Poisson approximation and $D(u_n)$ condition for extremes of transient random walks in random sceneries

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Abstract

Let $(S_n)_{n\geq 0}$ be a transient random walk in the domain of attraction of a stable law and let $(\xi(s))_{s\in\mathbb{Z}}$ be a sequence of random variables. Under suitable assumptions, we establish a Poisson approximation result for the point process of exceedances associated with $(\xi(S_n))_{n\geq 0}$ and demonstrate that it satisfies the $D(u_n)$ condition.

Keywords: extreme values, random walks, Poisson approximation.

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1 Introduction

Extreme Value Theory (EVT) was initially introduced in a univariate context for independent and identically distributed (i.i.d.) random variables. Subsequently, it was extended to sequences that do not exhibit independence but satisfy certain mixing conditions and an anti-clustering property. Among the weakest conditions are those by Leadbetter, referred to as the $D(u_n)$ and $D'(u_n)$ conditions [11]. In this paper, we deal with extremes for a sequence of real random variables which does not satisfy the $D'(u_n)$ condition. The sequence which is considered is a random walk in random scenery. This concept was initially introduced by Kesten and Spitzer [9], who established limit theorems for the sum of the first n terms and explored it extensively in various directions (see, e.g., [3, 13] and the survey in [7]). Franke

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and Saigo [4, 5] examined this process in the context of extremes. We detail below their problem.

Let $(X_k)_{k\geq 1}$ be a sequence of integer-valued i.i.d. random variables and let $S_0 = 0$ a.s. and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Assume that, for any $x \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{S_n}{n^{1/\alpha}} \leq x\right) \underset{n\to\infty}{\longrightarrow} F_\alpha(x),$$

where F_α is the distribution function of a stable law with characteristic function given by

$$\varphi(s) = \exp(-|s|^\alpha(C_1 + iC_2 \mathrm{sgn}\, s)), \quad \alpha \in (0, 2],$$

for some constants C_1, C_2, with $C_1 > 0$. Let $(\xi(s))_{s\in\mathbb{Z}}$ be a stationary sequence of $\mathbb{R}$-valued random variables which are independent of the sequence $(X_k)_{k\geq 1}$. The sequence $(\xi(S_n))_{n\geq 0}$ is referred to as a *random walk in a random scenery*. In [5], Franke and Saigo derive limit theorems for $\max_{i\leq n} \xi(S_i)$ as n goes to infinity when the $\xi(s)$'s are i.i.d.. The statements of their theorems depend on the value of α. When $\alpha < 1$ (resp. $\alpha > 1$), it is known that the random walk $(S_n)_{n\geq 0}$ is transient (resp. recurrent) [9, 10].

An important concept concerning random walks is the *range*. The latter is defined as the number of sites visited by the first n terms of the random walk, namely $R_n := \#\{S_1, \ldots, S_n\}$. When $\alpha < 1$, Le Gall and Rosen [10] proved that

$$\frac{R_{[nt]}}{n} \underset{n\to\infty}{\longrightarrow} qt \quad \mathbb{P} - a.s. \tag{1.1}$$

with $q := \mathbb{P}(S_k \neq 0, \forall k \geq 1) \in (0, 1]$ and $t \geq 0$. In [5] it is proved that, if u_n is a threshold such that $n\mathbb{P}(\xi > u_n) \underset{n\to\infty}{\longrightarrow} \tau$ for some $\tau > 0$, with $\xi = \xi(1)$, and if the $\xi(s)$'s are i.i.d., then

$$\mathbb{P}\left(\max_{i\leq n} \xi(S_i) \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau q} \tag{1.2}$$

for $\alpha < 1$. A consequence of this result, combined with and Theorem 1.2 in [11], is that the sequence $(\xi(S_n))_{n\geq 0}$ does not satisfy the $D'(u_n)$ condition if $q \neq 1$. Furthermore, Eq. (1.2) was then generalized in [1] for sequences $(\xi(s))_{s\in\mathbb{Z}}$ which are not necessarily i.i.d., but which satisfy a slight modification of the $D(u_n)$ and $D'(u_n)$ conditions.

