Cluster size distributions of extreme values for the Poisson-Voronoi tessellation

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July 14, 2016

Abstract

We consider the Voronoi tessellation based on a homogeneous Poisson point process in $\mathbf{R}^d$. For a geometric characteristic of the cells (e.g. the inradius, the circumradius, the volume), we investigate the point process of the nuclei of the cells with large values. Conditions are obtained for the convergence in distribution of this point process of exceedances to a homogeneous compound Poisson point process. We provide a characterization of the asymptotic cluster size distribution which is based on the Palm version of the point process of exceedances. This characterization allows us to compute efficiently the values of the extremal index and the cluster size probabilities by simulation for various geometric characteristics. The extension to the Poisson-Delaunay tessellation is also discussed.

Keywords: Extreme values; Voronoi tessellations; exceedance point processes.
AMS 2010 Subject Classifications: 60D05 . 62G32 . 60G70 . 60F05

1 Introduction

Stationary tessellations and the Poisson-Voronoi tessellation A tessellation in $\mathbf{R}^d$, $d \geq 1$, endowed with its Euclidean norm $|\cdot|$, is a countable collection of non-empty convex compact subsets, called *cells*, with disjoint interiors which subdivides the space and such that the number of cells intersecting any bounded subset of $\mathbf{R}^d$ is finite. The set $\mathbf{T}$ of tessellations is endowed with the σ-field generated by the sets $\{m \in \mathbf{T}, \cup_{C \in m} \partial C \cap K = \varnothing\}$, where ∂K is the boundary of K for any compact set K in $\mathbf{R}^d$. By a random tessellation m, we mean a random variable with values in $\mathbf{T}$. For a complete account on random tessellations and their applications, we refer to the books [28, 31].

A tessellation m is said to be stationary if its distribution is invariant under translations of the cells. Given a fixed realization of a stationary tessellation m, we associate with each cell $C \in m$, in a deterministic way, a point $z(C)$ which is called the *nucleus* of the cell, such that $z(C + x) = z(C) + x$ for all $x \in \mathbf{R}^d$. To describe the mean behavior of the tessellation, the notions of intensity and typical cell are introduced as follows. Let $A \subset \mathbf{R}^d$ be a Borel subset such that $\lambda_d(A) = 1$, where λ_d is the d-dimensional Lebesgue measure. The *intensity* of a stationary tessellation m is defined as

$$\gamma_m := \mathbb{E}\left[\#\{C \in m, z(C) \in A\}\right],$$

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where $\#\mathcal{S}$ denotes the cardinality of any finite set $\mathcal{S}$. Thanks to the stationarity of m, the intensity does not depend on the choice of A. Without loss of generality, we assume that $\gamma_m = 1$.

The *typical cell* $\mathcal{C}$ of a stationary tessellation m is a random polytope with distribution given by

$$\mathbb{E}[f(\mathcal{C})] = \mathbb{E}\left[\sum_{C \in m, z(C) \in A} f(C - z(C)) \right], \tag{1.1}$$

where $f : \mathcal{K}_d \to \mathbf{R}$ is any bounded measurable function on the set of convex bodies $\mathcal{K}_d$ (endowed with the Hausdorff topology).

Let χ be a locally finite subset of $\mathbf{R}^d$. The Voronoi cell with nucleus $x \in \chi$ is the set of all sites $y \in \mathbf{R}^d$ whose distance from x is smaller or equal than the distances to all other points of χ, i.e.

$$C_\chi(x) := \{y \in \mathbf{R}^d : |y - x| \leq |y - x'|, x' \in \chi\}.$$

When $\chi = \eta$ is a homogeneous Poisson point process, the family $m := \{C_\eta(x) : x \in \eta\}$ is the so-called *Poisson-Voronoi tessellation*. The intensity of such a tessellation equals the intensity of η. A consequence of the theorem of Slivnyak (see e.g. Theorem 3.3.5 in [28]) shows that

$$\mathcal{C} \overset{\mathcal{D}}{=} C_{\eta \cup \{0\}}(0), \tag{1.2}$$

where $\overset{\mathcal{D}}{=}$ denotes the equality in distribution. The study of this typical cell in the literature includes mean values calculations [20], second order properties [14] and distributional estimates [5, 21]. Voronoi tessellations are extensively used in many domains such as cellular biology [25], astrophysics [32], telecommunications [3] and finance [23]. For a complete account on Poisson-Voronoi tessellations and their applications, we refer to the book by Okabe et *al.* (see Chapter 5 in [22]).

Point process of exceedances for a stationary sequence of real random variables Let $(X_n)_{n \in \mathbb{Z}}$ be a strictly stationary sequence of real random variables. Assume that for each $\tau > 0$ there exists a sequence of levels $(u_n(\tau))$ such that $\lim_{n \to \infty} n\mathbb{P}(X_1 > u_n(\tau)) = \tau$. The point process of time normalized exceedances is defined by $\phi_B(\tau) := n^{-1} \cdot \{i \in B : X_i > u_n(\tau)\}$ for any Borel set $B \subset W_n := [-n/2, n/2]$. If (X_n) satisfies a long range dependence condition (known as condition $\Delta(u_n(\tau))$) and if the point process $\phi_{W_n}(\tau)$ weakly converges to a point process in $[-1/2, 1/2]$, then the limiting point process is necessarily a homogeneous compound Poisson process with intensity $\nu \geq 0$ and limiting *cluster size distribution* π (see Corollary 3.3 in [15]). According to Leadbetter [18], the constant $\theta = \nu/\tau$ is referred to as the *extremal index*. It may be shown that $0 \leq \theta \leq 1$ and that the compound Poisson limit becomes Poisson when $\theta = 1$.

If $\lim_{n \to \infty} \mathbb{P}(\#\phi_{W_n}(\tau) = 0) = e^{-\theta\tau}$, then a necessary and sufficient condition for the convergence of ϕ_{W_n} is the convergence of the conditional distribution of $\#\phi_{B_n}$, with $B_n = [0, q_n]$, given that there is at least one exceedance of $u_n(\tau)$ among $X_1, \ldots, X_{q_n}$, to the distribution $\pi = (\pi_k)_{k \geq 1}$, i.e.

$$\lim_{n \to \infty} \mathbb{P}\left(\#\phi_{B_n}(\tau) = k \mid \#\phi_{B_n}(\tau) > 0\right) := \pi_k, \qquad k \geq 1, \tag{1.3}$$

where (q_n) is a $\Delta(u_n(\tau))$-separating sequence, with $\lim_{n \to \infty} q_n/n = 0$ (see Theorem 4.2 in [15]). This condition is known as the *blocks* characterization of the cluster size distribution π. Under additional mild conditions (see e.g. [30]) the extremal index is equal to the reciprocal of the mean of π.

An equivalent condition to (1.3) is proposed in Theorem 4.1 in [27] (see also Theorem 2.5 in [24]) and is given by

$$\lim_{n \to \infty} \mathbb{P}\left(\#\phi_{B_n}(\tau) = k \mid X_0 > u_n(\tau)\right) := p'_k = \theta \sum_{m=k}^{\infty} \pi_m, \qquad k \geq 1. \tag{1.4}$$

In particular, we have $\theta = p'_1$. This second condition is useful to compute the values of the extremal index and the cluster size probabilities when the conditional distributions of the exceedances may be derived from the dynamics of $(X_n)_{n \in \mathbb{Z}}$, e.g. for the regularly varying multivariate time series [4] or the Markov sequences [24]. This condition may be called the *runs* characterization of the cluster size distribution since the runs estimator of the extremal index is based on the following result $\theta = \lim_{n \to \infty} \mathbb{P}\left(\cap_{i=1}^{q_n} \{X_i \leq u_n(\tau)\} \mid X_0 > u_n(\tau)\right)$. The runs characterization is natural for a random object as a time series where the direction of time is used to design the dynamics of the series. Estimators of the extremal index and the cluster size distribution, based on the blocks and runs characterizations, are extensively investigated, see e.g. [26, 29].

However, we claim that it could also be useful to consider a new condition where the conditional event $\{X_0 > u_n(\tau)\}$ is not used as the starting point of the considered cluster, but as a part of this cluster. We therefore introduce a new discrete probability distribution $p = (p_k)_{k \geq 1}$ and the following condition

$$\lim_{n \to \infty} \mathbb{P}\left(\#\phi_{C_n}(\tau) = k \mid X_0 > u_n(\tau)\right) := p_k, \qquad k \geq 1, \tag{1.5}$$

where $C_n = [-q_n/2, q_n/2]$. If p exists, an adaptation of our main result (see Theorem 4) shows that $p_k = \theta k \pi_k$ for $k \geq 1$, and therefore $\theta = \sum_{k=1}^{\infty} k^{-1} p_k$. Such a condition will be proposed for random tessellations for which there is no natural direction in the space $\mathbf{R}^d$. However, we think that our new condition could be fruitful for time series.

