# Compound Poisson approximation for simple transient random walks in random sceneries


Nicolas Chenavier *,1, Ahmad Darwiche †,1, and Arnaud Rousselle ‡,2

[1]Université du Littoral Côte d'Opale, UR 2597, LMPA, Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, 62100 Calais, France.
[2]Institut de Mathématiques de Bourgogne, UMR 5584, CNRS, Université Bourgogne Franche-Comté, F-21000 Dijon, France.


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**Abstract**

Given a simple transient random walk $(S_n)_{n\geq 0}$ in $\mathbb{Z}$ and a stationary sequence of real random variables $(\xi(s))_{s\in\mathbb{Z}}$, we investigate the extremes of the sequence $(\xi(S_n))_{n\geq 0}$. Under suitable conditions, we make explicit the extremal index and show that the point process of exceedances converges to a compound Poisson point process. We give two examples for which the cluster size distribution can be made explicit.



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## 1 Introduction

Extreme Value Theory (EVT) deals with rare events and has many applications in various domains such as hydrology [16], finance [10] and climatology [25]. It was first introduced in the context of independent and identically distributed (i.i.d.) random variables. It is straightforward that if $(\xi(s))_{s\in\mathbb{Z}}$ is a sequence of i.i.d. random variables then the following property holds: for any sequence of real numbers $(u_n)_{n\geq 0}$, and for $\tau > 0$,

$$n\,\mathbb{P}\left(\xi(0) > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau \Longrightarrow \mathbb{P}\left(\max_{0\leq k\leq n} \xi(k) \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau}.$$

The above property has been extended for sequences of dependent random variables satisfying two conditions. The first one, referred to as the $D(u_n)$ condition of Leadbetter, is a long range dependence property and the second one, known as the $D'(u_n)$ condition, ensures that, locally, there is no clusters of exceedances (see [19] for a statement of these conditions).


*nicolas.chenavier@univ-littoral.fr
†ahmad.darwiche@univ-littoral.fr
‡arnaud.rousselle@u-bourgogne.fr.

In 2009, Franke and Saigo [11, 12] considered the following problem. Let $(X_i)_{i\geq 1}$ be a sequence of centered, integer-valued i.i.d. random variables and let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Assume that $(X_i)_{i\geq 1}$ is in the domain of attraction of a stable law, i.e. for each $x \in \mathbb{R}$,

$$\mathbb{P}\left(n^{-\frac{1}{\alpha}} S_n \leq x\right) \underset{n\to\infty}{\longrightarrow} F_\alpha(x),$$

where $F_\alpha$ is the distribution function of a stable law with characteristic function given by

$$\varphi(\theta) = \exp(-|\theta|^\alpha (C_1 + iC_2 \mathrm{sgn}\theta)),\ \alpha \in (0, 2].$$

When $\alpha < 1$ (resp. $\alpha > 1$), it is known that the random walk $(S_n)_{n\geq 0}$ is transient (resp. recurrent) [17, 18]. Now, let $(\xi(s))_{s\in\mathbb{Z}}$ be a family of $\mathbb{R}$-valued i.i.d. random variables independent of the sequence $(X_i)_{i\geq 0}$. The sequence $(\xi(S_n))_{n\geq 0}$ is called a *random walk in a random scenery*. Such a concept was first introduced by Kesten and Spitzer [17] who established limit theorems on the sum of the first $n$ terms and was extensively investigated in various directions, see e.g. [5, 6] and the survey [15]. In [12], Franke and Saigo derive limit theorems for the maximum of the first $n$ terms of $(\xi(S_n))_{n\geq 0}$ as $n$ goes to infinity. An adaptation of Theorem 1 in [12] shows that in the transient case, i.e. $\alpha < 1$, the following property holds: if $n\,\mathbb{P}\,(\,\xi(0) > u_n\,) \underset{n\to\infty}{\longrightarrow} \tau$ for some sequence $(u_n)_{n\geq 0}$ and for some $\tau > 0$, then

$$\mathbb{P}\left(\max_{0\leq k\leq n} \xi(S_k) \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-q\tau}, \tag{1.1}$$

where

$$q = \mathbb{P}\,(\,S_i \neq 0, \forall i \geq 1\,)\,. \tag{1.2}$$

Notice that $q > 0$ because the random walk $(S_n)_{n\geq 0}$ is transient. The term $q$ can also be expressed as (see e.g. [18])

$$q = \lim_{n\to\infty} \frac{R_n}{n} \quad \text{a.s.}, \tag{1.3}$$

where $R_n = \#\{S_0, \ldots, S_n\}$ is the *range* of the random walk. The result (1.1) was recently extended to a random scenery which is not necessarily based on i.i.d. random variables but on a sequence satisfying a slight modification of the $D(u_n)$ condition [2]. One of the difficulties is that the sequence $(\xi(S_n))_{n\geq 0}$ does not satisfy the $D'(u_n)$ condition and clusters of exceedances can appear. One way to define clusters is based on the runs scheme: given a sequence $(k_n)_{n\geq 0}$ such that $k_n \to \infty$ and $k_n = o(n)$, two exceedances, i.e. two points of the form $i/n, j/n$ with $\xi(S_i) > u_n$ and $\xi(S_j) > u_n$, $i, j \leq n$, are said to belong to the same cluster if $|i - j| \leq k_n$ (see e.g. [23]).

In this paper, we establish more precise results on the long range dependence and on the so-called point process of exceedances, i.e. on the family of points $i/n$ such that $\xi(S_i)$ exceeds $u_n$. Such a point process is a classical object in EVT (see e.g. [13]) since it counts the number of occurrences of extreme events. To do it, we assume some additional properties on the random walk and on the random scenery. First, we let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$, where $(X_i)_{i\geq 1}$ is a family of independent random variables with distribution $\mathbb{P}\,(\,X_i = 1\,) = p$ and $\mathbb{P}\,(\,X_i = -1\,) = 1 - p$, $p \in ]0, 1[\setminus\{\frac{1}{2}\}$. Notice that, as opposed to [2] and [12], the random variable $X_i$ is not centered. In what follows, the term $q$ appearing in (1.2) also satisfies (1.3). It can easily be proved that $q = |2p - 1|$. Secondly, we assume that the random scenery $(\xi(s))_{s\in\mathbb{Z}}$ satisfies a long range dependence property, which is referred to as the $\Delta(u_n)$ condition. To state it, we give some notation. For each $n, m_1, m_2$ with $0 \leq m_1 \leq m_2 \leq n$, define $\mathcal{B}_{m_1}^{m_2}(u_n)$ as the $\sigma$-field generated by events $\{\xi(s) \leq u_n\}$, $m_1 \leq s \leq m_2$, where $(u_n)_{n\geq 0}$ is some sequence of positive numbers. Also for each $n$ and $1 \leq \ell \leq n - 1$, write

$$\alpha_{n,\ell} = \max_{1\leq k\leq n-\ell}\ \max_{A\in\mathcal{B}_0^k(u_n), B\in\mathcal{B}_{k+\ell}^n(u_n)} \{|\,\mathbb{P}\,(\,A \cap B\,) - \mathbb{P}\,(\,A\,)\,\mathbb{P}\,(\,B\,)\,|\}.$$

We are now prepared to state the $\Delta(u_n)$ condition.

