Multitopic Categories via Ordered Face Structures

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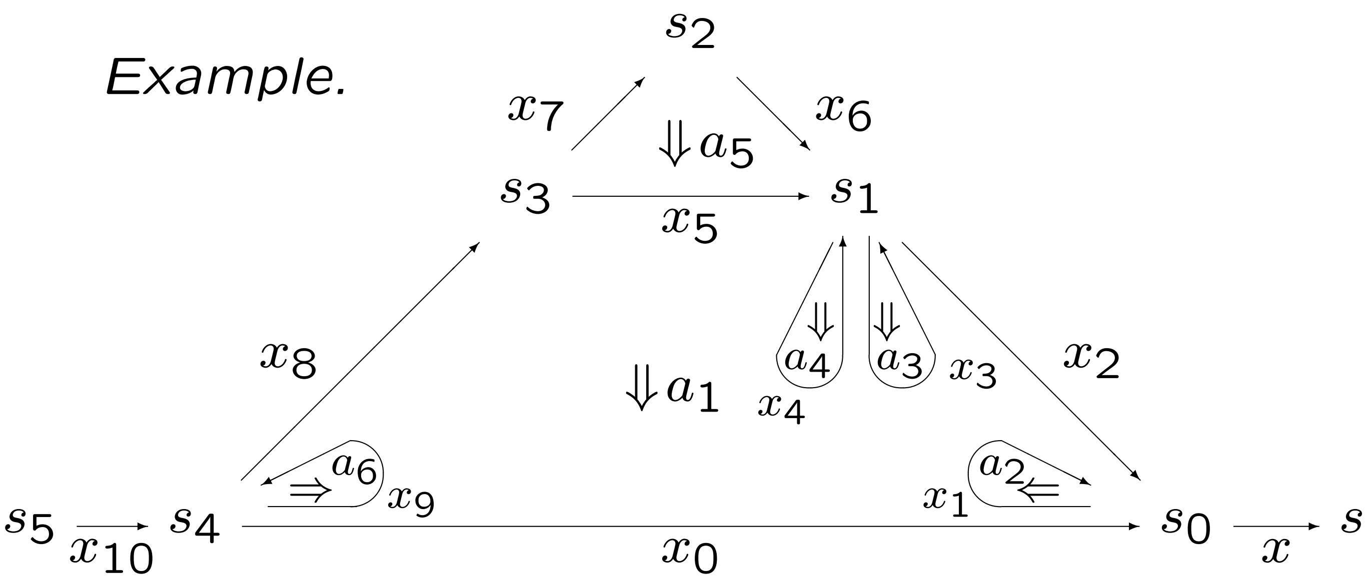
Plan of the talk

- Definition of ordered face structures (ofs), monotone and local maps
- Principal and normal ofs'es
- Operations on ofs'es ($\mathbf{d}^{(k)}$, $\mathbf{c}^{(k)}$, $\otimes_k$)
- Many-to-one computads vs ofs'es
- ω-maps and monotone ω-maps of ofs'es
- Multitopic category (substitution as a pushout, cell systems 'out of' a cell, a pasting diagram, strategies)

Ordered face structures are combinatorial structures describing the 'shapes' of (all) cells in many-to-one computads.

Example.



Primitive notions

Faces (finite sets): S_n - n-faces, $n \in \omega$

Codomains (functions): $\gamma : S_{n+1} \to S_n$
$\gamma(a_1) = x_0$, $\gamma(a_2) = x_1$

Domains (relations): $\delta : S_{n+1} \to S_n + 1_{S_{n-1}}$
$\delta(a_1) = \{x_1, x_2, x_3, x_4, x_5, x_8, x_9, \}$, $\delta(a_2) = 1_{s_0}$

Lower orders (relations): $<^\sim$ on $S_n \times S_n$
$a_5 <^\sim a_1$, $x_4 <^\sim x_3$