Point process of exceedances for a stationary tessellation Let m be a stationary tessellation in $\mathbf{R}^d$. We consider a geometric characteristic $g : \mathcal{K}_d \to \mathbb{R}$, which is a measurable translation-invariant function, i.e. $g(C + x) = g(C)$ for all $C \in \mathcal{K}_d$ and $x \in \mathbf{R}^d$, and such that, for any $\tau > 0$, there exists a threshold $v_\rho(\tau)$ satisfying

$$\lim_{\rho \to \infty} \rho \mathbb{P}\left(g(\mathcal{C}) > v_\rho(\tau)\right) = \tau, \tag{1.6}$$

where $\mathcal{C}$ is the typical cell. We observe only a part of the stationary tessellation m in the window $\mathbf{W}_\rho := \rho^{1/d} \cdot [-1/2, 1/2]^d$, $\rho > 0$, and we are interested in the point process of exceedances $\Phi_{\mathbf{W}_\rho}(\tau)$ where, for any Borel set $B \subset \mathbf{R}^d$, we let

$$\Phi_B(\tau) := \rho^{-1/d} \cdot \{z(C) : z(C) \in B, g(C) > v_\rho(\tau), C \in m\}.$$

In this paper, we investigate the weak convergence of the point process $\Phi_{\mathbf{W}_\rho}(\tau)$ in $[-1/2, 1/2]^d$ as ρ tends to infinity. In [10], a first result was obtained for geometric characteristics for which a short range dependence condition holds (equivalent to the so-called condition D' for stationary sequences of real random variables): it is shown that the point process $\Phi_{\mathbf{W}_\rho}(\tau)$ weakly converges to a homogeneous Poisson point process with intensity τ. In this paper, we are interested in finding weaker conditions for other geometric characteristics such that the point process $\Phi_{\mathbf{W}_\rho}(\tau)$ weakly converges to a homogeneous compound Poisson point process.

Let $\mathfrak{B}_\rho$ be a sub-cube of $\mathbf{W}_\rho$ such that $\lim_{\rho \to \infty} \lambda_d(\mathfrak{B}_\rho)/\rho = 0$. Condition (1.3) for the tessellation m will be written in the following way:

$$\lim_{\rho \to \infty} \mathbb{P}\left(\#\Phi_{\mathfrak{B}_\rho}(\tau) = k \mid \#\Phi_{\mathfrak{B}_\rho}(\tau) > 0\right) = \pi_k, \qquad k \geq 1, \tag{1.7}$$

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for a discrete probability distribution $\pi = (\pi_k)_{k \geq 1}$, which we also call the cluster size distribution. Additional assumptions on $\mathfrak{B}_\rho$ will be necessary and will depend on the mixing properties of the tessellation. Condition (1.4) cannot be transposed for stationary tessellations as explained previously. Condition (1.5) has to be modified since the cell which contains the origin (the Crofton cell) is not distributed as the typical cell. To overcome this difficulty, we consider a Palm version $\Phi^0_{\mathbf{R}^d}(\tau)$ of $\Phi_{\mathbf{R}^d}(\tau)$, i.e. a point process whose distribution is given by the Palm distribution of $\Phi_{\mathbf{R}^d}(\tau)$ (see Sections 3.3 and 3.4 in [28] for a complete account on Palm theory). For any $B \subset \mathbf{R}^d$, we also let $\Phi^0_B(\tau) = \Phi^0_{\mathbf{R}^d}(\tau) \cap B$. An analogous version of Condition (1.5) in the context of random tessellations can be stated as follows:

$$\lim_{\rho \to \infty} \mathbb{P}\left(\#\Phi^0_{\mathfrak{B}_\rho}(\tau) = k \right) := p_k, \qquad k \geq 1, \tag{1.8}$$

for a discrete probability distribution $p = (p_k)_{k \geq 1}$. In general the distributions π and p cannot be made explicit. It is necessary to use simulations to compute approximate values of the probabilities π_k and p_k.

The blocks method (1.7) competes with the Palm approach (1.8). The idea of the Palm approach is to consider clusters close to the origin given that the cell whose nucleus is the origin has an exceedance. Our approach provides better approximations of the extremal index and the cluster size distribution and requires less simulations. Indeed, we simulate only blocks that contain at least one exceedance (the one of the Crofton cell that contains the origin), while with the blocks approach, it is necessary to simulate a very large number of blocks (including those without any extreme value). More precisely, in our numerical illustrations in $\mathbf{R}^2$, we simulate tessellations only observed in the square $[-173, 173]^2$ to approximate θ and $p = (p_k)_{k \geq 1}$ thanks to our Palm approach. A blocks approach would have required to simulate tessellations in the square $[-5.18 \cdot 10^{21}, 5.18 \cdot 10^{21}]$, which is practically impossible.

In this paper, we only focus on the Poisson-Voronoi tessellation and we explain how our results can be extended to the Poisson-Delaunay tessellation. In Section 2, we give several preliminaries by introducing notation and conditions on our geometric characteristic. In Section 3, we investigate the convergence in distribution of the point process of exceedances to a homogeneous compound Poisson point process. This convergence is stated in our main result (Theorem 4). In Section 4, we give three examples and numerical illustrations. The extension to the Poisson-Delaunay tessellation is discussed in Section 5.

2 Preliminaries

In this section, we introduce several notation and conditions which will be used throughout the paper.

Notation

- Let $x \in \mathbf{R}^d$ and let $A, B \subset \mathbf{R}^d$ be two subsets. We write $x + A := \{x + a : a \in A\}$, $A \oplus B := \{a + b : a \in A, b \in B\}$ and $A \ominus B := \{x \in \mathbf{R}^d : x + B \subset A\}$. Moreover, we denote the complement of A by $A^c := \mathbf{R}^d \setminus A$.

- For any $A, B \subset \mathbf{R}^d$, we denote the distance between A and B by $\delta(A, B) := \inf_{(a,b) \in A \times B} |a - b|$.

- For any k-tuple of points $x_1, \ldots, x_k \in \mathbf{R}^d$, we write $x_{1:k} := (x_1, \ldots, x_k)$. With a slight abuse of notation, we also write $\{x_{1:k}\} := \{x_1, \ldots, x_k\}$.

- For any Borel subset $B \subset \mathbf{R}^d$, we write $\mathbf{B}_\rho := \rho^{1/d} \cdot B$.

- We denote by $\mathcal{F}_{lf}$ the set of locally finite subsets in $\mathbf{R}^d$. This set is endowed with the σ-field induced by the so-called Fell topology on $\mathcal{F}_{lf}$ (see e.g. p. 563 in [28]).

- Let $\chi \in \mathcal{F}_{lf}$.

  - For any $x_{1:k} \in \chi^k$ and for any $v \geq 0$, we write $g^\chi(x_{1:k}) > v$ to specify that $g(C_\chi(x_j)) > v$ for any $1 \leq j \leq k$. In particular, we let $g^\chi(x) := g(C_\chi(x))$.
  - For any $B \subset \mathbf{R}^d$, we write $M_B^\chi := \max_{x \in \chi \cap B} g^\chi(x)$. When $\chi \cap B = \varnothing$, we take $M_B^\chi := -\infty$.
  - For any $\rho > 0$ and $\tau > 0$, we denote by $\Phi^\chi(\tau)$ the point process of exceedances, i.e.
  $$\Phi^\chi(\tau) := \rho^{-1/d} \cdot \left\{ x \in \chi : g^\chi(x) > v_\rho(\tau) \right\}.$$
  Besides, for any $B \subset \mathbf{R}^d$, we write $\Phi_B^\chi(\tau) := \Phi^\chi(\tau) \cap (\rho^{-1/d}B)$.

- We denote by η a homogeneous Poisson point process in $\mathbf{R}^d$. Excepted in Section 5, we assume that the intensity of η is $\gamma_\eta = 1$.