In this paper, we give a more precise treatment of the extremes of $(\xi(S_n))_{n\geq 0}$. The conditions which are required are in the sense of [1] and presented below. To introduce the first one, we write for each $i_1 < \cdots < i_p$ and for each $u \in \mathbb{R}$,

$$F_{i_1,\dots,i_p}(u) = \mathbb{P}\left(\xi(i_1) \leq u, \dots, \xi(i_p) \leq u\right).$$

$\mathbf{D}(u_n)$ condition Let $(u_n)_{n\geq 0}$ be a sequence of real numbers. We say that the (stationary) sequence of real random variables $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ condition if there exist a sequence $(\alpha_{n,\ell})_{(n,\ell)\in\mathbb{N}^2}$ and a (non-decreasing) sequence (ℓ_n) of positive integers such that $\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$, $\ell_n = o(n)$, and

$$|F_{i_1,\dots,i_p,j_1,\dots,j_{p'}}(u_n) - F_{i_1,\dots,i_p}(u_n)F_{j_1,\dots,j_{p'}}(u_n)| \leq \alpha_{n,\ell}$$

for any integers $i_1 < \cdots < i_p < j_1 < \cdots < j_{p'}$ such that $j_1 - i_p \geq \ell$.

To introduce the $\mathbf{D}'(u_n)$ condition let us consider a sequence (k_n) such that

$$k_n \underset{n\to\infty}{\longrightarrow} \infty, \quad \frac{n^2}{k_n}\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0, \quad k_n\ell_n = o(n), \tag{1.3}$$

where (ℓ_n) and $(\alpha_{n,l})_{(n,l)\in\mathbb{N}^2}$ are the same as in the $\mathbf{D}(u_n)$ condition.

$\mathbf{D}'(u_n)$ condition In conjunction with the $\mathbf{D}(u_n)$ condition, we say that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}'(u_n)$ condition if there exists a sequence of integers (k_n) satisfying (1.3) such that

$$\lim_{n\to\infty} n \sum_{s=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\left(\xi(0) > u_n, \xi(s) > u_n\right) = 0.$$

In Eq. (3.2.1) in [12], the sequences $(\alpha_{n,l})_{(n,l)\in\mathbb{N}^2}$ and (k_n) only satisfy $k_n\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$ whereas in (1.3) we have assumed that $\frac{n^2}{k_n}\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$. In this sense, the $\mathbf{D}'(u_n)$ condition as written above is slightly more restrictive than the usual condition (see e.g. p29 in [12]) since $k_n \leq n$. The main topic of our paper is to extend [5] when $\alpha < 1$, i.e. in the transient case, to sequences which are not i.i.d. but which only satisfy the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions.

In Section 2, we prove that the so-called point process of exceedances converges to a Poisson point process in the transient case. In Section 3, we show that $(\xi(S_n))_{n\geq 0}$ satisfies the $D(u_n)$ condition.

2 Point process of exceedances

Throughout this section, we deal with the transient case, i.e. $\alpha < 1$. For any $k \geq 1$, let $\tau_k = \inf\{m \geq 1 : \#\{S_1, \dots, S_m\} \geq k\}$. Now, let $\tau > 0$ be fixed and u_n such that

$$n\mathbb{P}\left(\xi > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau. \tag{2.1}$$

Letting $m(n) = \lfloor qn \rfloor$, the *point process of exceedances* is defined as the random set

$$\Phi_n = \left\{\frac{\tau_k}{n} : \xi(S_{\tau_k}) > u_{m(n)},\ \tau_k \leq n\right\}_{k\geq 1} \subset [0,1]. \tag{2.2}$$

Proposition 2.1. *Let u_n be as in (2.1). Assume that the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions hold. Then Φ_n converges weakly to a Poisson point process Φ with intensity τ in $[0,1]$.*

A similar result was obtained in [5] but only for i.i.d. $\xi(s)$'s. According to Theorem 4.11 in [8], Proposition 2.1 can be rephrased as follows: for any Borel subsets $B_1, \dots, B_K \subset [0,1]$ with $m_{[0,1]}(\partial B_i) = 0$, $1 \leq i \leq K$,

$$\left(\#\Phi_n \cap B_1, \dots, \#\Phi_n \cap B_K\right) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \left(\#\Phi \cap B_1, \dots, \#\Phi \cap B_K\right),$$

where $m_{[0,1]}$ denotes the Lebesgue measure in $[0,1]$. Deriving Poisson approximation for the point process of exceedances is classical in Extreme Value Theory. In particular, Proposition 2.1 implies $\mathbb{P}\left(\max_{i\leq n} \xi(S_i) \leq u_{m(n)}\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau}$.