Derived notions

Lower preorder (relation): $<^-$ transitive closure of the relation

$$a \lhd^- b \text{ iff } \gamma(a) \in \delta(b)$$

Upper order (relation): $<^+$ transitive closure of the relation

$$a \lhd^+ b \text{ iff } \exists_{\alpha \text{ not a loop}} \quad a \in \delta(\alpha),\ \gamma(\alpha) = b$$

Axioms of ordered face structures

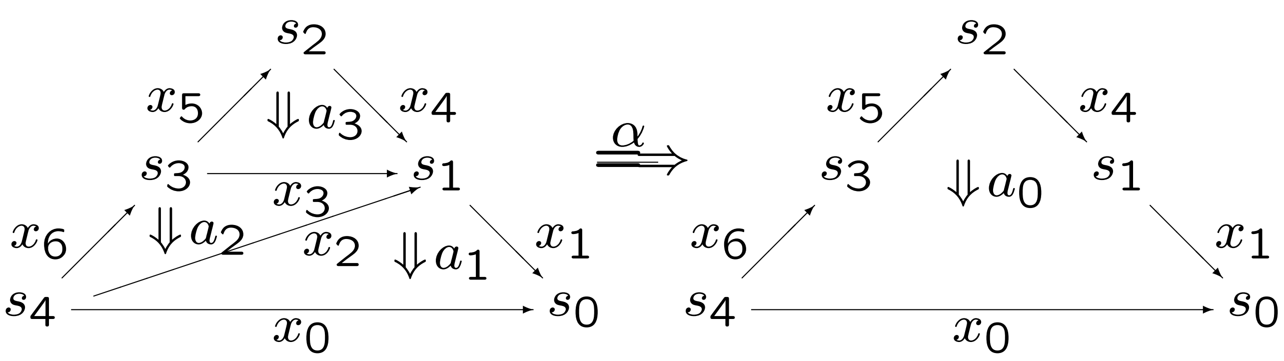
1. **Globularity axiom**

$$\gamma\gamma(\alpha) = \gamma\delta(\alpha) - \delta\dot{\delta}^{-\lambda}(\alpha)$$

$$\delta\gamma(\alpha) \equiv_1 \delta\delta(\alpha) - \gamma\dot{\delta}^{-\lambda}(\alpha)$$

$\equiv_1$ -'equality that almost ignores empty faces'.

Example.



we have

$\gamma\gamma(\alpha) = x_0$, $\delta\delta(\alpha) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$\delta\gamma(\alpha) = \{x_1, x_4, x_5, x_6\}$, $\gamma\delta(\alpha) = \{x_0, x_2, x_3\}$

... and five more axioms.

Two basic kinds of morphisms:

A **local morphism** $f : S \to T$ is a family of functions $f_k : S_k \to T_k$, for $k \in \omega$, such that the diagrams

$$\begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \gamma \downarrow & & \downarrow \gamma \\ S_k & \xrightarrow[f_k]{} & T_k \end{array} \qquad \begin{array}{ccc} S_{k+1} & \xrightarrow{f_{k+1}} & T_{k+1} \\ \delta \downarrow & & \downarrow \delta \\ S_k \dot{\sqcup} \mathbf{1}_{S_{k-1}} & \xrightarrow[f_k + \mathbf{1}_{f_{k-1}}]{} & T_k \dot{\sqcup} \mathbf{1}_{T_{k-1}} \end{array}$$

commute. For the right square it means more then commutation of relations, we demand that for any $a \in S_{\geq 1}$,

$$f_a : (\dot{\delta}(a), <^{\sim}) \longrightarrow (\dot{\delta}(f(a)), <^{\sim})$$

be an order isomorphism, where f_a is the restriction of f to $\dot{\delta}(a)$ (if $\delta(a) = 1_u$ we mean by that $\delta(f(a)) = 1_{f(u)}$).