- For each $\tau > 0$, we denote by $\Phi^{\eta,0}(\tau)$ the Palm version of $\Phi^\eta(\tau)$. In particular, for any $B \subset \mathbf{R}^d$ we let $\Phi_B^{\eta,0}(\tau) := \Phi^{\eta,0}(\tau) \cap (\rho^{-1/d}B)$. We also associate two probabilities defined as follows:
$$\pi_{k,B}(\tau) := \mathbb{P}\left(\#\Phi_B^\eta(\tau) = k | \#\Phi_B^\eta(\tau) > 0 \right) \quad \text{and} \quad p_{k,B}(\tau) := \mathbb{P}(\#\Phi_B^{\eta,0}(\tau) = k).$$
The quantity $\pi_{k,B}(\tau)$ is the probability that there are k exceedances in B conditional on the fact that there is at less an exceedance in B, whereas the quantity $p_{k,B}(\tau)$ is the probability that there are k exceedances in B conditional on the fact that the origin is a nucleus and that the cell with nucleus the origin is an exceedance. Notice that these probabilities also depend on ρ.

- For any pair of functions $h_1, h_2 \colon \mathbf{R} \to \mathbf{R}$, we write $h_1(\rho) \underset{\rho \to \infty}{\sim} h_2(\rho)$ and $h_1(\rho) = O(h_2(\rho))$ to respectively mean that $h_1(\rho)/h_2(\rho) \to 1$ as $\rho \to \infty$ and $h_1(\rho)/h_2(\rho)$ is bounded for ρ large enough.

- We denote by $q : \rho \mapsto q_\rho$ a generic function such that, for any $\alpha, \beta > 0$, we have simultaneously
$$q_\rho \cdot (\log \rho)^\alpha \cdot \rho^{-1} \underset{\rho \to \infty}{\longrightarrow} 0 \quad \text{and} \quad q_\rho^{-1} \cdot (\log \rho)^\beta \underset{\rho \to \infty}{\longrightarrow} 0. \tag{2.1}$$

- Let $\varepsilon > 0$ be fixed. For any $\rho > 0$, we denote by n_ρ and m_ρ the integers
$$n_\rho := \left\lfloor (\log \rho)^{-(1+\varepsilon)/d} \cdot \rho^{1/d} \right\rfloor \quad \text{and} \quad m_\rho := \left\lfloor q_\rho^{-1/d} \cdot (\log \rho)^{-(1+\varepsilon)/d} \cdot \rho^{1/d} \right\rfloor. \tag{2.2}$$
We also define two squares centered at 0 as follows:
$$\mathfrak{c}_\rho := \frac{\rho^{1/d}}{n_\rho} \cdot [-D, D]^d \quad \text{and} \quad \mathfrak{Q}_\rho := \frac{\rho^{1/d}}{m_\rho} \cdot [-1/2, 1/2]^d, \tag{2.3}$$
where $D := 2 \cdot (\lfloor \sqrt{d} \rfloor + 1)$. In particular, we have
$$\lambda_d(\mathfrak{c}_\rho) \underset{\rho \to \infty}{\sim} (\log \rho)^{1+\varepsilon}(2D)^d \quad \text{and} \quad \lambda_d(\mathfrak{Q}_\rho) \underset{\rho \to \infty}{\sim} q_\rho(\log \rho)^{1+\varepsilon}.$$

Throughout the paper, we use c to signify a universal positive constant not depending on ρ but which may depend on other quantities. When required, we assume that ρ is sufficiently large.

Conditional independence Let η be a homogeneous Poisson point process. We begin with a first lemma that characterizes the dependence structure of the Poisson-Voronoi tessellation induced by η.

We partition $\mathbf{W}_\rho$ into a set $\mathcal{V}_\rho$ of n_ρ^d sub-cubes of equal size, where n_ρ^d is defined in (2.2). These sub-cubes are indexed by the set of $\mathfrak{i} := (i_1, \ldots, i_d) \in [1, n_\rho]^d$. With a slight abuse of notation, we identify a cube with its index. Notice that for each $\mathfrak{i} \in \mathcal{V}_\rho$, we have $\lambda_d(\mathfrak{i}) = (2D)^{-d} \cdot \lambda_d(\mathfrak{c}_\rho)$. Besides, the distance between sub-cubes $\mathfrak{i}$ and $\mathfrak{j}$ is denoted by $d(\mathfrak{i}, \mathfrak{j}) := \max_{1 \le r \le d} |i_r - j_r|$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are two sets of sub-cubes, we let $d(\mathcal{A}, \mathcal{B}) = \min_{\mathfrak{i} \in \mathcal{A}, \mathfrak{j} \in \mathcal{B}} d(\mathfrak{i}, \mathfrak{j})$ and

$$\Sigma_\mathcal{A}^{\eta \cup \{x_{1:k}\}} := \sigma\{g^{\eta \cup \{x_{1:k}\}}(x) : x \in (\eta \cup \{x_{1:k}\}) \cap \mathfrak{i}, \mathfrak{i} \in \mathcal{A}\}. \tag{2.4}$$

Finally, to ensure several independence properties, we introduce the following event:

$$\mathscr{A}_\rho := \cap_{\mathfrak{i} \in \mathcal{V}_\rho} \{\eta \cap \mathfrak{i} \ne \varnothing\}.$$

The event $\mathscr{A}_\rho$ is extensively used in stochastic geometry to derive central limit theorems or to deal with extremes (see e.g. [2, 10]). It will play a crucial role in the rest of the paper. The following lemma is the heart of our development and captures the idea of "local dependence".

Lemma 1 *Let $x_1, \ldots, x_k \in \mathbf{R}^d$, with $k \ge 0$. Then*

  (i) *conditional on $\mathscr{A}_\rho$, the σ-fields $\Sigma_\mathcal{A}^{\eta \cup \{x_{1:k}\}}$ and $\Sigma_\mathcal{B}^{\eta \cup \{x_{1:k}\}}$ are independent when $d(\mathcal{A}, \mathcal{B}) > D$;*

  (ii) *for any $\alpha > 0$, we have $\rho^\alpha \cdot \mathbb{P}\left(\mathscr{A}_\rho^c\right) \xrightarrow[\rho \to \infty]{} 0$.*

Proof The first assertion is a simple adaptation of Lemma 5 in [10]. The second one comes from (2.2) and the fact that

$$\mathbb{P}\left(\mathscr{A}_\rho^c\right) = \mathbb{P}\left(\bigcup_{\mathfrak{i} \in \mathcal{V}_\rho} \{\eta \cap \mathfrak{i} = \varnothing\}\right) \le n_\rho^d e^{-\rho/n_\rho^d}.$$

$\square$

Condition on the geometric characteristic To state our main theorem, we assume some condition on the geometric characteristic g, referred to as Condition (C).

Condition (C) *For any $\tau > 0$, there exists a constant c such that, for any $(k-1)$-tuple of points $y_{2:k} \in \mathbf{R}^{d(k-1)}$, for any $z \in \{0, y_{2:k}\}$, we have*

$$\mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(z) > v_\rho(\tau)\right) \le c \cdot \rho^{-1},$$

where $v_\rho(\tau)$ satisfies Equation (1.6), with the convention $\{y_{2:k}\} = \varnothing$ when $k = 1$.

In particular, Condition (C) is satisfied when $g^{\eta \cup \{0, y_{2:k}\}}(x) \le g^{\eta \cup \{0\}}(x)$ for any $x \in \eta \cup \{0\}$ and for any $y_{2:k} \in \mathbf{R}^{d(k-1)}$, i.e. when the geometric characteristic of a cell always decreases if new points are added to the point process η.

3 Weak convergence of the point process of exceedances for a Poisson-Voronoi tessellation

3.1 An explicit representation for $p_{k,B}$

According to the theorem of Slivnyak, the Palm distribution of η is given by the distribution of $\eta \cup \{0\}$. As a consequence, the following lemma shows that for any $B \subset \mathbf{R}^d$, the distribution of $\#\Phi_B^{\eta,0}(\tau)$ is the same as the one of $\#\Phi_B^{\eta \cup \{0\}}(\tau)$ given that $g^{\eta \cup \{0\}}(0) > v_\rho(\tau)$.