Proof of Proposition 2.1. According to Kallenberg's theorem (see e.g. Proposition 3.22 in [14]), it is sufficient to prove the following properties:

(i) for any $0 \leq a < b \leq 1$, $\mathbb{E}\left[\#\Phi_n \cap (a,b]\right] \underset{n\to\infty}{\longrightarrow} \tau(b-a)$;

(ii) for any (finite) disjoint union of intervals $I = \sqcup_{i=1}^{L}(a_i, b_i] \subset (0,1]$, with $L \geq 1$ and $a_1 < b_1 < \cdots < b_L$, $\mathbb{P}\left(\#\Phi_n \cap I = 0\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau \sum_{i=1}^{L}(b_i - a_i)}$.

First, we prove (i). Given $a < b$, we have

$$\begin{aligned}
\mathbb{E}\left[\#\Phi_n \cap (a,b]\right] &= \mathbb{E}\left[\sum_{k\geq 1} \mathbf{1}_{\frac{\tau_k}{n}\in(a,b]}\mathbf{1}_{\xi(S_{\tau_k})>u_{m(n)}}\right] \\
&= \mathbb{E}\left[\sum_{k\geq 1} \mathbf{1}_{\frac{\tau_k}{n}\in(a,b]}\right] \times \mathbb{P}\left(\xi > u_{m(n)}\right) \\
&\underset{n\to\infty}{\sim} \mathbb{E}\left[R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}\right] \times \frac{\tau}{m(n)},
\end{aligned}$$

where the second line comes from the fact that $(\xi(s))_{s\in\mathbb{Z}}$ is independent of $(S_n)_{n\geq 0}$ and where the last one comes from (2.1). According to (1.1), we know that $\mathbb{E}\left[R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}\right] \underset{n\to\infty}{\sim} n(b-a)q$. This together with the fact that $m(n) = \lfloor qn\rfloor$ implies (i).

To prove (ii), we first assume that $I = (a,b]$, with $a < b$. Let (k_n), (ℓ_n) be as in (1.3) and

$$r_n = \left\lfloor \frac{n}{k_n - 1}\right\rfloor + 1, \tag{2.3}$$

for n large enough. Denoting by $\mathbb{P}^{(S_n)}$ the probability conditional on $(S_n)_{n\geq 0}$, we get

$$\begin{aligned}
\mathbb{P}\left(\#\Phi_n \cap (a,b] = 0\right) &= \mathbb{P}\left(\bigcap_{k\geq 1:\frac{\tau_k}{n}\in(a,b]} \left\{\xi(S_{\tau_k}) \leq u_{m(n)}\right\}\right) \\
&= \mathbb{E}\left[\mathbb{P}^{(S_n)}\left(\bigcap_{s\in\mathcal{S}_{(na,nb]}} \left\{\xi(s) \leq u_{m(n)}\right\}\right)\right],
\end{aligned} \tag{2.4}$$

where

$$\mathcal{S}_{(na,nb]} = \left\{S_{\tau_k} : k \geq 1, \frac{\tau_k}{n} \in (a,b]\right\}.$$

Notice that $\#\mathcal{S}_{(na,nb]} = R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}$. To capture the fact that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ condition and thus the $\mathbf{D}(u_{m(n)})$ condition, we construct blocks and stripes as follows. Let

$$K_n = \left\lfloor \frac{R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}}{r_n}\right\rfloor + 1.$$