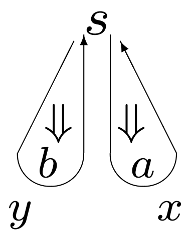
A **monotone morphism** $f : S \to T$ is a local morphism that preserves lower order $<^{\sim}$ (globally).

Examples. $f_1 : T_1 \to S$ is monotone:

T_1 :

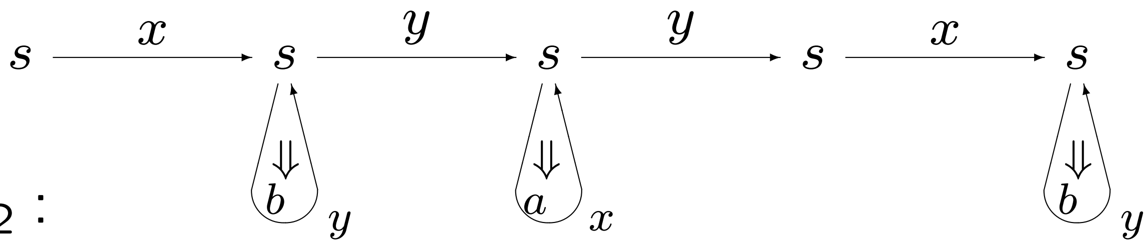
$$s_2 \xrightarrow{y} s_1 \xrightarrow{x} s_0$$

S :

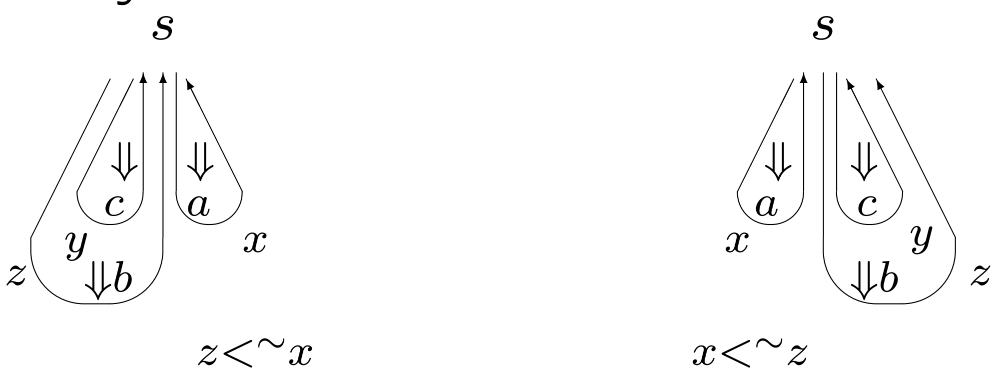


$f_2 : T_2 \to S$ is not monotone but it is local:

T_2 :



The following two ordered face structures are not isomorphic (globally) but they are isomorphic locally:



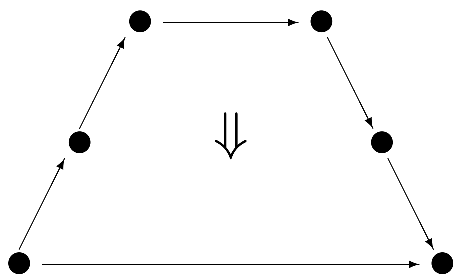
oFs ($\mathbf{oFs}_{loc}$) - is the category of ordered face structures and monotone (local) maps

The **size of an ordered face structure** S is the sequence natural numbers

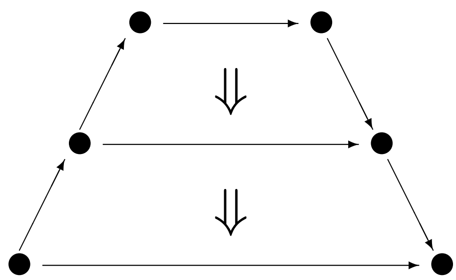
$$size(S) = \{|S_n - \delta(S^{-\lambda}_{n+1})|\}_{n\in\omega}$$

We have an order $<$ on such sequences, so that $\{x_n\}_{n\in\omega} < \{y_n\}_{n\in\omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and for all $l > k$, $x_l = y_l$.