Lemma 2 *For any $B \subset \mathbf{R}^d$, $\rho > 0$ and $k \geq 1$, we have*

$$p_{k,B}(\tau) = \mathbb{P}(\#\Phi_B^{\eta \cup \{0\}}(\tau) = k | g^{\eta \cup \{0\}}(0) > v_\rho(\tau)).$$

Proof Since $p_{k,B}(\tau) := \mathbb{P}\left(\#\Phi_B^{\eta,0}(\tau) = k\right) = \mathbb{P}\left(\#\Phi^{\eta,0}(\tau) \cap (\rho^{-1/d}B) = k\right)$, we obtain for any Borel subset $A \subset \mathbf{R}^d$, with $\lambda_d(A) = 1$, that

$$p_{k,B}(\tau) = \frac{1}{\gamma_{\Phi^\eta(\tau)}} \mathbb{E}\left[\sum_{z \in \Phi^\eta(\tau) \cap A} \mathbb{I}_{\#(\Phi^\eta(\tau)-z) \cap (\rho^{-1/d}B)=k} \right], \qquad (3.1)$$

where $\gamma_{\Phi^\eta(\tau)} := \mathbb{E}\left[\#(\Phi^\eta(\tau) \cap A)\right]$ is the intensity of $\Phi^\eta(\tau)$. According to (1.1), this intensity equals

$$\gamma_{\Phi^\eta(\tau)} = \mathbb{E}\left[\sum_{x \in \eta \cap \mathbf{A}_\rho} \mathbb{I}_{g^\eta(x)>v_\rho(\tau)} \right] = \rho\,\mathbb{P}\left(g(\mathcal{C}) > v_\rho(\tau) \right). \qquad (3.2)$$

Moreover, it results from the Slivnyak-Mecke formula (e.g. Corollary 3.2.3 in [28]) that for any $B \subset \mathbf{R}^d$,

$$\mathbb{E}\left[\sum_{z \in \Phi^\eta(\tau) \cap A} \mathbb{I}_{\#(\Phi^\eta(\tau)-z) \cap (\rho^{-1/d}B)=k} \right]$$

$$= \mathbb{E}\left[\sum_{x \in \eta \cap \mathbf{A}_\rho} \mathbb{I}_{\#\{y \in (\eta-x) \cap B : g^\eta(y)>v_\rho(\tau)\}=k}\, \mathbb{I}_{g^\eta(x)>v_\rho(\tau)} \right]$$

$$= \int_{\mathbf{A}_\rho} \mathbb{P}\left(\#\{y \in (\eta \cup \{x\} - x) \cap B : g^{\eta \cup \{x\}}(y) > v_\rho(\tau)\} = k, g^{\eta \cup \{x\}}(x) > v_\rho(\tau) \right) dx.$$

Thanks to the stationarity of η and because g is translation-invariant, the above integrand does not depend on x. By integrating over $x \in \mathbf{A}_\rho$ and using the fact that $\lambda_d(\mathbf{A}_\rho) = \rho$, it follows that

$$\mathbb{E}\left[\sum_{z \in \Phi^\eta(\tau) \cap A} \mathbb{I}_{\#(\Phi^\eta(\tau)-z) \cap (\rho^{-1/d}B)=k} \right]$$

$$= \rho \cdot \mathbb{P}\left(\#\{y \in (\eta \cup \{0\}) \cap B : g^{\eta \cup \{0\}}(y) > v_\rho(\tau)\} = k, g^{\eta \cup \{0\}}(0) > v_\rho(\tau) \right)$$

$$= \rho \cdot \mathbb{P}\left(\#\Phi_B^{\eta \cup \{0\}}(\tau) = k, g^{\eta \cup \{0\}}(0) > v_\rho(\tau) \right).$$

This together with (1.2), (3.1) and (3.2) concludes the proof of Lemma 2. $\qquad\square$

3.2 A technical result

The following technical proposition will be the key ingredient to prove our main theorem.

Proposition 3 *Assume that g satisfies Condition (C). Then, for any $k \geq 1$, we have*

$$k\mathbb{P}\left(\#\Phi^{\eta}_{\mathfrak{Q}_\rho}(\tau) = k\right) - \lambda_d(\mathfrak{Q}_\rho) \cdot \mathbb{P}\left(\#\Phi^{\eta \cup \{0\}}_{\mathfrak{Q}_\rho}(\tau) = k, g^{\eta \cup \{0\}}(0) > v_\rho(\tau)\right) = o\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right). \quad (3.3)$$

The previous proposition is obvious if we replace $o\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right)$ by $O\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right)$ in (3.3) since $\mathbb{P}\left(\#\Phi^{\eta}_{\mathfrak{Q}_\rho}(\tau) = k\right) = O\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right)$ and $\mathbb{P}\left(\#\Phi^{\eta \cup \{0\}}_{\mathfrak{Q}_\rho}(\tau) = k, g^{\eta \cup \{0\}}(0) > v_\rho(\tau)\right) = O\left(\rho^{-1}\right)$. Actually, the main difficulty is to prove that the left-hand side is negligible compared to $\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}$, which constitutes the main ingredient to prove Theorem 4.

Since q_ρ is any function such that (2.1) holds, we can take $q_\rho = (\log \log \rho)^{\log \log \rho}$. Actually, we think that Proposition 3 remains true when $q_\rho = \log \rho$, which is slightly more efficient for simulating estimators of θ and p_k.

Proof We begin with the case $k \geq 2$. First, we give an integral representation of the left-hand side of (3.3). Because of the stationarity of η and thanks to the Slivnyak-Mecke formula, we have

$$\mathbb{P}\left(\#\Phi^{\eta}_{\mathfrak{Q}_\rho}(\tau) = k\right) = \frac{1}{k!}\mathbb{E}\left[\sum_{x_{1:k} \in (\eta \cap \mathfrak{Q}_\rho)^k} \mathbb{I}_{g^\eta(x_{1:k}) > v_\rho(\tau)} \mathbb{I}_{M^{\eta}_{\mathfrak{Q}_\rho \setminus \{x_{1:k}\}} \leq v_\rho(\tau)}\right]$$

$$= \frac{1}{k!}\int_{\mathfrak{Q}_\rho}\int_{(\mathfrak{Q}_\rho - x_1)^{k-1}} p_{x_1}(y_{2:k})\mathrm{d}y_{2:k}\mathrm{d}x_1,$$

where, for any $x_1 \in \mathbf{R}^d$ and any $y_{2:k} \in \mathbf{R}^{(k-1)d}$, we write

$$p_{x_1}(y_{2:k}) := \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M^{\eta \cup \{0, y_{2:k}\}}_{(\mathfrak{Q}_\rho - x_1) \setminus \{0, y_{2:k}\}} \leq v_\rho(\tau)\right).$$

In the same spirit as above, we also obtain

$$\mathbb{P}\left(\#\Phi^{\eta \cup \{0\}}_{\mathfrak{Q}_\rho}(\tau) = k, g^{\eta \cup \{0\}}(0) > v_\rho(\tau)\right) = \frac{1}{(k-1)!}\int_{\mathfrak{Q}_\rho^{k-1}} p_0(y_{2:k})\mathrm{d}y_{2:k}.$$

Integrating the right-hand side of the above equation over $\mathfrak{Q}_\rho$, it follows that

$$k\mathbb{P}\left(\#\Phi^{\eta}_{\mathfrak{Q}_\rho}(\tau) = k\right) - \lambda_d(\mathfrak{Q}_\rho) \cdot \mathbb{P}\left(\#\Phi^{\eta \cup \{0\}}_{\mathfrak{Q}_\rho}(\tau) = k, g^{\eta \cup \{0\}}(0) > v_\rho(\tau)\right)$$

$$= \frac{1}{(k-1)!}\int_{\mathfrak{Q}_\rho}\left(\int_{(\mathfrak{Q}_\rho - x_1)^{k-1}} p_{x_1}(y_{2:k})\mathrm{d}y_{2:k} - \int_{\mathfrak{Q}_\rho^{k-1}} p_0(y_{2:k})\mathrm{d}y_{2:k}\right)\mathrm{d}x_1.$$

The main difficulty to prove that the right-hand side equals $o\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right)$ comes from the dependence between the $(k+1)$-events considered in the probability $p_x(y_{2:k})$, with $x = x_1$ and $x = 0$. Actually, the more the distances between the $y_2, \ldots, y_k$ is large, the more the dependence is weak. To overcome this difficulty, we introduce the following event:

$$E_m := \left\{y_{2:k} \in \mathbf{R}^{(k-1)d} : S_0(y_{2:k}) \text{ has } m \text{ connected components}\right\},$$

where, for any $y_{2:k} \in \mathbf{R}^{d(k-1)}$, the set $S_0(y_{2:k}) \subset \mathbf{R}^d$ is defined as (see Figure 1)