We subdivide the set $\mathcal{S}_{(na,nb]}$ into subsets $B_i \subset \mathcal{S}_{(na,nb]}$, $1 \leq i \leq K_n$, referred to as *blocks*, in such a way that $\#B_i = r_n$ and $\max B_i < \min B_{i+1}$ for all $i \leq K_n - 1$. Notice that $K_n \leq k_n$ and $\#B_{K_n} = R_{\lfloor nb \rfloor} - R_{\lfloor na \rfloor} - (K_n - 1) \cdot r_n$ a.s.. For each $j \leq K_n$, we denote by L_j the family consisting of the ℓ_n largest terms of B_j. When $j = K_n$, we take the convention $L_{K_n} = \emptyset$ if $\#B_{K_n} < \ell_n$. The set L_j is referred to as a *stripe*, and the union of the stripes is denoted by $\mathcal{L}_n = \bigcup_{j \leq K_n} L_j$. Proceeding as in the proofs of Lemmas 1 and 2 in [1], we can show that for almost all realization of $(S_n)_{n \geq 0}$,

- $\mathbb{P}^{(S_n)}\left(\bigcap_{s \in \mathcal{S}_{(na,nb]}} \left\{\xi(s) \leq u_{m(n)}\right\}\right) - \mathbb{P}^{(S_n)}\left(\bigcap_{s \in \mathcal{S}_{(na,nb]} \setminus \mathcal{L}_n} \left\{\xi(s) \leq u_{m(n)}\right\}\right) \underset{n \to \infty}{\longrightarrow} 0$;
- $\mathbb{P}^{(S_n)}\left(\bigcap_{s \in \mathcal{S}_{(na,nb]} \setminus \mathcal{L}_n} \left\{\xi(s) \leq u_{m(n)}\right\}\right) - \prod_{i \leq K_n} \mathbb{P}^{(S_n)}\left(\bigcap_{s \in B_i \setminus \mathcal{L}_n} \left\{\xi(s) \leq u_{m(n)}\right\}\right) \underset{n \to \infty}{\longrightarrow} 0$;
- $\prod_{i \leq K_n} \mathbb{P}^{(S_n)}\left(\bigcap_{s \in B_i \setminus \mathcal{L}_n} \left\{\xi(s) \leq u_{m(n)}\right\}\right) - \prod_{i \leq K_n} \mathbb{P}^{(S_n)}\left(\bigcap_{s \in B_i} \left\{\xi(s) \leq u_{m(n)}\right\}\right) \underset{n \to \infty}{\longrightarrow} 0$;
- $\prod_{i \leq K_n} \mathbb{P}^{(S_n)}\left(\bigcap_{s \in B_i} \left\{\xi(s) \leq u_{m(n)}\right\}\right) - \exp\left(-\frac{R_{\lfloor nb \rfloor} - R_{\lfloor na \rfloor}}{m(n)} \tau\right) \underset{n \to \infty}{\longrightarrow} 0$.

The first and the third assertions come from the fact that $\ell_n = o(r_n)$. The second assertion is a consequence of the fact that the sequence $(\xi(s))_{s \in \mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ condition and the last one is obtained by using the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions. Since $\frac{R_{\lfloor nb \rfloor} - R_{\lfloor na \rfloor}}{m(n)} \tau \underset{n \to \infty}{\longrightarrow} \tau(b-a)$ a.s., we deduce that, for almost all realization of $(S_n)_{n \geq 0}$,

$$\mathbb{P}^{(S_n)}\left(\bigcap_{s \in \mathcal{S}_{(na,nb]}} \left\{\xi(s) \leq u_{m(n)}\right\}\right) \underset{n \to \infty}{\longrightarrow} e^{-\tau(b-a)}. \tag{2.5}$$

This, together with (2.4) implies (ii) in the particular case when $I = (a, b]$.

Now, if I is of the form $I = \bigsqcup_{i=1}^{L} (a_i, b_i]$, we write

$$\mathbb{P}\left(\#\Phi_n \cap I = 0\right) = \mathbb{E}\left[\mathbb{P}^{(S_n)}\left(\bigcap_{i=1}^{L} \bigcap_{s \in \mathcal{S}_{(na_i, nb_i]}} \{\xi(s) \leq u_{m(n)}\}\right)\right].$$