An ordered face structure P is **principal** iff $size(P)_n \leq 1$, for $n \in \omega$.



An ordered face structure N is k-**normal** iff $dim(N) \leq k$ and $size(N)_n = 1$, for $n < k$.

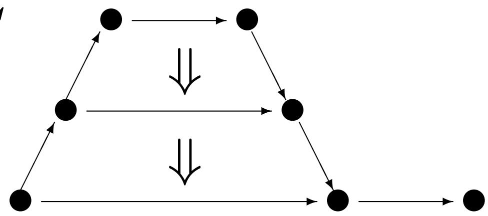


In **oFs** we have operations of the k-**domain** $\mathbf{d}^{(k)}$ and k-**codomain** $\mathbf{c}^{(k)}$, i.e. we have monotone morphisms:

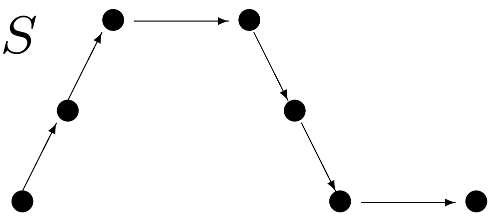
$$\mathbf{d}^{(k)}S \xrightarrow{\mathbf{d}^{(k)}_S} S \xleftarrow{\mathbf{c}^{(k)}_S} \mathbf{c}^{(k)}S$$

Example

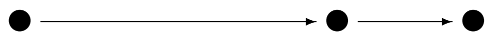
S



$\mathbf{d}^{(1)}S$



$\mathbf{c}^{(1)}S$

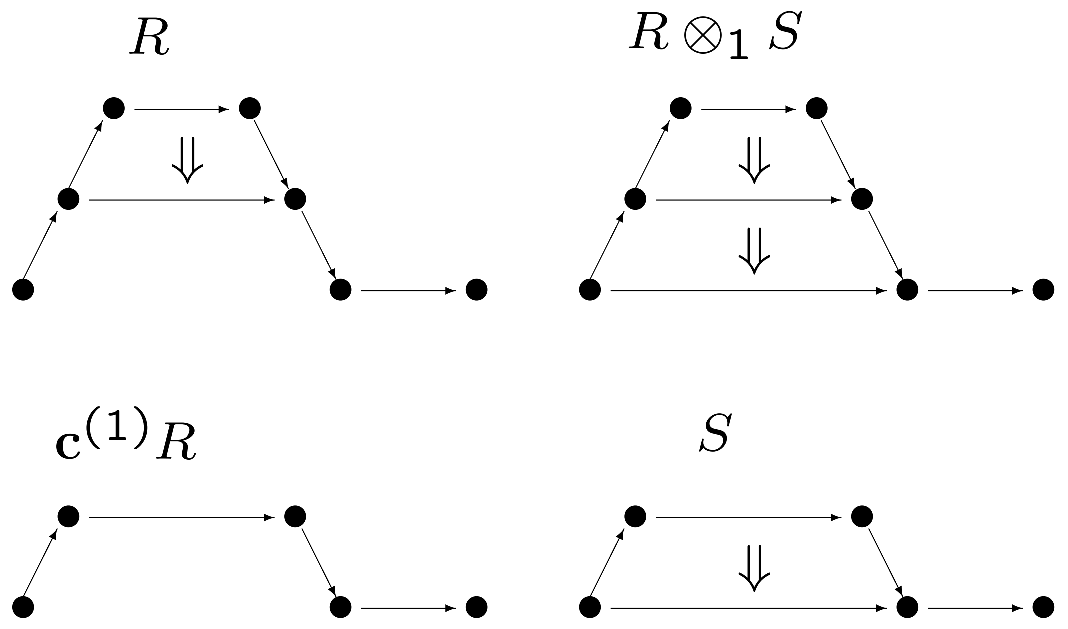


When the k-codomain of R agrees with the k-domain of S we have a commuting k-**tensor square**