**Definition 1** *We say that the stationary sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\Delta(u_n)$ condition if there exists some sequence $(\ell_n)_{n\geq 0}$ such that $\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$ and $\ell_n = o(n)$.*

The above condition is slightly more restrictive than the $D(u_n)$ condition and was introduced by Hsing, Hüsler and Leadbetter [14] in the context of stationary sequence of random variables indexed by the set of *positive* integers. As an example, any stationary sequence $(\xi(s))_{s\in\mathbb{Z}}$ which is $\alpha$-mixing, i.e. such that

$$\sup_{(A,B)\in\mathcal{F}_{-\infty}^{0}\times\mathcal{F}_{\ell}^{\infty}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \underset{\ell\to\infty}{\longrightarrow} 0,$$

where $(\mathcal{F}_j^L)_{j\leq L}$ is the natural filtration of $(\xi(s))_{s\in\mathbb{Z}}$, satisfies the $\Delta(u_n)$ condition and therefore the $D(u_n)$ condition. This includes, for instance, $k$-dependent sequences, irreducible ergodic Markov chains, ARMA models and Gibbs processes (see Chapter 2 in [7] for various examples and [1] for a survey on mixing conditions).

**The extremal index** Assume from now on that, for any $\tau > 0$, there exists a threshold $u_n = u_n^{(\tau)}$ such that

$$n\,\mathbb{P}(\xi(0) > u_n) \underset{n\to\infty}{\longrightarrow} \tau. \tag{1.4}$$

The existence of the threshold $u_n = u_n^{(\tau)}$ is ensured when $\lim_{x\to x_f} \frac{\overline{F}(x)}{\overline{F}(x-)} = 1$, where $x_f = \sup\{u : F(u) < 1\}$, $F(u) = \mathbb{P}(\xi(0) \leq u)$ and $\overline{F} = 1 - F$ (see Theorem 1.1.13 in [20]).

To state our first main result, we recall the $D^{(k)}(u_n)$ condition, introduced by Chernick, Hsing and McCormick [4].

**Definition 2** *Let $k \geq 1$. Assume that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\Delta(u_n)$ condition. We say that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $D^{(k)}(u_n)$ condition if there exist sequences of integers $(s_n)_{n\geq 0}$ and $(\ell_n)_{n\geq 0}$ such that $s_n \underset{n\to\infty}{\longrightarrow} \infty$, $s_n\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$, $s_n\ell_n/n \underset{n\to\infty}{\longrightarrow} 0$, and*

$$\lim_{n\to\infty} n\,\mathbb{P}(\xi(0) > u_n \geq M_{1,k-1}, M_{k,r_n} > u_n) = 0,$$

*where $M_{i,j} = -\infty$ for $i > j$, $M_{i,j} = \max_{0\leq i\leq t\leq j} \xi(t)$ for $i \leq j$, and $r_n = \lfloor n/s_n \rfloor$.*

As noticed in [4], the $D'(u_n)$ condition is slightly more restrictive than the $D^{(1)}(u_n)$ condition. Observe that the $D^{(k)}(u_n)$ condition is satisfied when the sequence $(\xi(s))_{s\in\mathbb{Z}}$ is $k$-dependent. Recall also that the (stationary) sequence $(\xi(s))_{s\in\mathbb{Z}}$ has an *extremal index* $\sigma \in [0,1]$ if, in conjunction to (1.4), we have

$$\mathbb{P}\left(\max_{0\leq s\leq n} \xi(s) \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\sigma\tau},$$

for any $\tau > 0$. The extremal index can be interpreted as the reciprocal of the mean size of a cluster of exceedances. According to Corollary 1.3. in [4], under the assumptions that the $\Delta(u_n)$ and $D^{(k)}(u_n)$ conditions hold for $u_n = u_n^{(\tau)}$ for any $\tau > 0$, the extremal index exists and is equal to $\sigma$ if and only if $\mathbb{P}(M_{1,k} \leq u_n | \xi(0) > u_n) \underset{n\to\infty}{\longrightarrow} \sigma$ for any $\tau > 0$. In particular, when the $D'(u_n)$ condition is satisfied, the extremal index exists and is equal to $\sigma = 1$ (Theorem 1.2. in [19]).

The following proposition ensures that, under suitable conditions, the extremal index of the sequence $(\xi(S_n))_{n\geq 0}$ exists and can be made explicit.

**Proposition 1** *Let $(\xi(s))_{s\in\mathbb{Z}}$ be a stationary sequence satisfying the $D^{(k)}(u_n)$ condition with $k \geq 1$, for $u_n = u_n^{(\tau)}$ for any $\tau > 0$ and such that $(\xi(S_n))_{n\geq 0}$ satisfies the $\Delta(u_n)$ condition. Assume that the extremal index of $(\xi(s))_{s\in\mathbb{Z}}$ exists and is equal to $\sigma \in (0,1]$, i.e.*

$$n\,\mathbb{P}\left(\,\xi(0) > u_n\,\right) \underset{n\to\infty}{\longrightarrow} \tau \quad \text{and} \quad \mathbb{P}\left(\max_{0\leq s\leq n}\{\xi(s)\} \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\sigma\tau}.$$

*Then the sequence $(\xi(S_n))_{n\geq 0}$ admits an extremal index which is equal to $\theta = \sigma q$, where $q$ is as in (1.2).*

In other words, Proposition 1 claims that $\mathbb{P}\left(\max_{0\leq k\leq n}\xi(S_k) \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\sigma q\tau}$.

**The $D^{(k)}(u_n)$ condition** In [12], Franke and Saigo proved that, when the $\xi(s)$'s are i.i.d., the sequence $(\xi(S_n))_{n\geq 0}$ does not satisfy the $D'(u_n)$ condition. Since the $D^{(1)}(u_n)$ condition is slightly less restrictive than the $D'(u_n)$ condition and since Equation (1.5) (as stated below) is satisfied when the $\xi(s)$'s are i.i.d., the following result can be compared to Proposition 3 in [12].

**Proposition 2** *Assume that $(\xi(s))_{s\in\mathbb{Z}}$ is a stationary sequence satisfying*

$$\limsup_{n\to\infty} n\mathbb{P}\Big(\xi(0) > u_n \text{ and there exists } s \in \{-k+1,\ldots,k-1\}\setminus\{0\} \text{ s.t. } \xi(s) > u_n\Big) < \tau. \quad (1.5)$$

*Then $(\xi(S_n))_{n\geq 0}$ does not satisfy the $D^{(k)}(u_n)$ condition for any $k \geq 1$.*

**The point process of exceedances** A point process in $[0,1]$ is a random variable in the space $\mathbf{N}$ of all finite counting measures on $[0,1]$. The space $\mathbf{N}$ is endowed to the corresponding $\sigma$-algebras that are induced by the mappings $\omega \mapsto \omega(B)$ for all Borel subset $B$ in $[0,1]$ (see e.g. Section 2.1 in [21]).