By considering stripes and blocks again, we can show that

$$\mathbb{P}^{(S_n)}\left(\bigcap_{i=1}^{L} \bigcap_{s \in \mathcal{S}_{(na_i, nb_i]}} \{\xi(s) \leq u_{m(n)}\}\right) - \prod_{i=1}^{L} \mathbb{P}^{(S_n)}\left(\bigcap_{s \in \mathcal{S}_{(na_i, nb_i]}} \{\xi(s) \leq u_{m(n)}\}\right) \underset{n \to \infty}{\longrightarrow} 0,$$

for almost all realization of $(S_n)_{n\geq 0}$. It follows that

$$
\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(\#\Phi_n \cap I = 0\right) &= \lim_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{L} \mathbb{P}^{(S_n)}\left(\bigcap_{s\in\mathcal{S}_{(na_i,nb_i]}} \{\xi(s) \leq u_{m(n)}\}\right)\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} \prod_{i=1}^{L} \mathbb{P}^{(S_n)}\left(\bigcap_{s\in\mathcal{S}_{(na_i,nb_i]}} \{\xi(s) \leq u_{m(n)}\}\right)\right] \\
&= e^{-\tau \sum_{i=1}^{L}(b_i - a_i)},
\end{aligned}
$$

where the last line comes from (2.5). This concludes the proof of Proposition 2.1. □

We end this section with several remarks. First, Proposition 2.1 provides a more detailed analysis of the extremes considered in [1] as it implies that $\mathbb{P}\left(\max_{i\leq n} \xi(S_i) \leq u_{m(n)}\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau}$. A natural question is whether we can consider the points i/n such that $\xi(S_i)$ is larger than u_n instead of those which are larger than $u_{m(n)}$. In this case, we think that the underlying point process converges to a compound Poisson point process (a specific example is dealt in [2] but does not constitute a general approach). Regarding the transient case, i.e. $\alpha > 1$, Franke and Saigo (Theorem 4 in [5]) demonstrate that the normalized point process of exceedances converges to a Cox point process. However, their result relies on the assumption that the $\xi(s)$'s are i.i.d.. This result could be extended to the case where the $\xi(s)$'s only satisfy the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions. Such an extension would be possible by adapting the proof of Proposition 2.1 above.

3 The $\mathbf{D}(u_n)$ condition

For technical reasons, we assume in this section only that

$$
\frac{1}{n^2} \sup_{p\leq n+1} \sup_{0\leq i_1<\cdots<i_p\leq n} \mathbb{V}\left[R_{i_1,\ldots,i_p}\right] \underset{n\to\infty}{\longrightarrow} 0, \tag{3.1}
$$

where $R_{i_1,\ldots,i_p} = \#\{S_{i_1}, \ldots, S_{i_p}\}$. Such an assumption holds if the X_i's are a.s. positive (as an example, we can take $X_i = \lfloor Z_i \rfloor + 1$, where the Z_i's are i.i.d., positive and have a one-sided stable distribution, i.e. with characteristic function $\varphi(s) = e^{-|s|^\alpha (C_1 - i\tan(\pi\alpha/2))}$).

Proposition 3.1. *Assume that* $(\xi(s))_{s\in\mathbb{Z}}$ *satisfies the* $\mathbf{D}(u_n)$ *and* $\mathbf{D}'(u_n)$ *conditions for* u_n *such that* $n\mathbb{P}(\xi > u_n) \underset{n\to\infty}{\longrightarrow} \tau$*, with* $\tau > 0$*. Then* $(\xi(S_n))_{n\geq 0}$ *satisfies the* $\mathbf{D}(u_n)$ *condition.*

In [5], the authors establish a similar result (Proposition 2) under the assumption that the $\xi(s)$'s are i.i.d.. However, a key equality on page 463, namely

$$\mathbb{E}\left[(F(u_n))^{R_{j_1,\dots,j_q}}\right]\mathbb{E}\left[(F(u_n))^{R_{i_1,\dots,i_p}}\right] = \mathbb{E}\left[\mathbb{E}\left[(F(u_n))^{R_{j_1,\dots,j_q}}|S_{i_1},\dots,S_{i_p}\right](F(u_n))^{R_{i_1,\dots,i_p}}\right],$$

raises question as the justification provided lacks explicit detail. We propose a more general alternative proof that remedies this issue.