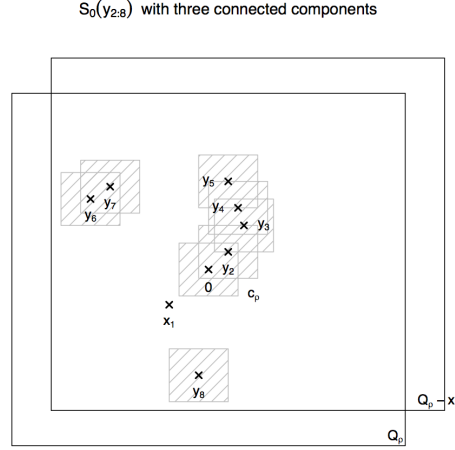$$S_0(y_{2:k}) := \mathfrak{c}_\rho \cup \bigcup_{j=2}^{k}(y_j + \mathfrak{c}_\rho).$$

Figure 1: A configuration of points $y_{2:8}$, where $S_0(y_{2:k})$ has three connected components

It results from the above that

$$k\mathbb{P}\left(\#\Phi^\eta_{\mathfrak{Q}_\rho}(\tau) = k\right) - \lambda_d(\mathfrak{Q}_\rho) \cdot \mathbb{P}\left(\#\Phi^{\eta\cup\{0\}}_{\mathfrak{Q}_\rho}(\tau) = k, g^{\eta\cup\{0\}}(0) > v_\rho(\tau)\right)$$

$$= \frac{1}{(k-1)!}\sum_{m=1}^{k}\int_{\mathfrak{Q}_\rho} P_{x_1}[m]\mathrm{d}x_1,$$

where, for any $x_1 \in \mathbf{R}^d$, we write

$$P_{x_1}[m] := \int_{(\mathfrak{Q}_\rho - x_1)^{k-1} \cap E_m} p_{x_1}(y_{2:k})\mathrm{d}y_{2:k} - \int_{\mathfrak{Q}_\rho^{k-1} \cap E_m} p_0(y_{2:k})\mathrm{d}y_{2:k}. \tag{3.4}$$

It is enough to show that for each $1 \le m \le k$, we have $\int_{\mathfrak{Q}_\rho} P_{x_1}[m]\mathrm{d}x_1 = o\left(\lambda_d(\mathfrak{Q}_\rho)\rho^{-1}\right)$. To do it, we begin with $m = 1$ which deals with the case where there is exactly one connected component of size k in $S_0(y_{2:k})$. Then we extend our proof for $m \ge 2$ by dividing the set $S_0(y_{2:k})$ into its connected components.

First case ($S_0(y_{2:k})$ has one connected component) Assume that $m = 1$ and let $x_1 \in \mathbf{R}^d$ be fixed. We trivially obtain that

$$\begin{aligned}
P_{x_1}[1] = {} & \int_{((\mathfrak{Q}_\rho - x_1) \cap \mathfrak{Q}_\rho)^{k-1} \cap E_1} \left(p_{x_1}(y_{2:k}) - p_0(y_{2:k})\right)\mathrm{d}y_{2:k} \\
& + \int_{(\mathfrak{Q}_\rho - x_1)^{k-1} \cap E_1} p_{x_1}(y_{2:k})\mathbb{I}_{\exists j \le k : y_j \in \mathfrak{Q}_\rho^c}\,\mathrm{d}y_{2:k} \\
& - \int_{\mathfrak{Q}_\rho^{k-1} \cap E_1} p_0(y_{2:k})\mathbb{I}_{\exists j \le k : y_j \in (\mathfrak{Q}_\rho - x_1)^c}\,\mathrm{d}y_{2:k}.
\end{aligned} \tag{3.5}$$

We provide below a suitable upper bound for each term considered in the right-hand side of the above equation.

Upper bound for the first term in (3.5) Using the fact that for any events A, B, C, we have $|\mathbb{P}(A \cap B) - \mathbb{P}(A \cap C)| \leq \mathbb{P}(A \cap B^c \cap C) + \mathbb{P}(A \cap B \cap C^c)$, we obtain

$$
\left| \int_{((\mathfrak{Q}_\rho - x_1) \cap \mathfrak{Q}_\rho)^{k-1} \cap E_1} (p_{x_1}(y_{2:k}) - p_0(y_{2:k})) \, \mathrm{d}y_{2:k} \right|
$$

$$
\leq \int_{((\mathfrak{Q}_\rho - x_1) \cap \mathfrak{Q}_\rho)^{k-1} \cap E_1} \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right) \mathrm{d}y_{2:k}
$$

$$
+ \int_{((\mathfrak{Q}_\rho - x_1) \cap \mathfrak{Q}_\rho)^{k-1} \cap E_1} \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M_{(\mathfrak{Q}_\rho - x_1) \setminus \mathfrak{Q}_\rho}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right) \mathrm{d}y_{2:k}. \tag{3.6}
$$

To deal with the first term of the right-hand side of (3.6), we introduce the event:

$$
E^{\varnothing}(x_1) := \left\{ y_{2:k} \in \mathbf{R}^{(k-1)d} : S_0(y_{2:k}) \cap (\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)) = \varnothing \right\}.
$$

First, we assume that $y_{2:k} \in E^{\varnothing}(x_1) \cap E_1$. From (2.4), we know that $\left\{ g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau) \right\} \in \Sigma_{\mathcal{A}}^{\eta \cup \{0, y_{2:k}\}}$ and $\{ M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \} \in \Sigma_{\mathcal{B}}^{\eta \cup \{0, y_{2:k}\}}$, where

$$
\mathcal{A} := \{ \mathfrak{i} \in \mathcal{V}_\rho : \{0, y_{2:k}\} \cap \mathfrak{i} \neq \varnothing \} \quad \text{and} \quad \mathcal{B} := \{ \mathfrak{j} \in \mathcal{V}_\rho : (\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)) \cap \mathfrak{j} \neq \varnothing \}.
$$

Since $y_{2:k} \in E^{\varnothing}(x_1)$, we have $d(\mathcal{A}, \mathcal{B}) > D$. It follows from Lemma 1, (i) that, conditional on $\mathscr{A}_\rho$, the events $\left\{ g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau) \right\}$ and $\left\{ M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right\}$ are independent. This implies that

$$
\mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right) \mathbb{I}_{y_{2:k} \in E^{\varnothing}(x_1)}
$$

$$
\leq \mathbb{P}(\mathscr{A}_\rho)^{-1} \cdot \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau) \right) \cdot \mathbb{P}\left(M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right) + \mathbb{P}((\mathscr{A}_\rho)^c),
$$

where we have bounded the indicator function $\mathbb{I}_{y_{2:k} \in E^{\varnothing}(x_1)}$ by 1. According to Lemma 1, (ii), we know that $\mathbb{P}(\mathscr{A}_\rho) \geq 1 - c \cdot \rho^{-\alpha}$ for any $\alpha > 0$. Moreover, since g satisfies Condition (C), we have $\mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau) \right) \leq c \cdot \rho^{-1}$. Besides, it results from the Slivnyak-Mecke formula that

$$
\mathbb{P}\left(M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right)
$$

$$
= \mathbb{P}\left(\exists z \in (\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)) \cap (\eta \cup \{0, y_{2:k}\}) : g^{\eta \cup \{0, y_{2:k}\}}(z) > v_\rho(\tau) \right)
$$

$$
\leq \mathbb{E}\left[\sum_{z \in \eta \cap \mathfrak{Q}_\rho} \mathbb{I}_{g^{\eta \cup \{0, y_{2:k}\}}(z) > v_\rho(\tau)} \right] + \sum_{j=2}^{k} \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(y_j) > v_\rho(\tau) \right)
$$

$$
= \int_{\mathfrak{Q}_\rho} \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}, z\}}(z) > v_\rho(\tau) \right) \mathrm{d}z + \sum_{j=2}^{k} \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(y_j) > v_\rho(\tau) \right)
$$

$$
\leq c \cdot \lambda_d(\mathfrak{Q}_\rho) \rho^{-1},
$$

where the last line is also a consequence of Condition (C). This implies that

$$
\mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau), M_{\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)}^{\eta \cup \{0, y_{2:k}\}} > v_\rho(\tau) \right) \mathbb{I}_{y_{2:k} \in E^{\varnothing}(x_1) \cap E_1} \leq c \cdot \lambda_d(\mathfrak{Q}_\rho) \rho^{-2}. \tag{3.7}
$$

Secondly, we assume that $y_{2:k} \in (E^{\varnothing}(x_1))^c \cap E_1$. In particular, we have

$$
\delta(0, \mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)) \leq \mathrm{diam}(S_0(y_{2:k})) \leq c \cdot \lambda_d(\mathfrak{c}_\rho)^{1/d}.
$$