Let $\tau > 0$ and $n \geq 1$. The *point process of exceedances* is defined as

$$\Phi_n(B) = \Phi_n^{(\tau)}(B) = \sum_{i\leq n} \mathbb{I}_{\xi(S_i)>u_n}\,\delta_{i/n}(B), \quad (1.6)$$

for any Borel subset $B \subset [0,1]$. With a slight abuse of notation, we identify $\Phi_n$ to its support, i.e. the random (closed) subset $\left\{\frac{i}{n} : \xi(S_i) > u_n, i \leq n\right\} \subset [0,1]$. Now, let $(k_n)_{n\geq 0}$ be a sequence of positive integers with $k_n \underset{n\to\infty}{\longrightarrow} \infty$ and $k_n = o(n)$. In what follows, for any $j \geq 1$, we let

$$p_n(j) = p_n^{(\tau)}(j) = \mathbb{P}\left(\,\#\Phi_{k_n} = j|\xi(0) > u_n\,\right),$$

where $\Phi_{k_n}$ is defined in the same spirit as $\Phi_n$ by considering this time integers $i \leq k_n$, with the abuse of notation $\#\Phi_{k_n} := \Phi_{k_n}([0,1])$.

Recall that a *compound Poisson point process* (see Section 15.1 in [21]) in $[0,1]$ of intensity $\lambda > 0$ and cluster size distribution $\pi = (\pi_i)_{i\geq 1}$ is a point process $\Phi$ of the form

$$\Phi(B) = \sum_{i\geq 1} i\zeta(B \times \{i\}),$$

where $\zeta$ is a Poisson point process in $[0,1]\times\mathbb{N}^*$ with intensity measure given by $\mathbb{E}\left[\,\zeta(B\times\{i\})\,\right] = \lambda|B|\pi_i$, for any Borel subset $B \subset [0,1]$ and $i \in \mathbb{N}^*$. Intuitively, $\pi_i$ is the measure for clumps of size $i$. The point process $\Phi$ can be identified to the random (closed) subset $\{(x_j, n_j), j \geq 1\} \subset [0,1]\times\mathbb{N}^*$, where $\{x_j, j \geq 1\}$ is a stationary Poisson point process in $[0,1]$ of intensity $\lambda$ and where $(n_j)_{j\geq 1}$ is a family of i.i.d. random variables with distribution $\pi$ which is independent of the $x_j$'s. The following proposition states that, under suitable conditions, the point process of exceedances converges to a compound Poisson point process.

**Proposition 3** *Let $(\xi(s))_{s\in\mathbb{Z}}$ be a stationary sequence satisfying the $D^{(k)}(u_n)$ condition with $k \geq 1$, for $u_n = u_n^{(\tau)}$ for any $\tau > 0$ and such that $(\xi(S_n))_{n\geq 0}$ satisfies the $\Delta(u_n)$ condition. Assume that $p_n^{(\tau_0)}(j)$ converges to some number $p(j)$ for any $j \geq 1$ and for some $\tau_0$. Then the point process $\Phi_n^{(\tau)}$ converges in distribution to a compound Poisson point process with intensity $\theta\tau$, where $\theta = \sigma q$, and cluster size distribution $\pi(j) = \frac{1}{\theta}(p(j) - p(j+1))$, for any $\tau > 0$.*

The above proposition is classical in EVT and is a simple application of Theorem 2.5 in [22] and Theorem 5.1 in [14]. Informally, it claims that asymptotically a cluster of exceedances is identified to a point of the compound Poisson point process and that the number of exceedances in the cluster is a random variable which is distributed w.r.t. $\pi$. In a different context, asymptotic results on point processes associated with extremes in random sceneries are also established in [3].

Our paper is organized as follows. In Section 2, we prove Propositions 1-3. In Section 3, we give some examples illustrating Proposition 3. In particular, we make explicit the cluster size distribution of the limiting point process of exceedances. In section 4, we shortly discuss possible ways to ensure that $(\xi(S_n))_{n\geq 0}$ satisfies the $\Delta(u_n)$ condition to apply Propositions 1 and 3.

# 2 Proofs of the main results

## 2.1 Proof of Proposition 1

Let $R_n = \#\{S_0, \dots, S_n\}$ be the range associated with the random walk $(S_n)_{n\geq 0}$. Then

$$\mathbb{P}\left(\max_{0\leq i\leq n} \xi(S_i) \leq u_n\right) = \mathbb{E}\left[\mathbb{P}\left(\max_{1\leq s\leq R_n}\{\xi(s)\} \leq u_n | R_n\right)\right].$$

Moreover,

$$\begin{aligned}\left|\mathbb{P}\left(\max_{1\leq s\leq R_n}\{\xi(s)\} \leq u_n | R_n\right) - \mathbb{P}\left(\max_{1\leq s\leq \lfloor qn\rfloor}\{\xi(s)\} \leq u_n\right)\right| &\leq 2\,\mathbb{P}\left(\exists s \in (R_n, \lfloor qn\rfloor) : \xi(s) > u_n\right)\\ &\leq 2|R_n - \lfloor qn\rfloor|\,\mathbb{P}\left(\xi(0) > u_n\right),\end{aligned}$$

where $(R_n, \lfloor qn\rfloor)$ denotes the interval with (non-necessarily ordered) extremities $R_n$ and $\lfloor qn\rfloor$. According to (1.3) and (1.4), we deduce that

$$\mathbb{P}\left(\max_{1\leq s\leq R_n}\{\xi(s)\} \leq u_n | R_n\right) - \mathbb{P}\left(\max_{1\leq s\leq \lfloor qn\rfloor}\{\xi(s)\} \leq u_n\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, to prove that $(\xi(S_n))_{n\geq 0}$ has an extremal index which is equal to $\theta = \sigma q$, it is sufficient to prove that

$$\mathbb{P}\left(\max_{1\leq s\leq \lfloor qn\rfloor}\{\xi(s)\} \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\sigma q\tau}. \tag{2.1}$$

We prove below (2.1).

When $k=1$, the identity follows from Corollary 1.3 in [4] which also shows that the extremal index is $\sigma = 1$. Assume from now on that $k \geq 2$. Because $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $D^{(k)}(u_n)$ condition, it follows from Corollary 1.3 in [4] that

$$\mathbb{P}\left( \max_{1\leq s\leq k-1}\{\xi(s)\} \leq u_n | \xi(0) > u_n \right) \underset{n\to\infty}{\longrightarrow} \sigma.$$

In particular

$$\mathbb{P}\left( \max_{1\leq s\leq k-1}\{\xi(s)\} \leq u_n^{(q)} | \xi(0) > u_n^{(q)} \right) \underset{n\to\infty}{\longrightarrow} \sigma,$$

where $u_n^{(q)} = u_{\lfloor n/q \rfloor}$. Observe that $(\xi(s))_{s\in\mathbb{Z}}$ also satisfies the $\Delta(u_n^{(q)})$ and $D^{(k)}(u_n^{(q)})$ condition. Because $n\,\mathbb{P}\left( \xi(0) > u_n^{(q)} \right) \underset{n\to\infty}{\longrightarrow} q\tau$, it follows again from Corollary 1.3 in [4] that

$$\mathbb{P}\left( \max_{1\leq s\leq n}\{\xi(s)\} \leq u_n^{(q)} \right) \underset{n\to\infty}{\longrightarrow} e^{-\sigma q \tau}.$$

Taking $n = \lfloor qn' \rfloor$, we deduce that

$$\mathbb{P}\left( \max_{1\leq s\leq \lfloor qn' \rfloor}\{\xi(s)\} \leq u_{\lfloor qn' \rfloor}^{(q)} \right) \underset{n'\to\infty}{\longrightarrow} e^{-\sigma q \tau}.$$

Because $\frac{n'}{\lfloor \lfloor qn' \rfloor / q \rfloor} \underset{n'\to\infty}{\longrightarrow} 1$, the latter expression gives

$$\mathbb{P}\left( \max_{1\leq s\leq \lfloor qn' \rfloor}\{\xi(s)\} \leq u_{n'} \right) \underset{n'\to\infty}{\longrightarrow} e^{-\sigma q \tau},$$

which proves (2.1).