$$\begin{array}{ccc} R & \xrightarrow{\kappa_R} & R \otimes_k S \\ {\scriptstyle \mathbf{c}_R^{(k)}} \uparrow & & \uparrow {\scriptstyle \kappa_S} \\ \mathbf{c}^{(k)} R & \xrightarrow[\mathbf{d}_S^{(k)}]{} & S \end{array}$$

in $\mathbf{oFs}$ which is a pushout in $\mathbf{oFs}_{loc}$.

Example



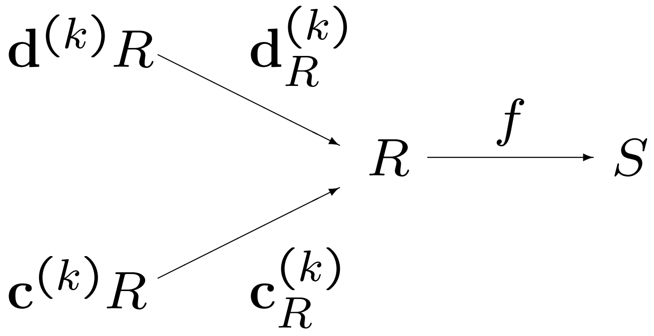
Embedding

$$(-)^* : \mathbf{oFs} \longrightarrow \mathbf{Comp}^{m/1}$$

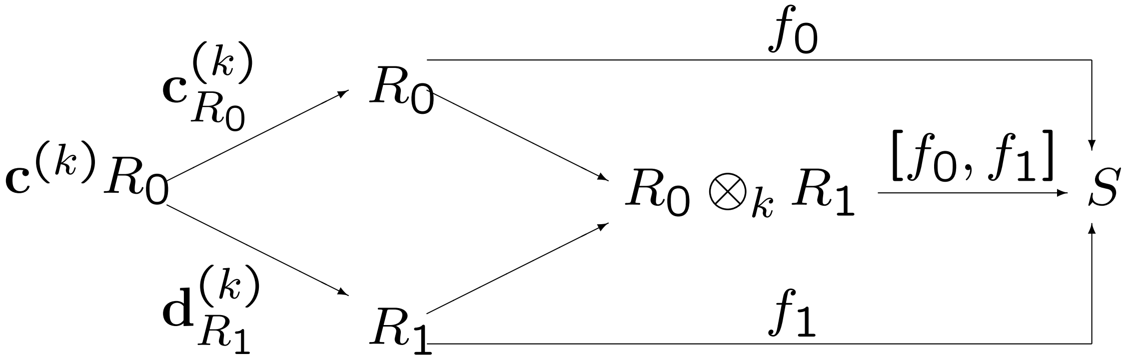
$$S \mapsto S^*$$

$$S^*_k = \coprod_{dim(R) \leq k} \mathbf{oFs}_{loc}(R, S)$$

k-domains and k-codomains S^*:



k-compositions in S^*: if $c^{(k)}(f_0) = d^{(k)}(f_1)$



then $f_1 \circ_k f_0 = [f_0, f_1]$

$Mod_{\bigotimes}(\mathbf{oFs}^{op}, Set)$ is the category models of $\mathbf{oFs}$, i.e. of the functors from $\mathbf{oFs}^{op}$ to Set sending tensor squares to pullbacks.

Theorem.

$$(-)^* : \mathbf{oFs} \longrightarrow \mathbf{Comp}^{m/1}$$

induces the functor

$$\mathbf{Comp}^{m/1} \longrightarrow Mod_{\bigotimes}(\mathbf{oFs}^{op}, Set)$$

$$C \longmapsto \mathbf{Comp}((-)^*, C)$$

which is an equivalence of categories. The full image of $(-)^*$ *is the category* $\mathbf{oFs}_{loc}$.