Proof of Proposition 3.1. We adapt several arguments of [5] to our context. Let $0 \leq i_1 < \dots < i_p < j_1 < \dots < j_{p'} \leq n$ be a family of integers, with $j_1 - i_p > \ell_n$ and $\ell_n = o(n)$. To prove that $(\xi(S_n))_{n\geq 0}$ satisfies the $\mathbf{D}(u_n)$ condition, we have to show that

$$|F'_{i_1,\dots,i_p,j_1,\dots,j_{p'}}(u_n) - F'_{i_1,\dots,i_p}(u_n)F'_{j_1,\dots,j_{p'}}(u_n)| \leq \tilde{\alpha}_{n,\ell_n},$$

for some sequence $(\tilde{\alpha}_{n,\ell})_{(n,\ell)\in\mathbb{N}^2}$ such that $\tilde{\alpha}_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$, with

$$F'_{i_1,\dots,i_p}(u_n) = \mathbb{P}\left(\xi(S_{i_1}) \leq u_n, \dots, \xi(S_{i_p}) \leq u_n\right).$$

We have

$$\begin{aligned}
&|F'_{i_1,\dots,i_p,j_1,\dots,j_{p'}}(u_n) - F'_{i_1,\dots,i_p}(u_n)F'_{j_1,\dots,j_{p'}}(u_n)| \\
&\leq \left|F'_{i_1,\dots,i_p,j_1,\dots,j_{p'}}(u_n) - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right)\right]\right| \\
&+ \left|\mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right)\right] - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p} + R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p} + R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right] - F'_{i_1,\dots,i_p}(u_n)F'_{j_1,\dots,j_{p'}}(u_n)\right|. \qquad (3.2)
\end{aligned}$$

To deal with the first and the third terms, we will use the following lemma.

Lemma 3.2. *We have*

$$\sup_{0\leq i_1<\cdots<i_p\leq n} \mathbb{E}\left[\left|F'_{i_1,\ldots,i_p}(u_n) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right)\right|\right] \underset{n\to\infty}{\longrightarrow} 0.$$

Proof of Lemma 3.2. We adapt key arguments from the proofs of Lemmas 1 and 2 in [1] to our context. Let (k_n) and (r_n) be as in (1.3) and (2.3). Given $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, we subdivide $\{S_{i_1}, \ldots, S_{i_p}\}$ into K_n blocks, with $K_n = \lfloor \frac{R_{i_1,\ldots,i_p}}{r_n} \rfloor + 1$, as in the proof of Proposition 2.1. More precisely, there exists a unique K_n-tuple of subsets $B_i \subset \mathcal{S}_n := \mathcal{S}_{(0,n]}$, $i \leq K_n$, such that: $\bigcup_{j\leq K_n} B_j = \{S_{i_1}, \ldots, S_{i_p}\}$, $\#B_i = r_n$ and $\max B_i < \min B_{i+1}$ for all $i \leq K_n - 1$. In particular, $K_n \leq k_n$ and $\#B_{K_n} = R_{i_1,\ldots,i_p} - (K_n - 1)\cdot r_n$ a.s.. Without loss of generality, we assume that $\#B_{K_n} = \#B_i = r_n$ for all $i \leq K_n - 1$, so that $R_{i_1,\ldots,i_p} = K_n r_n$. For each $j \leq K_n$, we denote by L_j the family consisting of the ℓ_n largest terms of B_j and let $\mathcal{L}_n = \bigcup_{j\leq K_n} L_j$. In the rest of the paper, we write $M_B = \max_{s\in B} \xi(s)$ for all $B \subset \mathbb{Z}$.

Adapting the proof of Lemma 1 in [1], we can show that, for almost all realization of $(S_n)_{n\geq 0}$ and for n larger than some deterministic integer n_0,

$$\left|\mathbb{P}^{(S_n)}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}} \leq u_n\right) - \mathbb{P}^{(S_n)}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}\setminus\mathcal{L}_n} \leq u_n\right)\right| \leq k_n \ell_n \mathbb{P}\left(\xi > u_n\right);$$

$$\left|\mathbb{P}^{(S_n)}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}\setminus\mathcal{L}_n} \leq u_n\right) - \prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j\setminus\mathcal{L}_n} \leq u_n\right)\right| \leq k_n \alpha_{n,\ell_n};$$

$$\left|\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j\setminus\mathcal{L}_n} \leq u_n\right) - \prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right)\right| \leq 2\frac{\tau k_n \ell_n}{n}.$$