## 2.2 Proof of Proposition 2

It is sufficient to prove that, for any sequence $(r_n)_{n\geq 0}$ with $r_n \underset{n\to\infty}{\longrightarrow} \infty$, we have

$$\liminf_{n\to\infty} n\,\mathbb{P}\left( \xi(S_0) > u_n \geq \tilde{M}_{1,k-1}, \tilde{M}_{k,r_n} > u_n \right) > 0, \tag{2.2}$$

where, similarly to Definition 2, we let $\tilde{M}_{i,j} = -\infty$ for $i > j$, $\tilde{M}_{i,j} = \max_{0\leq i\leq t\leq j} \xi(S_t)$ for $i \leq j$.

Assume that $k$ is even. We have

$$\begin{aligned} n\,\mathbb{P}\left( \xi(S_0) > u_n \geq \tilde{M}_{1,k-1}, \tilde{M}_{k,r_n} > u_n \right) & \\ &\geq n\,\mathbb{P}\left( \xi(0) > u_n \geq \tilde{M}_{1,k-1}, S_k = 0 \right) \\ &= n\,\mathbb{P}\left( \xi(0) > u_n, S_k = 0 \right) - n\,\mathbb{P}\left( \xi(0) > u_n, \tilde{M}_{1,k-1} > u_n, S_k = 0 \right). \end{aligned} \tag{2.3}$$

First, because $(\xi(s))_{s\in\mathbb{Z}}$ and $(S_n)_{n\geq 0}$ are independent, we obtain from (1.4) that

$$n\,\mathbb{P}\left( \xi(0) > u_n, S_k = 0 \right) \underset{n\to\infty}{\sim} \tau\,\mathbb{P}\left( S_k = 0 \right), \tag{2.4}$$

where $\mathbb{P}(S_k = 0) \neq 0$ since $k$ is even. Secondly, we have

$$n\,\mathbb{P}\Big(\xi(0) > u_n, \tilde{M}_{1,k-1} > u_n, S_k = 0\Big)$$
$$= n\,\mathbb{P}\left(\{\xi(0) > u_n, S_k = 0\} \cap \bigcup_{1\leq i\leq k-1}\{S_i = 0\}\right)$$
$$+ n\,\mathbb{P}\left(\Big\{\xi(0) > u_n, \tilde{M}_{1,k-1} > u_n, S_k = 0\Big\} \cap \bigcap_{1\leq i\leq k-1}\{S_i \neq 0\}\right). \quad (2.5)$$

The first term of the right hand-side of (2.5) is equal to

$$n\,\mathbb{P}(\xi(0) > u_n)\,\mathbb{P}\left(\{S_k = 0\} \cap \bigcup_{1\leq i\leq k-1}\{S_i = 0\}\right) \underset{n\to\infty}{\sim} \tau\,\mathbb{P}\left(\{S_k = 0\} \cap \bigcup_{1\leq i\leq k-1}\{S_i = 0\}\right).$$

To deal with the second term of the right hand-side of (2.5), observe that

$$n\,\mathbb{P}\left(\Big\{\xi(0) > u_n, \tilde{M}_{1,k-1} > u_n, S_k = 0\Big\} \cap \bigcap_{1\leq i\leq k-1}\{S_i \neq 0\}\right)$$
$$\leq n\,\mathbb{P}(\xi(0) > u_n, \exists s \in \{-k+1, \ldots, k-1\} \setminus \{0\} \text{ s.t. } \xi(s) > u_n)$$
$$\times\,\mathbb{P}\left(\bigcap_{1\leq i\leq k-1}\{S_i \neq 0\} \cap \{S_k = 0\}\right).$$

According to (1.5), it follows that

$$\limsup_{n\to\infty} n\,\mathbb{P}\left(\Big\{\xi(0) > u_n, \tilde{M}_{1,k-1} > u_n, S_k = 0\Big\} \cap \bigcap_{1\leq i\leq k-1}\{S_i \neq 0\}\right)$$
$$< \tau\,\mathbb{P}\left(\bigcap_{1\leq i\leq k-1}\{S_i \neq 0\} \cap \{S_k = 0\}\right). \quad (2.6)$$

Then (2.3) - (2.6) implies (2.2) when $k$ is even.

In a similar way, we prove that the condition $D^{(k)}(u_n)$ is not satisfied when $k$ is odd by considering this time the event $\{S_{k+1} = 0\}$.

## 2.3 Proof of Proposition 3

First, notice that, according to Proposition 1, the extremal index of $(\xi(S_n))_{n\geq 0}$ exists and equals $\theta = \sigma q > 0$. Now, let $\tau_0$ be such that $p_n^{(\tau_0)}(j)$ converges to some number $p(j)$. In particular $p_n^{(\tau_0)}(j) - p_n^{(\tau_0)}(j-1) \underset{n\to\infty}{\longrightarrow} \theta\pi(j)$ for any $j \geq 1$, where $\pi(j) = \frac{1}{\theta}(p(j) - p(j+1))$. Such a property ensures that Equation (2.5) in [22] is satisfied. It follows from Theorem 2.5 in [22] that $\Phi_n^{(\tau_0)}$

converges to a point process $\Phi^{(\tau_0)}$ with Laplace transform

$$L_{\Phi^{(\tau_0)}}(f) := \mathbb{E}\left[\exp\left(-\sum_{x\in\Phi^{(\tau_0)}} f(x)\right)\right] = \exp\left(-\theta\tau_0\int_0^\infty (1-L(f(t)))\mathrm{d}t\right),$$

for any positive and measurable function $f$, where $L$ denotes the Laplace transform of $\pi$. In particular, $\Phi^{(\tau_0)}$ is a compound Poisson point process of intensity $\theta\tau_0$ with cluster size distribution $\pi$. Moreover, according to Theorem 5.1 in [14], the fact that $\Phi_n^{(\tau_0)}$ converges to $\Phi^{(\tau_0)}$ for some $\tau_0 > 0$ ensures that $\Phi_n^{(\tau)}$ converges to $\Phi^{(\tau)}$ for any $\tau > 0$. This concludes the proof of Proposition 3.

