A, p, j, \alpha, \lambda, \rho)$ map monoidal if p and j are maps ($=$ left adjoints). Then α and α^{-1} have mates

$$\begin{array}{ccc} A \otimes A \otimes A = A^3 & \xleftarrow{Ap^*} & A^2 \\ {\scriptstyle pA}\big\downarrow & \overset{\phi}{\Longrightarrow} & \big\downarrow{\scriptstyle p} \\ A^2 & \xleftarrow[p^*]{} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} A^3 & \xleftarrow{p^*A} & A^2 \\ {\scriptstyle Ap}\big\downarrow & \overset{\psi}{\Longrightarrow} & \big\downarrow{\scriptstyle p} \\ A^2 & \xleftarrow[p^*]{} & A \end{array}$$

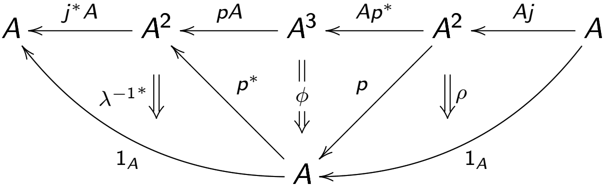
Say $(A, \cdots)$ is naturally Frobenius if both ϕ and ψ invertible.

$$n = p^*.j : I \to A^2$$
$$e = j^*.p : A^2 \to I$$

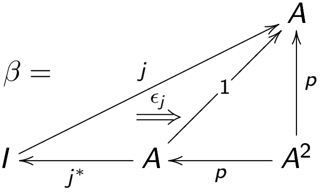
$\eta =$



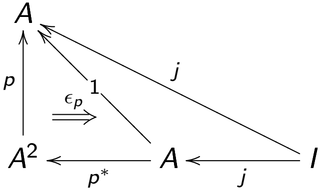
$\epsilon =$



$\beta =$



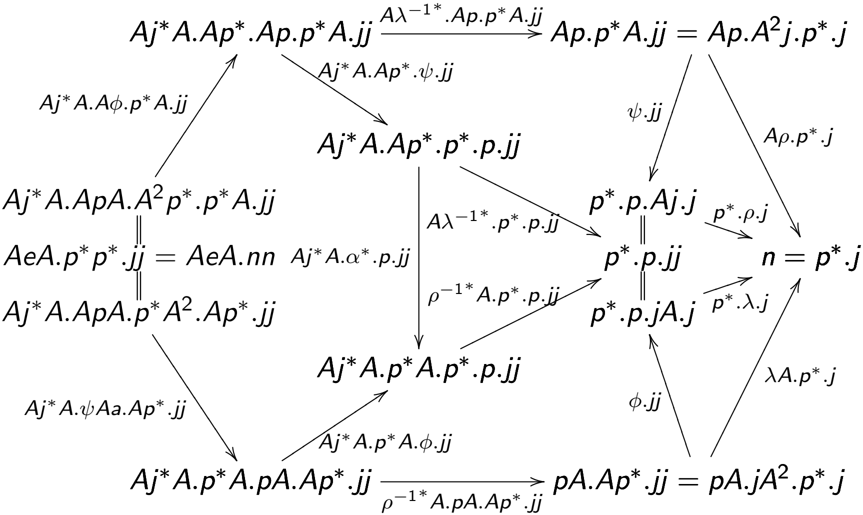
$\alpha =$



THEOREM For $(A, p, j, \alpha, \lambda, \rho)$ a naturally Frobenius map monoidal object in a monoidal bicategory,

i) the arrows $n = p^*.j : I \to A \otimes A$ and $e = j^*.p : A \otimes A \to I$, together with the 2-cells η and ϵ, exhibit A as a right bidual for itself;

ii) the 2-cells β and α exhibit $(1_A, 1_A; \alpha, \beta)$ as an exact pairing, so that A has left dualization, given by the identity, as a monoidal object of $\mathcal{M}$.


$Aj^*A.Ap^*.Ap.p^*A.jj$
$A\lambda^{-1*}.Ap.p^*A.jj$
$Ap.p^*A.jj = Ap.A^2j.p^*.j$
$Aj^*A.A\phi.p^*A.jj$
$Aj^*A.Ap^*.\psi.jj$
$\psi.jj$
$A\rho.p^*.j$
$Aj^*A.Ap^*.p^*.p.jj$
$Aj^*A.ApA.A^2p^*.p^*A.jj$
$AeA.p^*p^*.jj = AeA.nn$
$Aj^*A.ApA.p^*A^2.Ap^*.jj$
$A\lambda^{-1*}.p^*.p.jj$
$Aj^*A.\alpha^*.p.jj$
$\rho^{-1*}A.p^*.p.jj$
$p^*.p.Aj.j$
$p^*.p.jj$
$p^*.p.jA.j$
$p^*.\rho.j$
$p^*.\lambda.j$
$n = p^*.j$
$Aj^*A.p^*A.p^*.p.jj$
$Aj^*A.\psi Aa.Ap^*.jj$
$\phi.jj$
$\lambda A.p^*.j$
$Aj^*A.p^*A.\phi.jj$
$Aj^*A.p^*A.pA.Ap^*.jj$
$\rho^{-1*}A.pA.Ap^*.jj$
$pA.Ap^*.jj = pA.jA^2.p^*.j$

- For $\mathbf{B}$ a cartesian bicategory, every object X has, essentially uniquely, the structure of a strict comonoidal object.
- In the monoidal bicategory $\mathbf{B}^{\mathrm{coop}}$, every object is a (strict) map monoidal object, naturally Frobenius iff Frobenius in $\mathbf{B}$.
- In $\mathbf{B}$ every arrow is comonoidal. In $\mathbf{B}^{\mathrm{coop}}$ every arrow r is monoidal, strong monoidal iff r is a map.
- Any 2-cell $\tau : r \Rightarrow s : Y \twoheadrightarrow X$ in $\mathbf{B}$ is comonoidal. If r and s are maps then in $\mathbf{B}^{\mathrm{coop}}$, τ is both monoidal and comonoidal.
- If A is Frobenius then $\mathrm{Map}\mathbf{B}(X, A)^{\mathrm{op}} = \mathrm{Map}\mathbf{B}^{\mathrm{coop}}(A, X)$ is a groupoid (by Rivano's Theorem) (and so is $\mathrm{Map}\mathbf{B}(X, A)$).