$\mathbf{oFs}_\omega$ is the full image of $\mathbf{oFs}$ in ωCat.

A morphism $\xi : R \to S$ is $\mathbf{oFs}_\omega$ is an ω-**map** that is a transformation between presheaves

$$\xi : \mathbf{oFs}_{loc}(-, R) \longrightarrow \mathbf{oFs}_{loc}(-, S)$$

$$a : V \to R \mapsto \xi_a : V_a \to S$$

that 'preserves' dimension of domains, k-domains, k-codomains, and k-tensors, i.e.

1. $dim(V_a) \leq dim(V)$;

2. $\xi(a \circ \mathbf{d}^{(k)}_V) = \xi(a) \circ \mathbf{d}^{(k)}_{V_a}$, similar for codomains

3. if $V = V^1 \otimes_k V^2$ then $V_a = V^1_{a \circ \kappa^1_{V^1}} \otimes_k V^2_{a \circ \kappa^2_{V^2}}$ and

$$\xi(a \circ \kappa^1) = \xi(a) \circ \bar{\kappa}^1, \qquad \xi(a \circ \kappa^2) = \xi(a) \circ \bar{\kappa}^2$$

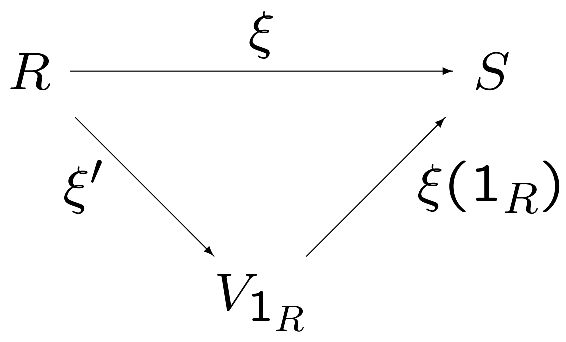
We have two embeddings

$$\mathbf{oFs}_{loc} \longrightarrow \mathbf{oFs}_{\omega} \longrightarrow \omega Cat$$

first is essentially surjective (defined by composition) and the second is full.

Let $\xi : R \to S$ be an ω-map. ξ is an **inner** ω-**map** iff $\xi(1_R) = 1_S$.

Proposition. *Every o-map $\xi : R \to S$ in $\mathbf{oFs}_{\omega}$ can be factored as a inner map followed by a local map.*



ξ is a **monotone ω-map** iff $\xi(1_R)$ is a monotone morphism.

$\mathbf{oFs}_\mu$ - the category of ordered face structures and monotone ω-maps

We have a commuting square of categories and functors

$$\begin{array}{ccccc}
\mathbf{pFs} & \longrightarrow & \mathbf{oFs} & \xrightarrow{\ \mathcal{G}_{\mathbf{oFs}}\ } & \mathbf{oFs}_{loc} \\
 & & \Big\downarrow{\scriptstyle \iota_\mu} & & \Big\downarrow{\scriptstyle \iota_\omega} \\
 & & \mathbf{oFs}_\mu & \xrightarrow[\ \mathcal{G}_{\mathbf{oFs}_\mu}\ ]{} & \mathbf{oFs}_\omega
\end{array}$$

All functors are essentially surjective embeddings. The vertical ones are full on isomorphisms, and the horizontal ones send tensor squares to pushouts.

Multitopic category (M.Makkai) = model of $\mathbf{oFs}$,

$$X : \mathbf{oFs}^{op} \to Set$$

in which every cell (pasting diagram) $\alpha \in X(N)$ with N normal has a composition $a \in X(P)$ with P principal, $P \| N$.

Thus we need to say what does it mean that a pasting diagram α do compose to a principal cell a.