# 3 Examples

In this section, we give two examples illustrating Proposition 3. The second one extends the first one. Being slightly easier to establish, we have chosen to present Example 1 separately for sake of simplicity.

## 3.1 Example 1

Assume that the $\xi(s)$'s are i.i.d.. Let $N(0) = \#\{i \geq 0 : S_i = 0\}$ be the number of visits of the random walk $(S_n)_{n\geq 0}$ to site 0 and recall that

$$q = \mathbb{P}(\, S_i \neq 0,\ \forall i \geq 1\,) = \mathbb{P}(\, N(0) = 0\,).$$

Notice that $q \in (0,1)$ since the random walk is transient ($p \neq 1/2$) and that $N(0)$ has a geometric distribution with parameter $q$. Moreover, since the $\xi(s)$'s are i.i.d., the extremal index of $(\xi(s))_{s\in\mathbb{Z}}$ equals 1. Thus, according to Proposition 1, the extremal index of $(\xi(S_n))_{n\geq 0}$ exists and is equal to $\theta = q$.

Now, let $\Phi$ be a compound Poisson point process in $[0,1]$ with intensity $\theta\tau$ and cluster size distribution

$$\begin{aligned}\pi(j) &= \frac{\mathbb{P}(\, N(0) = j\,) - \mathbb{P}(\, N(0) = j+1\,)}{\theta} \\ &= q(1-q)^{j-1}.\end{aligned}$$

**Proposition 4** *For any $\tau > 0$, the point process of exceedances $\Phi_n$, as defined in* (1.6), *converges in distribution to $\Phi$.*

**Proof of Proposition 4.** Since the $\xi(s)$'s are i.i.d., the $\Delta(u_n)$ and the $D^{(1)}(u_n)$ conditions clearly hold for $(\xi(s))_{s\in\mathbb{Z}}$. Let us justify that $(\xi(S_n))_{n\geq 0}$ also satisfies the $\Delta(u_n)$ condition. By looking carefully at the proof of [9, Theorem 2.2], one can see that the argument, based on a suitable coupling, adapts *verbatim* - without assuming that the $\xi(s)$'s take their values in a finite set - to prove that $(\xi(S_n))_{n\geq 0}$ is $\alpha-$mixing. In particular, the $\Delta(u_n)$ condition holds for $(\xi(S_n))_{n\geq 0}$.

According to Proposition 3, it is sufficient to compute the cluster size distribution. First, let $(k_n)_{n\geq 0}$ be a family of integers such that $k_n \underset{n\to\infty}{\longrightarrow} \infty$ and $k_n = o(n)$ and let

$$\Phi_{k_n} = \left\{\frac{i}{n} : \xi(S_i) > u_n, 0 \leq i \leq k_n\right\}.$$

For any $j \geq 1$, recall that

$$p_n(j) = \mathbb{P}\left( \#\Phi_{k_n} = j | \xi(0) > u_n \right).$$

We proceed into two steps: first, we compute the limit of $p_n(j)$; then we compute $\pi(j)$.

*Step 1.* We write

$$p_n(j) = \mathbb{P}\left( \#\Phi_{k_n} \geq j | \xi(0) > u_n \right) - \mathbb{P}\left( \#\Phi_{k_n} \geq j+1 | \xi(0) > u_n \right).$$

Moreover,

$$\begin{aligned}\mathbb{P}\left( \#\Phi_{k_n} \geq j | \xi(0) > u_n \right) = \mathbb{P}\left( \#\Phi_{k_n} \geq j, N_{k_n}(0) \geq j | \xi(0) > u_n \right) \\ + \mathbb{P}\left( \#\Phi_{k_n} \geq j, N_{k_n}(0) < j | \xi(0) > u_n \right),\end{aligned}$$

where $N_{k_n}(0) = \#\{0 \leq i \leq k_n : S_i = 0\}$ is the number of visits in 0 until time $k_n$. We show below that the first term of the right-hand side converges to $\mathbb{P}\left( N(0) = j \right)$ and that the second term converges to 0. First, we notice that, conditional on the event $\{\xi(0) > u_n\}$, we have

$$\{\#\Phi_{k_n} \geq j\} \cap \{N_{k_n}(0) \geq j\} = \{N_{k_n}(0) \geq j\}.$$

Therefore

$$\begin{aligned}\mathbb{P}\left( \#\Phi_{k_n} \geq j, N_{k_n}(0) \geq j | \xi(0) > u_n \right) &= \mathbb{P}\left( N_{k_n}(0) \geq j | \xi(0) > u_n \right) \\ &= \mathbb{P}\left( N_{k_n}(0) \geq j \right) \\ &\underset{n\to\infty}{\longrightarrow} \mathbb{P}\left( N(0) \geq j \right),\end{aligned} \tag{3.1}$$

where the second line comes from the fact that the random walk is independent of the scenery. Moreover, writing $\mathcal{S}_{k_n} = \{S_0, \ldots, S_{k_n}\}$, we have

$$\begin{aligned}\mathbb{P}\left( \#\Phi_{k_n} \geq j, N_{k_n}(0) < j | \xi(0) > u_n \right) &\leq \mathbb{P}\left( \exists s \in \mathcal{S}_{k_n} \setminus \{0\} : \xi(s) > u_n | \xi(0) > u_n \right) \\ &= \frac{\mathbb{P}\left( \{\exists s \in \mathcal{S}_{k_n} \setminus \{0\} : \xi(s) > u_n\} \cap \{\xi(0) > u_n\} \right)}{\mathbb{P}\left( \xi(0) > u_n \right)} \\ &\leq \frac{\mathbb{E}\left[ \sum_{s \in \mathcal{S}_{k_n} \setminus \{0\}} \mathbb{P}\left( \xi(s) > u_n, \xi(0) > u_n \right) \right]}{\mathbb{P}\left( \xi(0) > u_n \right)} \\ &\leq \mathbb{P}\left( \xi(0) > u_n \right) \mathbb{E}\left[ \#\mathcal{S}_{k_n} \right],\end{aligned}$$

where the last line comes from the fact that the $\xi(s)$'s are i.i.d.. Moreover, according to [18], we know that $\frac{\#\mathcal{S}_{k_n}}{k_n} \underset{n\to\infty}{\longrightarrow} q$ a.s.. Thus, according to the Lebesgue's dominated convergence theorem (which can be applied since $\#\mathcal{S}_{k_n} \leq k_n$), we have $\mathbb{E}\left[ \frac{\#\mathcal{S}_{k_n}}{k_n} \right] \underset{n\to\infty}{\longrightarrow} q$. Because $\mathbb{P}\left( \xi(0) > u_n \right) \underset{n\to\infty}{\sim} \frac{\tau}{n}$, we have

$$\mathbb{P}\left( \#\Phi_{k_n} \geq j, N_{k_n}(0) < j | \xi(0) > u_n \right) = O\left( k_n/n \right),$$

which converges to 0 since $k_n = o(n)$. This together with (3.1) gives that

$$\mathbb{P}\left( \#\Phi_{k_n} \geq j | \xi(0) > u_n \right) \underset{n\to\infty}{\longrightarrow} \mathbb{P}\left( N(0) \geq j \right)$$

and consequently that

$$p_n(j) \underset{n\to\infty}{\longrightarrow} \mathbb{P}\left( N(0) \geq j \right) - \mathbb{P}\left( N(0) \geq j+1 \right) = \mathbb{P}\left( N(0) = j \right) =: p(j).$$