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Since g satisfies Condition (C), this implies that

$$\mathbb{P}\left(g^{\eta\cup\{0,y_{2:k}\}}(0,y_{2:k})>v_\rho(\tau),M^{\eta\cup\{0,y_{2:k}\}}_{\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1)}>v_\rho(\tau)\right)\mathbb{I}_{y_{2:k}\in(E^\varnothing(x_1))^c\cap E_1}$$
$$\le c\cdot\rho^{-1}\mathbb{I}_{\delta(0,\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}. \quad (3.8)$$

Integrating over $y_{2:k}\in((\mathfrak{Q}_\rho-x_1)\cap\mathfrak{Q}_\rho)^{k-1}\cap E_1$, it follows from (3.7) and (3.8) that

$$\int_{((\mathfrak{Q}_\rho-x_1)\cap\mathfrak{Q}_\rho)^{k-1}\cap E_1}\mathbb{P}\left(g^{\eta\cup\{0,y_{2:k}\}}(0,y_{2:k})>v_\rho(\tau),M^{\eta\cup\{0,y_{2:k}\}}_{\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1)}>v_\rho(\tau)\right)\mathrm{d}y_{2:k}$$
$$\le c\cdot\lambda_{(k-1)d}(E_1)\cdot\left(\lambda_d(\mathfrak{Q}_\rho)\rho^{-2}+\rho^{-1}\mathbb{I}_{\delta(0,\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}\right)$$
$$\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{k-1}\cdot\left(\lambda_d(\mathfrak{Q}_\rho)\cdot\rho^{-2}+\rho^{-1}\mathbb{I}_{\delta(0,\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}\right), \quad (3.9)$$

where the last line comes from the fact that $\lambda_{(k-1)d}(E_1)\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{k-1}$.

Proceeding exactly along the same lines as above, by considering here the event:

$$F^\varnothing(x_1):=\left\{y_{2:k}\in\mathbf{R}^{(k-1)d}:S_0(y_{2:k})\cap((\mathfrak{Q}_\rho-x_1)\setminus\mathfrak{Q}_\rho)=\varnothing\right\},$$

we can show that the second term of the right-hand side of (3.6) can be bounded in a similar way, i.e.

$$\int_{((\mathfrak{Q}_\rho-x_1)\cap\mathfrak{Q}_\rho)^{k-1}\cap E_1}\mathbb{P}\left(g^{\eta\cup\{0,y_{2:k}\}}(0,y_{2:k})>v_\rho(\tau),M^{\eta\cup\{0,y_{2:k}\}}_{(\mathfrak{Q}_\rho-x_1)\setminus\mathfrak{Q}_\rho}>v_\rho(\tau)\right)\mathrm{d}y_{2:k}$$
$$\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{k-1}\cdot\left(\lambda_d(\mathfrak{Q}_\rho)\cdot\rho^{-2}+\rho^{-1}\mathbb{I}_{\delta(0,(\mathfrak{Q}_\rho-x_1)\setminus\mathfrak{Q}_\rho))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}\right).$$

This together with (3.6) and (3.9) implies that

$$\int_{((\mathfrak{Q}_\rho-x_1)\cap\mathfrak{Q}_\rho)^{k-1}\cap E_1}(p_{x_1}(y_{2:k})-p_0(y_{2:k}))\,\mathrm{d}y_{2:k}\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{k-1}\cdot\lambda_d(\mathfrak{Q}_\rho)\cdot\rho^{-2}$$
$$+c\cdot\lambda_d(\mathfrak{c}_\rho)^{k-1}\cdot\rho^{-1}\cdot\left(\mathbb{I}_{\delta(0,\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}+\mathbb{I}_{\delta(0,(\mathfrak{Q}_\rho-x_1)\setminus\mathfrak{Q}_\rho))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}\right).$$

This deals with the first term of the right-hand side in (3.5).

Upper bound for the second term in (3.5)   Trivially, this term equals 0: this comes from the fact that for ρ large enough, we have

$$E_1\cap\{y_{2:k}\in\mathbf{R}^{(k-1)d}:\exists j\le k\text{ s.t. }y_j\in\mathfrak{Q}_\rho^c\}=\varnothing$$

since $\lambda_d(\mathfrak{c}_\rho)=o\left(\lambda_d(\mathfrak{Q}_\rho)\right)$.

Upper bound for the third term in (3.5)   To deal with this term, we notice that if $E_1\cap\{y_{2:k}\in\mathfrak{Q}_\rho^{k-1}:\exists j\le k\text{ s.t. }y_j\in(\mathfrak{Q}_\rho-x_1)^c\}\ne\varnothing$, then $\delta\left(0,\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1)\right)\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}$ because $\mathrm{diam}(S_0(y_{2:k}))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}$. Besides, since $p_0(y_{2:k})\le c\cdot\rho^{-1}$ according to Condition (C), we obtain by integrating over $y_{2:k}\in\mathfrak{Q}_\rho^{k-1}\cap E_1$ that

$$\int_{\mathfrak{Q}_\rho^{k-1}\cap E_1}p_0(y_{2:k})\mathbb{I}_{\exists j\le k:y_j\in(\mathfrak{Q}_\rho-x_1)^c}\,\mathrm{d}y_{2:k}\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{k-1}\cdot\rho^{-1}\cdot\mathbb{I}_{\delta(0,\mathfrak{Q}_\rho\setminus(\mathfrak{Q}_\rho-x_1))\le c\cdot\lambda_d(\mathfrak{c}_\rho)^{1/d}}.$$

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This deals with the third term of the right-hand side in (3.5).

By considering the three upper bounds discussed above and by integrating over $x_1 \in \mathfrak{Q}_\rho$, we get

$$\left| \int_{\mathfrak{Q}_\rho} P_{x_1}[1] \mathrm{d}x_1 \right| \leq c \cdot \lambda_d(\mathfrak{c}_\rho)^{k-1} \cdot \lambda_d(\mathfrak{Q}_\rho)^2 \cdot \rho^{-2}$$

$$+ c \cdot \lambda_d(\mathfrak{c}_\rho)^{k-1} \cdot \rho^{-1} \cdot \lambda_d \left(\left\{ x_1 \in \mathfrak{Q}_\rho : \delta(0, \mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1) \leq c \cdot \lambda_d(\mathfrak{c}_\rho)^{1/d} \right\} \right)$$

$$+ c \cdot \lambda_d(\mathfrak{c}_\rho)^{k-1} \cdot \rho^{-1} \cdot \lambda_d \left(\left\{ x_1 \in \mathfrak{Q}_\rho : \delta(0, (\mathfrak{Q}_\rho - x_1) \setminus \mathfrak{Q}_\rho \leq c \cdot \lambda_d(\mathfrak{c}_\rho)^{1/d} \right\} \right).$$

We can easily prove that if $x_1 \in \mathfrak{Q}_\rho$ is such that $\delta(0, \mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)) \leq c \cdot \lambda_d(\mathfrak{c}_\rho)^{1/d}$, then $x_1 \in \mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho \ominus c^{1/d} \cdot \mathfrak{c}_\rho)$. Hence

$$\lambda_d \left(\left\{ x_1 \in \mathfrak{Q}_\rho : \delta(0, \mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho - x_1)) \leq c \cdot \lambda_d(\mathfrak{c}_\rho)^{1/d} \right\} \right) \leq \lambda_d \left(\mathfrak{Q}_\rho \setminus (\mathfrak{Q}_\rho \ominus c^{1/d} \cdot \mathfrak{c}_\rho) \right)$$

$$= O \left(\lambda_d(\mathfrak{Q}_\rho)^{(d-1)/d} \cdot \lambda_d(\mathfrak{c}_\rho)^{1/d} \right).$$

Moreover, for ρ large enough, we have

$$\lambda_d \left(\{ x_1 \in \mathfrak{Q}_\rho : \delta(0, (\mathfrak{Q}_\rho - x_1) \setminus \mathfrak{Q}_\rho \leq c \cdot \lambda_d(\mathfrak{c}_0)^{1/d} \} \right) = 0$$

since $\lambda_d(\mathfrak{c}_\rho) = o\left(\lambda_d(\mathfrak{Q}_\rho)\right)$. From (2.1), we deduce that

$$\left| \int_{\mathfrak{Q}_\rho} P_{x_1}[1] \mathrm{d}x_1 \right| \leq c \cdot \left(\lambda_d(\mathfrak{c}_\rho)^{k-1+1/d} \cdot \lambda_d(\mathfrak{Q}_\rho)^{-1/d} \right) \cdot \left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1} \right) = o\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1} \right).$$

This concludes the proof for the case $m = 1$.

Second case ($S_0(y_{2:k})$ has m connected component with $m \geq 2$)   Assume that $m \geq 2$ and $y_{2:k} \in E_m$.