Since $\mathbb{P}\left(\xi > u_n\right) \underset{n\to\infty}{\sim} \frac{\tau}{n}$ and $F'_{i_1,\ldots,i_p}(u_n) = \mathbb{E}\left[\mathbb{P}^{(S_n)}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}} \leq u_n\right)\right]$, we get

$$\sup_{0\leq i_1<\cdots<i_p\leq n} \left|F'_{i_1,\ldots,i_p}(u_n) - \mathbb{E}\left[\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0.$$

Without restriction, we assume from now on that $\mathbb{P}\left(\xi > u_n\right) = \frac{\tau}{n}$. We show below that

$$\sup_{0\leq i_1<\cdots<i_p\leq n} \left|\mathbb{E}\left[\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right)\right] - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0. \tag{3.3}$$

To do it, we adapt several arguments of Lemma 2 in [1]. Using the facts that $\log(1-x) \geq$

$-x-x^2$ for $|x|$ small enough and that $r_n\mathbb{P}(\xi > u_n) \underset{n\to\infty}{\longrightarrow} 0$, we get for n large enough

$$\begin{aligned}\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right) \\ &\geq \exp\left(K_n \log(1 - r_n\mathbb{P}(\xi > u_n))\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right) \\ &\geq \exp\left(-K_n r_n \mathbb{P}(\xi > u_n) - K_n (r_n\mathbb{P}(\xi > u_n))^2\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right).\end{aligned}$$

Because $K_n r_n = R_{i_1,\dots,i_p}$ and $\mathbb{P}(\xi > u_n) = \frac{\tau}{n}$, we have

$$\begin{aligned}\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right) \\ &\geq \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right)\left(\exp\left(-K_n(r_n\mathbb{P}(\xi > u_n))^2\right) - 1\right) \\ &\geq \exp(-k_n(r_n\mathbb{P}(\xi > u_n))^2) - 1,\end{aligned}$$

since $K_n \leq k_n$ a.s.. Because $k_n r_n \underset{n\to\infty}{\sim} n$, we obtain for some $c > 0$,

$$\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right) \geq -c \cdot \frac{1}{k_n}.$$

Moreover, because $\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right) \leq \exp\left(-\sum_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} > u_n\right)\right)$, it follows from the Bonferroni inequalities that

$$\begin{aligned}\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right) \\ &\leq \exp\left(-(K_n - 1)r_n\mathbb{P}(\xi > u_n) + \sum_{j\leq K_n} \sum_{\alpha<\beta;\alpha,\beta\in B_j} \mathbb{P}(\xi(\alpha) > u_n, \xi(\beta) > u_n)\right).\end{aligned}$$

Since $K_n r_n = R_{i_1,\dots,i_p}$ and $\mathbb{P}(\xi > u_n) = \frac{\tau}{n}$, we have

$$\begin{aligned}\prod_{j\leq K_n} \mathbb{P}^{(S_n)}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right) &\leq \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right) \\ &\times \left(\exp\left(r_n\mathbb{P}(\xi > u_n) + \sum_{j\leq K_n} \sum_{\alpha<\beta;\alpha,\beta\in B_j} \mathbb{P}(\xi(\alpha) > u_n, \xi(\beta) > u_n)\right) - 1\right)\end{aligned}$$

and therefore

$$\prod_{j\le K_n} \mathbb{P}^{(S_n)}\left(M_{B_j}\le u_n\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right)$$
$$\le \exp\left(r_n\mathbb{P}\left(\xi>u_n\right) + \sum_{j\le K_n}\sum_{\alpha<\beta;\alpha,\beta\in B_j}\mathbb{P}\left(\xi(\alpha)>u_n,\xi(\beta)>u_n\right)\right) - 1.$$

Proceeding along the same lines as in the proof of Lemma 2 in [1], we can show that

$$\exp\left(r_n\mathbb{P}\left(\xi>u_n\right) + \sum_{j\le K_n}\sum_{\alpha<\beta;\alpha,\beta\in B_j}\mathbb{P}\left(\xi(\alpha)>u_n,\xi(\beta)>u_n\right)\right) - 1$$
$$\le c\left(\frac{1}{k_n} + n\sum_{s=1}^{\lfloor n/k_n\rfloor}\mathbb{P}\left(\xi(0)>u_n,\xi(s)>u_n\right)\right).$$

Thus, for almost all realization of $(S_n)_{n\ge 0}$,

$$\left|\prod_{j\le K_n} \mathbb{P}^{(S_n)}\left(M_{B_j}\le u_n\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right)\right|$$
$$\le c\left(\frac{1}{k_n} + n\sum_{s=1}^{\lfloor n/k_n\rfloor}\mathbb{P}\left(\xi(0)>u_n,\xi(s)>u_n\right)\right),$$

This shows (3.3) by taking the expectations and the triangular inequality.