*Step 2.* We provide below an explicit formula for $\pi$. According to Theorem 4.1 in [24] (see also Theorem 2.5 in [22]), we have

$$p(j) = \theta \sum_{m=j}^{\infty} \pi(m).$$

Therefore

$$\pi(j) = \frac{p(j) - p(j+1)}{\theta} = \frac{\mathbb{P}\left( N(0) = j \right) - \mathbb{P}\left( N(0) = j+1 \right)}{\theta}.$$

□

**Remark 1** *It is known that, under suitable conditions, the extremal index can be interpreted as the reciprocal of the mean size of a cluster of exceedances, i.e.* $\theta^{-1} = \sum_{j=1}^{\infty} j\pi(j)$, *see e.g. [14]. Such an identity holds in the above example since* $\frac{1}{q} = \sum_{j=1}^{\infty} jq(1-q)^{j-1}$.

## 3.2 Example 2

Let $k \geq 0$. Assume that $\xi(s) = \max\{Y_s, Y_{s+1}, \ldots, Y_{s+k}\}$, where the family of random variables $(Y_s)_{s\in\mathbb{Z}}$ is assumed to be i.i.d.. For any subset $A \subset \mathbb{Z}$, let $N(A)$ be the number of visits of the random walk in $A$, i.e.

$$N(A) = \#\{i \geq 0 : S_i \in A\}.$$

The following proposition gives an explicit formula for the extremal index of $(\xi(S_n))_{n\geq 0}$ and for the cluster size distribution.

**Proposition 5** *The point process of exceedances* $\Phi_n$*, as defined in* (1.6)*, converges in distribution to a compound Poisson point process of intensity* $\theta\tau$*, with* $\theta = \frac{q}{k+1}$*, and cluster size distribution*

$$\pi(j) = \frac{1}{q} \sum_{A \ni 0 : |A| = k+1} \left( \mathbb{P}\left( N(A) = j \right) - \mathbb{P}\left( N(A) = j+1 \right) \right),$$

*for any* $\tau > 0$.

**Proof of Proposition 5.** Replacing the use of [9, Theorem 2.2] and its proof by the one of [8, Theorem 2], one can show that $(\xi(S_n))_{n\geq 0}$ satisfies the $\Delta(u_n)$ condition. Moreover, we notice that the extremal index of $(\xi(s))_{s\in\mathbb{Z}}$ is equal to $\sigma = \frac{1}{k+1}$. Moreover, because the $\xi(s)$'s are $k$-dependent, the sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $D^{(k)}(u_n)$. Therefore, according to Proposition 1, the extremal index of $(\xi(S_n))_{n\geq 0}$ exists and is equal to $\theta = \frac{q}{k+1}$.

Let us compute the cluster size distribution. We only deal with the case $k = 1$ since the general case can be dealt in a similar way. To do it, we proceed in the same spirit as in Section 3.1.

*Step 1.* First, we make explicit the limit of

$$p_n(j) = \mathbb{P}\left( \#\Phi_{k_n} \geq j | \xi(0) > u_n \right) - \mathbb{P}\left( \#\Phi_{k_n} \geq j+1 | \xi(0) > u_n \right).$$

We notice that

$$\mathbb{P}\left( \#\Phi_{k_n} \geq j | \xi(0) > u_n \right) = \frac{\mathbb{P}\left( \#\Phi_{k_n} \geq j, Y_0 > u_n \right)}{\mathbb{P}\left( \xi(0) > u_n \right)} + \frac{\mathbb{P}\left( \#\Phi_{k_n} \geq j, Y_1 > u_n \right)}{\mathbb{P}\left( \xi(0) > u_n \right)} - \frac{\mathbb{P}\left( \#\Phi_{k_n} \geq j, Y_0 > u_n, Y_1 > u_n \right)}{\mathbb{P}\left( \xi(0) > u_n \right)}. \quad (3.2)$$

To deal with the first term of the right-hand side, we write

$$\frac{\mathbb{P}\left(\#\Phi_{k_n} \geq j, Y_0 > u_n\right)}{\mathbb{P}\left(\xi(0) > u_n\right)} = \mathbb{P}\left(\#\Phi_{k_n} \geq j | Y_0 > u_n\right) \times \frac{\mathbb{P}\left(Y_0 > u_n\right)}{\mathbb{P}\left(\xi(0) > u_n\right)}.$$

Proceeding in the same spirit as in Section 3.1, we can prove that $\mathbb{P}\left(\#\Phi_{k_n} \geq j | Y_0 > u_n\right)$ converges to $\mathbb{P}\left(N(\{-1,0\}) \geq j\right)$. Moreover, because $\xi(0) = \max\{Y_0, Y_1\}$, where $Y_0$ and $Y_1$ are independent, and because $n\,\mathbb{P}\left(\xi(0) > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau$, we have $n\,\mathbb{P}\left(Y_0 > u_n\right) \underset{n\to\infty}{\longrightarrow} \frac{\tau}{2}$. Therefore,

$$\frac{\mathbb{P}\left(\#\Phi_{k_n} \geq j, Y_0 > u_n\right)}{\mathbb{P}\left(\xi(0) > u_n\right)} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\mathbb{P}\left(N(\{-1,0\}) \geq j\right).$$

In a similar way, we get

$$\frac{\mathbb{P}\left(\#\Phi_{k_n} \geq j, Y_1 > u_n\right)}{\mathbb{P}\left(\xi(0) > u_n\right)} \underset{n\to\infty}{\longrightarrow} \frac{1}{2}\mathbb{P}\left(N(\{0,1\}) \geq j\right).$$

Moreover, for the last term of (3.2), we have

$$\frac{\mathbb{P}\left(\#\Phi_{k_n} \geq j, Y_0 > u_n, Y_1 > u_n\right)}{\mathbb{P}\left(\xi(0) > u_n\right)} \leq \frac{\left(\mathbb{P}\left(Y_0 > u_n\right)\right)^2}{\mathbb{P}\left(\xi(0) > u_n\right)} \underset{n\to\infty}{\sim} \frac{\tau}{4n}.$$

This, together with (3.2), gives

$$p_n(j) \underset{n\to\infty}{\longrightarrow} p(j),$$

with

$$p(j) = \frac{1}{2}\left(\mathbb{P}\left(N(\{-1,0\}) = j\right) + \mathbb{P}\left(N(\{0,1\}) = j\right)\right).$$

*Step 2.* In the same spirit as in Section 3.1, we have $\pi(j) = \frac{p(j)-p(j+1)}{\theta}$, that is, since $q = 2\theta$

$$\pi(j) = \frac{1}{q}\Big(\mathbb{P}\left(N(\{-1,0\}) = j\right) - \mathbb{P}\left(N(\{-1,0\}) = j+1\right) + \mathbb{P}\left(N(\{0,1\}) = j\right) - \mathbb{P}\left(N(\{0,1\}) = j+1\right)\Big).$$

□

## 4 $\Delta(u_n)$ condition for $(\xi(S_n))_{n\geq 0}$ and related questions

In the examples provided in Section 3, the field $(\xi(s))_{s\in\mathbb{Z}}$ has a weak dependence property and $(\xi(S_n))_{n\geq 0}$ gets stronger mixing properties than the one required in order to apply Propositions 1 and 3. One may wonder how to proceed to verify that $(\xi(S_n))_{n\geq 0}$ satisfies the $\Delta(u_n)$ condition if the field $(\xi(s))_{s\in\mathbb{Z}}$ has a stronger dependence property. In this section, we state a (theoretical) result which ensures that $(\xi(S_n))_{n\geq 0}$ satisfies the $\Delta(u_n)$ condition and give some open questions.