First, we provide below a uniform upper bound for $p_x(y_{2:k})$, with $x \in \mathfrak{Q}_\rho$. Since $y_{2:k} \in E_m$, we can divide $S_0(y_{2:k})$ into its m connected components, say $C_1(y_{2:k}), \ldots, C_m(y_{2:k})$. For each $1 \leq l \leq m$, let $J_l \subset \{1, \ldots, k\}$ be the set of indices j such that $C_l(y_{2:k}) = \bigcup_{j \in J_l}(y_j + \mathfrak{c}_\rho)$, with $y_1 := 0$. In particular, we have

$$p_x(y_{2:k}) \leq \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(0, y_{2:k}) > v_\rho(\tau) \right) = \mathbb{P}\left(\bigcap_{l=1}^{m} \left\{ g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau) \right\} \right),$$

where we recall that $g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau)$ means that $g^{\eta \cup \{0, y_{2:k}\}}(y) > v_\rho(\tau)$ for any $y \in J_l$. In the same spirit as in the case where $S_0(y_{2:k})$ has one connected component, we deduce from Lemma 1, (i) that, conditional on the event $\mathscr{A}_\rho$, the events $\{g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau)\}$, $1 \leq l \leq m$, are independent. This gives

$$p_x(y_{2:k}) \leq \mathbb{P}(\mathscr{A}_\rho)^{-(m-1)} \cdot \prod_{l=1}^{m} \mathbb{P}\left(g^{\eta \cup \{0, y_{2:k}\}}(y_{J_l}) > v_\rho(\tau) \right) + \mathbb{P}\left(\mathscr{A}_\rho^c \right).$$

Since g satisfies Condition (C), it follows from Lemma 1, (ii) that there exists a constant $c > 0$ such that, for any $x \in \mathfrak{Q}_\rho$ and for any $y_{2:k} \in E_m$, we have $p_x(y_{2:k}) \leq c \cdot \rho^{-m}$.

Now, we are able to provide an upper bound for $\left|\int_{\mathfrak{Q}_\rho} P_{x_1}[m]\mathrm{d}x_1\right|$. Indeed, integrating over $y_{2:k}$ in the right-hand side of (3.4), we have

$$|P_{x_1}[m]| \le c \cdot \rho^{-m} \cdot \sup_{x \in \mathfrak{Q}_\rho} \lambda_{(k-1)d}((\mathfrak{Q}_\rho - x)^{k-1} \cap E_m) \le c \cdot \rho^{-m} \cdot \lambda_d(\mathfrak{Q}_\rho)^{m-1} \cdot \lambda_d(\mathfrak{c}_\rho)^{k-m}.$$

Integrating over $x_1 \in \mathfrak{Q}_\rho$, we get

$$\left|\int_{\mathfrak{Q}_\rho} P_{x_1}[m]\mathrm{d}x_1\right| \le c \cdot \left(\rho^{-1} \cdot \lambda_d(\mathfrak{Q}_\rho) \cdot \lambda_d(\mathfrak{c}_\rho)^{(k-m)/(m-1)}\right)^{m-1} \cdot \left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right) = o\left(\lambda_d(\mathfrak{Q}_\rho) \cdot \rho^{-1}\right).$$

This concludes the proof of Proposition 3 for any $k \ge 2$. The case $k = 1$ is much more simple than the case $k \ge 2$ and can be dealt by following the same lines as above and by noting that

$$\mathbb{P}\left(\#\Phi^\eta_{\mathfrak{Q}_\rho}(\tau) = 1\right) - \lambda_d(\mathfrak{Q}_\rho) \cdot \mathbb{P}\left(\#\Phi^{\eta\cup\{0\}}_{\mathfrak{Q}_\rho}(\tau) = 1, g^{\eta\cup\{0\}}(0) > v_\rho(\tau)\right) = \int_{\mathfrak{Q}_\rho} (p_{x_1} - p_0)\mathrm{d}x_1,$$

where, for any $x \in \mathfrak{Q}_\rho$, we write $p_x := \mathbb{P}\left(g^{\eta\cup\{0\}}(0) > v_\rho(\tau), M^{\eta\cup\{0\}}_{(\mathfrak{Q}_\rho - x)\setminus\{0\}} \le v_\rho(\tau)\right)$. $\qquad\square$

3.3 Our main theorem

Let g be a geometric characteristic such that (1.6) holds for some $\tau_0 > 0$. According to Leadbetter [18], we say that the extremal index $\theta \in [0, 1]$ of the Poisson-Voronoi tessellation exists if $\lim_{\rho\to\infty} \mathbb{P}(\#\Phi^\eta_{\mathbf{W}_\rho}(\tau_0) = 0) = e^{-\theta\tau_0}$. We are now prepared to state our main theorem on the weak convergence of the point process $\Phi^\eta_{\mathbf{W}_\rho}(\tau)$ for each $\tau > 0$.

Theorem 4 *Let g be a geometric characteristic satisfying Condition (C). Assume that there exist $\tau_0 > 0$ such that (1.6) holds and $(a_k)_{k\ge1}$ such that $\pi_{k,\mathfrak{Q}_\rho}(\tau_0) \le a_k$ for any $k \ge 1$ and any $\rho > 0$, with $\sum_{k=1}^\infty a_k < \infty$.*

(i) The following assertions are equivalent:

(A) there exists $\theta \in (0, 1]$ such that $\lim_{\rho\to\infty} \mathbb{P}(\#\Phi^\eta_{\mathbf{W}_\rho}(\tau_0) = 0) = e^{-\theta\tau_0}$ and the following limit exist $p_k := \lim_{\rho\to\infty} p_{k,\mathfrak{Q}_\rho}(\tau_0)$ for any $k \ge 1$;

(B) for any $\tau > 0$, the point process $\Phi^\eta_{\mathbf{W}_\rho}(\tau)$ converges to a homogeneous compound Poisson point process in $W := [-1/2, 1/2]^d$ with intensity $\nu(\tau) > 0$ and cluster size distributions $\pi_k := \lim_{\rho\to\infty} \pi_{k,\mathfrak{Q}_\rho}(\tau_0)$, with $k \ge 1$.

(ii) If one of the above assertions holds, we have $p_k = k\theta\pi_k$ for any $k \ge 1$ and $\theta = \sum_{k=1}^\infty k^{-1}p_k$.

Our theorem provides a new characterization of the extremal index. Indeed, this index was previously interpreted as the reciprocal of the mean of the cluster size distribution π. Now, it can be viewed as the mean of the reciprocal of the Palm version of the cluster size. Besides, our new characterization: $\theta = \sum_{k=1}^\infty k^{-1}p_k$ will be extensively used in Section 4 to estimate the extremal indices for various geometric characteristics.

To prove Theorem 4, we associate with the point process $\Phi^\eta_{\mathbf{W}_\rho}(\tau)$ its Laplace transform $\mathbb{L}_\rho$ defined as follows: for any continuous function $f : W \to \mathbf{R}_+$, we have

$$\mathbb{L}_\rho(f) := \mathbb{E}\left[\exp\left(-\sum_{y\in\Phi^\eta_{\mathbf{W}_\rho}(\tau)} f(y)\right)\right].$$

It is well-known that the weak convergence of $\Phi_{\mathbf{W}_\rho}^\eta(\tau)$ is equivalent to the convergence of its Laplace transform for any positive and continuous function f. For a sequence of real random variables, the weak convergence of the point process of exceedances has been investigated in [15] and generalized to random fields on $\mathbf{N}_+^d$ in [12]. We use below the same type of approach. However, we have to take into account specific features of random tessellations in $\mathbf{R}^d$.

The first step consists in showing that exceedances over disjoint sub-cubes behave asymptotically as if they were independent. To do it, we divide $\mathbf{W}_\rho$ into m_ρ^d disjoint sub-cubes $\mathfrak{B}[l]$, $l = 1, \ldots, m_\rho^d$, with the same volume as $\mathfrak{Q}_\rho$, where m_ρ is defined in (2.2).