This is expressed by saying that

the cell system $X(a)$ of 'cells going out' of a

is equivalent over X with

the cell system $X(\alpha)$ of 'cells going out' of α.

We fix $X : \mathbf{oFs}^{op} \to Set$, $\alpha \in X(N)$, $a \in X(P)$ such that $P \| N$, P principal of dimension k, N k-normal, for the rest of the talk.

The **composition ω-map**

$$m_N : P \to N$$

is the inner ω-map which sends 1_P to 1_N and is identity on lower dimension cell (=local maps).

We say that the morphism $f : P \to S$ in $\mathbf{oFs}$ is of **d-type** iff $f(P_k) \subseteq S_k - \gamma(S_{k+1})$.

Lemma. *(substitute N for P in S along f) If $m_N : P \to N$ is composition ω-map, $f : P \to S$ d-type morphisms then there is a pushout in* $\mathbf{oFs}_\mu$

$$\begin{array}{ccc} N & \xrightarrow{f_N} & N \diamond_f S \\ {\scriptstyle m_N}\uparrow & & \uparrow{\scriptstyle m_{N,f,S}} \\ P & \xrightarrow[f]{} & S \end{array}$$

with $m_{N,f,S}$ inner ω-map and f_N monotone.

$P \Downarrow \mathbf{oFs}$ - the category of P-pointed shapes is defined as follows.

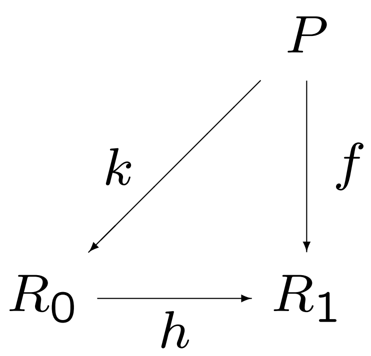
There are two kinds of objects

- the *non-pointed* objects are the objects of $\mathbf{oFs}$
- the *pointed* objects are d-type morphisms from P to objects of $\mathbf{oFs}$

There are three kinds of morphisms

- between non-pointed objects are the usual morphisms in $\mathbf{oFs}$,
- between pointed objects are the morphisms in comma category $P \downarrow \mathbf{oFs}$,

- from the non-pointed objects to pointed object are 'monotone maps that omit the point', i.e. for $R_0 \in \mathbf{oFs}$ and d-type morphism $f : P \to R_1$ in $P \downarrow_d \mathbf{oFs}$, the monotone morphism $h : R_0 \to R_1$ is a morphism in $P \Downarrow \mathbf{oFs}$ if f does not factorize through h, i.e. there is no local map k making the triangle



commutes.

- There are no morphisms from pointed objects to non-pointed objects in $P \Downarrow \mathbf{nFs}_{loc}$.

We have an embedding on non-pointed objects

$$\iota_P : \mathbf{oFs} \longrightarrow P \Downarrow \mathbf{oFs}$$

'cells out of a (α) in X'

$$X_a,\, X_\alpha : P \Downarrow \mathbf{oFs} \longrightarrow Set$$

The functors X_a, X_α agree with X on $\mathbf{oFs}$ part of $P \Downarrow \mathbf{oFs}$, and on d-type morphisms it picks those cells that have a, α in the specified place. More specifically, for $S \in \mathbf{oFs}$