**Proposition 6** *Let $(S_n)_{n\geq 0}$ be a simple transient random walk, i.e. $p \neq 1/2$ and let $(u_n)_{n\geq 0}$ be a sequence of positive integers. Assume that the following conditions hold.*

*(i) The sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\Delta(u_n)$ condition.*

*(ii)* *For some* $\beta \in (1/2, 1)$,

$$\max_{1\le k\le n-\tilde{\ell}_n}\ \max_{A\in\tilde{\mathcal{B}}_0^k(u_n),B\in\tilde{\mathcal{B}}^n_{k+\tilde{\ell}_n}(u_n)} \{|\mathbb{E}\left[\mathbb{P}\left(A|\mathcal{S}_{0:n}\right)\mathbb{P}\left(B|\mathcal{S}_{0:n}\right)\right] - \mathbb{P}\left(A\right)\mathbb{P}\left(B\right)|\} \underset{n\to\infty}{\longrightarrow} 0,$$

*where*

$$\tilde{\ell}_n = \left\lceil \frac{\ell_{2n+1} + 2n^\beta}{|2p-1|} \right\rceil. \tag{4.1}$$

*Then* $(\xi(S_n))_{n\ge 0}$ *satisfies the* $\Delta(u_n)$ *condition.*

In the above proposition, the notation $\tilde{\mathcal{B}}_0^n(u_n)$ stands for the $\sigma$-algebra generated by events of the form $\{\xi(S_i)\le u_n\}$, $0\le i\le n$, and $\mathcal{S}_{0:n}=\{S_0,\dots,S_n\}$.

From a practical point of view, it is difficult to apply Proposition 6 since it requires to deal with the conditional probabilites appearing in *(ii)*. However, due to the fact that the random walk has a drift, we think that this condition is unnecessary. A first open question is to determine if the assumption *(i)* is sufficient to ensure that $(\xi(S_n))_{n\ge 0}$ satisfies the $\Delta(u_n)$ condition. In the case where *(ii)* cannot be relaxed, another natural question is to apply Proposition 6 for examples which are different from those considered in this paper.

**Proof of Proposition 6.** Without loss of generality, we only deal with the case $p>1/2$. Let $(\ell_n)_{n\ge1}$ be such that $\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$ and $\ell_n = o(n)$. Let $\tilde{\ell}_n$ be as in (4.1), with $\beta\in(1/2,1)$. Notice that $\tilde{\ell}_n \underset{n\to\infty}{\longrightarrow}\infty$ and $\tilde{\ell}_n=o(n)$. Similarly to $\alpha_{n,\ell}$, we introduce the term

$$\tilde{\alpha}_{n,\ell} = \max_{1\le k\le n-\ell}\ \max_{A\in\tilde{\mathcal{B}}_0^k(u_n),B\in\tilde{\mathcal{B}}^n_{k+\ell}(u_n)} \{|\,\mathbb{P}\left(A\cap B\right) - \mathbb{P}\left(A\right)\mathbb{P}\left(B\right)|\},$$

where $\tilde{\mathcal{B}}_{m_1}^{m_2}(u_n)$ denotes the $\sigma$-field generated by events $\{\xi(S_i)\le u_n\}$, $m_1\le i\le m_2$. We prove below that $\tilde{\alpha}_{n,\tilde{\ell}_n} \underset{n\to\infty}{\longrightarrow} 0$. To do it, we will use the following lemma.

**Lemma 7** *Assume that* $p>1/2$. *Let us consider the following event:*

$$E_n = \bigcap_{k\le n-\tilde{\ell}_n} \left\{\max\{S_0,\dots,S_k\}\le k(2p-1)+n^\beta\right\}\cap\left\{\min\{S_{k+\tilde{\ell}_n},\dots,S_n\}\ge (k+\tilde{\ell}_n)(2p-1)-n^\beta\right\}.$$

*Then there exist two positive constants* $c_1$ *and* $c_2$ *such that* $\mathbb{P}\left(E_n^c\right)\le c_1 e^{-c_2 n^{2\beta-1}}$.

**Proof of Lemma 7.** It is sufficient to prove that the events

$$\bigcup_{k\le n-\tilde{\ell}_n}\left\{\max\{S_0,\dots,S_k\} > k(2p-1)+n^\beta\right\}$$

and

$$\bigcup_{k\le n-\tilde{\ell}_n}\left\{\min\{S_{k+\tilde{\ell}_n},\dots,S_n\} < (k+\tilde{\ell}_n)(2p-1)-n^\beta\right\}$$

occur with probability smaller than $c_1e^{-c_2n^{2\beta-1}}$. We only deal with the first one since the second one can be dealt in a similar way. To do it, we notice that

$$\begin{aligned}\mathbb{P}\left(\bigcup_{k\le n-\tilde{\ell}_n}\left\{\max\{S_0,\dots,S_k\} > k(2p-1)+n^\beta\right\}\right) &\le \sum_{k\le n-\tilde{\ell}_n}\sum_{i\le k}\mathbb{P}\left(S_i > k(2p-1)+n^\beta\right)\\ &\le \sum_{k\le n-\tilde{\ell}_n} k\,\mathbb{P}\left(S_k > k(2p-1)+n^\beta\right),\end{aligned}$$

where the last line comes from the fact that $S_i$ is stochastically dominated by $S_k$, with $i \leq k$, since $p > 1/2$. Now, let $k \leq n - \tilde{\ell}_n$ be fixed, and let $S'_k$ be a binomial random variable with parameter $(k, p)$, so that $S_k \overset{sto}{=} 2S'_k - k$. We have

$$\mathbb{P}\left( S_k > k(2p-1) + n^\beta \right) = \mathbb{P}\left( S'_k > kp + \frac{1}{2} n^\beta \right) \leq e^{-\frac{1}{2} \cdot \frac{n^{2\beta}}{k}},$$

according to the Hoeffding's inequality. Since $k \leq n$, the last term is lower than $e^{-\frac{1}{2} n^{2\beta - 1}}$. Summing over $k$, we get

$$\mathbb{P}\left( \bigcup_{k \leq n - \tilde{\ell}_n} \left\{ \max\{S_0, \ldots, S_k\} > k(2p-1) + n^\beta \right\} \right) \leq n^2 e^{-\frac{1}{2} n^{2\beta - 1}}.$$