It remains to prove that

$$\sup_{0\le i_1<\dots<i_p\le n}\mathbb{E}\left[\left|\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right) - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right)\right]\right|\right] \underset{n\to\infty}{\longrightarrow} 0.$$

To do it, we write for any $\varepsilon>0$,

$$\mathbb{E}\left[\left|\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right) - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right)\right]\right|\right]$$
$$= \mathbb{E}\left[\left|\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right) - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right)\right]\right|\mathbf{1}_{\left|R_{i_1,\dots,i_p}/n-\mathbb{E}\left[R_{i_1,\dots,i_p}/n\right]\right|\le\varepsilon}\right]$$
$$+\mathbb{E}\left[\left|\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right) - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p}\tau}{n}\right)\right]\right|\mathbf{1}_{\left|R_{i_1,\dots,i_p}/n-\mathbb{E}\left[R_{i_1,\dots,i_p}/n\right]\right|>\varepsilon}\right].$$

The first term of the right-hand of the equality is smaller than some function $f(\varepsilon)$, with $f(\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow} 0$ whereas the second one is smaller than $2\,n^{-2}\,\varepsilon^{-2}\sup_{0\le i_1<\dots<i_p\le n}\mathbb{V}\left[R_{i_1,\dots,i_p}\right]$, which

converges to 0 as n goes to infinity according to (3.1). This concludes the proof of Lemma 3.2 by taking first the limit over $n \to \infty$ and then the limit over $\varepsilon \to 0$. $\square$

As a consequence of Lemma 3.2, the first and the third terms of the right-hand side of (3.2) converge to 0 as n goes to infinity. To deal with the second one, we write

$$\begin{aligned}\left|\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right| \\ = \exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\left(\exp\left(\frac{R^{j_1,\dots,j_{p'}}_{i_1,\dots,i_p}}{n}\tau\right)-1\right) \\ \leq \exp\left(\frac{R^{i_p+\ell_n+1,\dots,n}_{1,\dots,i_p}}{n}\tau\right)-1,\end{aligned}$$

where the last line comes from the fact that $j_1 - i_p > \ell_n$, with

$$R^{j_1,\dots,j_{p'}}_{i_1,\dots,i_p} = \#\left(\{S_{i_1},\dots,S_{i_p}\}\cap\{S_{j_1},\dots,S_{j_{p'}}\}\right) = R_{i_1,\dots,i_p} + R_{j_1,\dots,j_{p'}} - R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}.$$

Since $\ell_n \geq 0$, we get

$$\begin{aligned}\sup\left|\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right) - \exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right| \\ \leq \sup_{i\leq n}\exp\left(\frac{R^{i+1,\dots,n}_{1,\dots,i}}{n}\tau\right)-1, \quad (3.4)\end{aligned}$$

where the supremum in the left-hand side is taken over all integers $0 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_{p'} \leq n$, with $j_1 - i_p > \ell_n$. Using the fact that $R^{i+1,\dots,n}_{1,\dots,i} = R_{1,\dots,i} + R_{i+1,\dots,n} - R_{1,\dots,n}$ and following [10], we have $\sup_{i\leq n}\frac{R^{i+1,\dots,n}_{1,\dots,i}}{n} \underset{n\to\infty}{\longrightarrow} 0$ a.s.. This, together with (3.4) implies

$$\sup\left|\mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right)\right] - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0$$

and concludes the proof of Proposition 3.1. $\square$

When $\alpha > 1$ and when the $\xi(s)$'s are i.i.d., it is proved in [5] (Proposition 1) that $(\xi(S_n))_{n\geq 0}$ does not satisfy the $\mathbf{D}(u_n)$ condition. The same holds when the $\xi(s)$'s only satisfy the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions.

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