$$X_a(S) = X(S) = X_\alpha(S)$$

for $f : P \to S \in P \Downarrow \mathbf{oFs}$

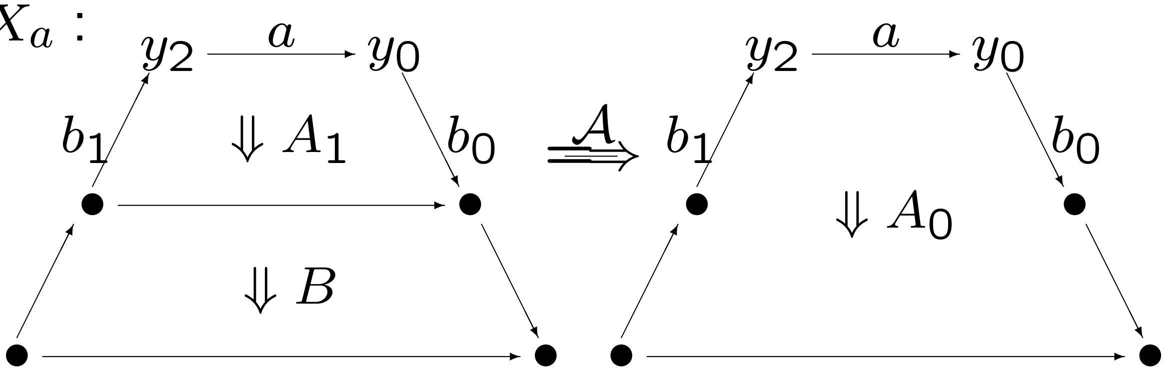
$$X_a(f) = X(f)^{-1}(\{a\}) \subseteq X(S)$$

$$X_\alpha(f) = X(f_N)^{-1}(\{\alpha\}) \subseteq X(N \diamond_f S)$$

recall that $f_N : N \longrightarrow N \diamond_f S$ is a monotone morphism.

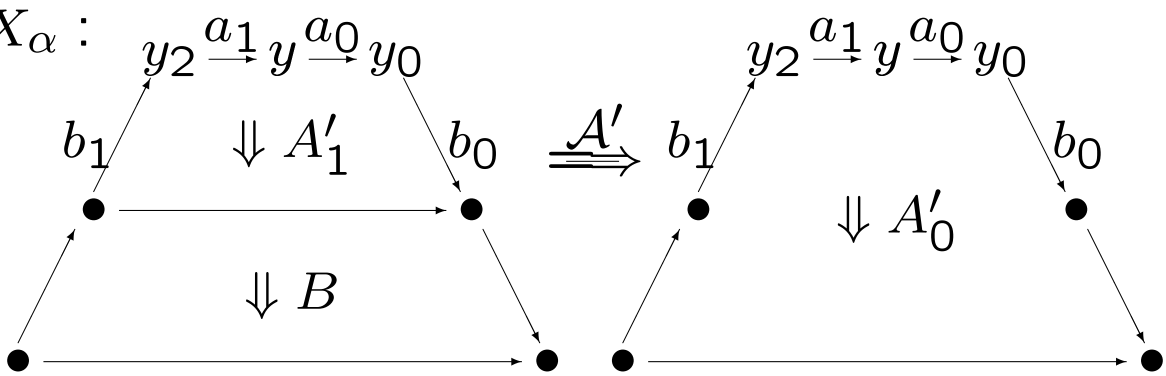
Example

X_a :

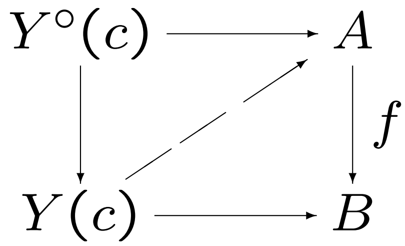


$$\alpha : \quad y_2 \xrightarrow{a_1} y \xrightarrow{a_0} y_0$$

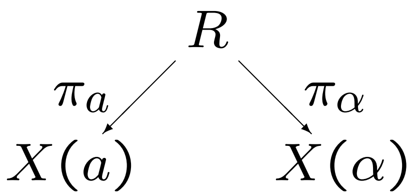
X_α :



A functor $f : A \to B \in Set^{\mathcal{C}^{op}}$ is **fiberwise surjective** iff for any $c \in \mathcal{C}$



$X(a) \simeq_X X(\alpha)$ iff there is a **strategy** i.e. a span of fiberwise surjective functors



that restricts to a commuting diagram