This concludes the proof of Lemma 7. □

Lemma 7 ensures that the event $E_n$ occurs with high probability since $\beta > 1/2$. Now, let $k \leq n - \tilde{\ell}_n$ be fixed. Let us consider two events $A \in \tilde{\mathcal{B}}_0^k(u_n)$ and $B \in \tilde{\mathcal{B}}_{k+\tilde{\ell}_n}^n(u_n)$. First, according to Lemma 7, we notice that

$$\begin{aligned} \mathbb{P}\left( A \cap B \right) &= \mathbb{P}\left( A \cap B \cap E_n \right) + o(1) \\ &= \mathbb{E}\left[ \mathbb{I}_{E_n} \, \mathbb{P}\left( A \cap B | \mathcal{S}_{0:n} \right) \right] + o(1), \end{aligned}$$

where $o(1)$ only depends on $n$. Moreover,

$$\left| \mathbb{I}_{E_n} \, \mathbb{P}\left( A \cap B | \mathcal{S}_{0:n} \right) - \mathbb{I}_{E_n} \, \mathbb{P}\left( A | \mathcal{S}_{0:n} \right) \mathbb{P}\left( B | \mathcal{S}_{0:n} \right) \right| \leq \alpha_{2n+1, \ell_{2n+1}}. \tag{4.2}$$

Indeed, since $A \in \tilde{\mathcal{B}}_0^k(u_n)$ and since we are on $E_n$, conditional on $\mathcal{S}_{0:n}$, the event $A$ only depends on events of the form $\{\xi(s) \leq u_n\}$, with $s \leq k(2p-1) + n^\beta$. In the same way, the event $B$ only depends on events of the form $\{\xi(s) \leq u_n\}$, with $s \geq (k + \tilde{\ell}_n)(2p-1) - n^\beta$. Equation (4.2) follows since the difference between $k(2p-1) + n^\beta$ and $(k + \tilde{\ell}_n)(2p-1) - n^\beta$ is at least $\ell_{2n+1}$. Now, because $(\xi(s))_{s \in \mathbb{Z}}$ satisfies the $\Delta(u_n)$ condition, we know that $\alpha_{2n+1, \ell_{2n+1}}$ converges to 0 as $n$ goes to infinity. Thus, thanks again to Lemma 7, we have

$$\mathbb{P}\left( A \cap B \right) = \mathbb{E}\left[ \mathbb{P}\left( A | \mathcal{S}_{0:n} \right) \mathbb{P}\left( B | \mathcal{S}_{0:n} \right) \right] + o(1).$$

This together with assumption *(ii)* shows that $\tilde{\alpha}_{n, \tilde{\ell}_n} \underset{n \to \infty}{\longrightarrow} 0$ and concludes the proof of Proposition 6. □

# References


[1] Bradley, R. Basic Properties of Strong Mixing Conditions. A Survey and Some Open Questions. *Probability Surveys.* **2**, 107–144 (2005).

[2] Chenavier, N. & Darwiche, A. Extremes for transient random walks in random sceneries under weak independence conditions. *Statist. Probab. Lett.*. **158** pp. 108657, 6 (2020).

[3] Chenavier, N. & Darwiche, A. Some properties on extremes for transient random walks in random sceneries. *Available In Https://arxiv.org/pdf/2210.04854.pdf* (2022)

[4] Chernick, M., Hsing, T. & McCormick, W. Calculating the extremal index for a class of stationary sequences. *Adv. In Appl. Probab.*. **23**, 835-850 (1991).

[5] Deuschel, J. D. & Fukushima, R. Quenched tail estimate for the random walk in random scenery and in random layered conductance. *Stoch. Proc. Appl..* **129**(1), 102–128 (2019).

[6] Dombry, C. & Guillotin-Plantard, N. A functional approach for random walks in random sceneries. *Electron. J. Probab..* **14**, 1495–1512 (2009).

[7] Doukhan, P. Mixing. Properties and examples. *Lect. Notes Stat., Springer-Verlag, New York.* **85** (1994).

[8] Hollander, F., Keane, M., Serafin, J. & Steif, J. Weak Bernoullicity of random walk in random scenery. *Japan. J. Math. (N.S.).* **29**, 389-406 (2003).

[9] Hollander, F. & Steif, J. Mixing properties of the generalized $T, T^{-1}$-process. *J. Anal. Math..* **72** pp. 165-202 (1997).

[10] Embrechts, P., Klüppelberg, C. & Mikosch, T. Modelling extremal events, volume 33 of Applications of Mathematics (New York). *Springer-Verlag*, Berlin, For insurance and finance (1997).

[11] Franke, B. & Saigo, T. The extremes of a random scenery as seen by a random walk in a random environment. *Statist. Probab. Lett..* **79**, 1025-1030 (2009).

[12] Franke, B. & Saigo, T. The extremes of random walks in random sceneries. *Advances In Applied Probability.* **41**, 452-468 (2009).

[13] Freitas, A. C. M., Freitas, J. M. & Todd, M. The compound Poisson limit ruling periodic extreme behaviour of non-uniformly hyperbolic dynamics. *Comm. Math. Phys..* **2**, 483–527 (2013).

[14] Hsing, T., Hüsler, J. & Leadbetter, M. On the exceedance point process for a stationary sequence. *Probab. Theory Related Fields.* **78**, 97-112 (1988).

[15] F. den Hollander & J. E. Steif. Random walk in random scenery: A survey of some recent results. *IMS Lecture Notes–Monograph Series Dynamics & Stochastics.* **48**, 53–65 (2006).

[16] Katz, R., Parlange, M. & Naveau, P. Statistics of extremes in hydrology. *Advances In Water Resources.* **25** pp. 1287-1304 (2002).

[17] Kesten, H. & Spitzer, F. A limit theorem related to a new class of sel-similar processes. *Z. Wahrsch. Verw. Gebiete.* **50** pp. 5-25 (1979).

[18] Le Gall, J. & Rosen, J. The range of stable random walks. *Ann Proba.* **19**, 650-705 (1991).

[19] Leadbetter, M. Extremes and local dependence in stationary sequences. *Z. Wahrsch. Verw. Gebiete.* **65**, 291-306 (1983).

[20] Leadbetter, M., Lindgren, G. & Rootzén, H. Extremes and related properties of random sequences and processes. *Springer-Verlag* (1983).

[21] Last, G. & Penrose, M. Lectures on the Poisson process. *Cambridge University Press, Cambridge.* **7** (2018).

[22] Perfekt, R. Extremal behaviour of stationary Markov chains with applications. *Ann. Appl. Probab..* **4**, 529-548 (1994).

[23] Robert, C. Y. Inference for the limiting cluster size distribution of extreme values. *Ann. Stat.*. **37**(1), 271-310 (2009).

[24] Rootzén, H. Maxima and exceedances of stationary Markov chains. *Adv. In Appl. Probab.*. **20**, 371-390 (1988).

[25] Yiou, P., Goubanova, K., Li, Z. & Nogaj, M. Weather regime dependence of extreme value statistics for summer temperature and precipitation. *Nonlinear Processes In Geophysics*. **15**, 365-378 (2008).