Lemma 5   *(i) For any measurable function $f : W \to \mathbf{R}_+$, we have*

$$\mathbb{L}_\rho(f) - \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[\exp\left(-\sum_{y \in \Phi_{\mathbf{W}_\rho}^\eta(\tau) \cap \rho^{-1/d}\mathfrak{B}[l]} f(y)\right)\right] \underset{\rho \to \infty}{\longrightarrow} 0.$$

(ii) Moreover, we have

$$\mathbb{P}\left(M_{\mathbf{W}_\rho}^\eta \le v_\rho(\tau)\right) - \prod_{l=1}^{m_\rho^d} \mathbb{P}\left(M_{\mathfrak{B}[l]}^\eta \le v_\rho(\tau)\right) \underset{\rho \to \infty}{\longrightarrow} 0.$$

Proof We begin with the first assertion. For any $l \le m_\rho^d$, we write $\mathfrak{B}^\circ[l] := \mathfrak{B}[l] \ominus \mathfrak{c}_\rho$. Let

$$L_{\rho,l}(f) = \exp\left(-\sum_{y \in \Phi_{\mathbf{W}_\rho}^\eta(\tau) \cap \rho^{-1/d}\mathfrak{B}[l]} f(y)\right) \quad \text{and} \quad L_{\rho,l}^\circ(f) = \exp\left(-\sum_{y \in \Phi_{\mathbf{W}_\rho}^\eta(\tau) \cap \rho^{-1/d}\mathfrak{B}^\circ[l]} f(y)\right).$$

We write

$$\mathbb{L}_\rho(f) - \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}(f)\right] = \Delta L_{\rho,1}(f) + \Delta L_{\rho,2}(f) + \Delta L_{\rho,3}(f) + \Delta L_{\rho,4}(f),$$

where

$$\Delta L_{\rho,1}(f) = \mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}(f)\right] - \mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}^\circ(f)\right],$$

$$\Delta L_{\rho,2}(f) = \mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}^\circ(f)\right] - \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}^\circ(f)\right],$$

$$\Delta L_{\rho,3}(f) = \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}^\circ(f)\right] - \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}(f)\right],$$

$$\Delta L_{\rho,4}(f) = \mathbb{E}\left[\exp\left(-\sum_{y \in \Phi_{\mathbf{W}_\rho}^\eta(\tau)} f(y)\right)\right] - \mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}(f)\right].$$

We prove below that each term converges to 0. For the third term, using the fact that $|\prod x_i - \prod y_i| \le \sum |x_i - y_i|$ for $0 \le x_i, y_i \le 1$ and the fact that $|\exp(-x) - \exp(-y)| \le |x - y|$ for all

$x, y \geq 0$, we get

$$|\Delta L_{\rho,3}(f)| \leq m_\rho^d \cdot \sup_{l \leq m_\rho^d} \mathbb{E}\left[\sum_{y \in \Phi_{\mathbf{W}_\rho}^\eta(\tau) \cap \rho^{-1/d}(\mathfrak{B}[l] \setminus \mathfrak{B}^\circ[l])} f(y) \right]$$

$$\leq m_\rho^d \cdot \sup_{l \leq m_\rho^d} \mathbb{E}\left[\sum_{x \in \eta \cap (\mathfrak{B}[l] \setminus \mathfrak{B}^\circ[l])} f(\rho^{-1/d}x) \mathbb{I}_{g^\eta(x) > v_\rho(\tau)} \right]$$

$$= c \cdot m_\rho^d \cdot \sup_{l \leq m_\rho^d} \int_{\mathfrak{B}[l] \setminus \mathfrak{B}^\circ[l]} f(\rho^{-1/d}x) \mathbb{P}\left(g^{\eta \cup \{x\}}(x) > v_\rho(\tau) \right) \mathrm{d}x$$

$$\leq c \cdot m_\rho^d \cdot \lambda_d\left(\mathfrak{B}[l] \setminus \mathfrak{B}^\circ[l] \right) \cdot \mathbb{P}\left(g(\mathcal{C}) > v_\rho(\tau) \right),$$

where the third line comes from the Slivnyak-Mecke formula and where the fourth line comes from (1.1) and the fact that f is bounded because it is continuous on the compact set W. Since $m_\rho^d \underset{\rho \to \infty}{\sim} \rho \cdot q_\rho^{-1} \cdot (\log \rho)^{-(1+\varepsilon)}$ and

$$\lambda_d\left(\mathfrak{B}[l] \setminus \mathfrak{B}^\circ[l] \right) \leq c \cdot q_\rho^{(d-1)/d} \cdot (\log \rho)^{(1+\varepsilon)},$$

we deduce that

$$|\Delta L_{\rho,3}(f)| = O\left(q_\rho^{-1/d} \right).$$

In the same spirit as above, we prove that $\Delta L_{\rho,1}(f)$ and $\Delta L_{\rho,4}(f)$ converges to 0.

For $\Delta L_{\rho,2}(f)$, we notice that conditional on $\mathscr{A}_\rho$, the random variables considered in the expectations are independent. Then, we have

$$\mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}^\circ(f) \right] = \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho \right] + \mathbb{P}\left(\mathscr{A}_\rho^c \right) \left(\mathbb{E}\left[\prod_{l=1}^{m_\rho^d} L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho^c \right] - \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho \right] \right).$$

Moreover

$$\prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}^\circ(f) \right] = \prod_{l=1}^{m_\rho^d} \left(\mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho \right] + \mathbb{P}\left(\mathscr{A}_\rho^c \right) \left(\mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho^c \right] - \mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho \right] \right) \right)$$

$$:= \prod_{l=1}^{m_\rho^d} \mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho \right] + m_\rho^d \mathbb{P}\left(\mathscr{A}_\rho^c \right) H_\rho(f).$$

The term $H_\rho(f)$ appearing in the above equation is such that $|H_\rho(f)| \leq c$: this is a consequence of Lemma 1, (ii) and the fact that $0 \leq \mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho^c \right] \leq 1$ and $0 \leq \mathbb{E}\left[L_{\rho,l}^\circ(f) \middle| \mathscr{A}_\rho \right] \leq 1$. By applying again Lemma 1, (ii), it follows that $\Delta L_{\rho,2}(f)$ converges to 0. We proceed in a similar way for the proof of the second assertion. $\qquad \square$

We now adapt two theorems due to Leadbetter, Lindgren and Rootzén in our context. The following result is an adaptation of Theorem 4.2 in [15] (resp. Proposition 4.2 in [12]) and gives sufficient conditions to derive the convergence of $\Phi_{\mathbf{W}_\rho}^\eta(\tau)$ to a homogeneous compound Poisson point process.

Proposition 6 *Assume that* $\mathbb{P}\left(\#\Phi_{\mathbf{W}_\rho}^\eta(\tau_0) = 0 \right) \underset{\rho \to \infty}{\longrightarrow} e^{-\nu}$ *for some* $\tau_0 > 0$ *and* $\nu > 0$. *If* $\left(\pi_{k,\mathfrak{Q}_\rho} \right)_{k \geq 1}$ *converges to a probability distribution* π *on* $\mathbf{N}_+$, *then* $\Phi_{\mathbf{W}_\rho}^\eta(\tau_0)$ *converges in distribution to a homogeneous compound Poisson point process with intensity* ν *and limiting cluster size distribution* π.

The following result adapted from Theorem 5.1 in [15] (resp. Proposition 4.3. in [12]) shows that if $\Phi^\eta_{\mathbf{W}_\rho}(\tau_0)$ has a limit for some $\tau_0 > 0$, it has a limit for all $\tau > 0$.

Proposition 7 *Assume that $\Phi^\eta_{\mathbf{W}_\rho}(\tau_0)$ converges to a homogeneous compound Poisson point process in W with intensity $\nu > 0$ and cluster size distribution π, for some $\tau_0 > 0$. Then $\Phi^\eta_{\mathbf{W}_\rho}(\tau)$ converges to a homogeneous compound Poisson point process with intensity $\nu \cdot \tau/\tau_0$ and limiting cluster size distribution π, for each $\tau > 0$.*

We do not give the proofs of Propositions 6 and 7 since they are readily obtained through [12] substituting Lemma 2.1 by our Lemma 5. We are now prepared to give a proof of Theorem 4.

Proof of Theorem 4 Proof of (i). First we show that (A)$\Rightarrow$(B). By Lemma 5, we have

$$\mathbb{P}\left(M^\eta_{\mathbf{W}_\rho} \leq v_\rho(\tau_0) \right) = \left(\mathbb{P}\left(M^\eta_{\mathfrak{Q}_\rho} \leq v_\rho(\tau_0) \right) \right)^{\rho \cdot (\lambda_d(\mathfrak{Q}_\rho))^{-1}} + o(1).$$

Since $\lim_{\rho \to \infty} \mathbb{P}\left(M^\eta_{\mathbf{W}_\rho} \leq v_\rho(\tau_0) \right) = e^{-\theta\tau_0}$ and since $\{M^\eta_{\mathfrak{Q}_\rho} \leq v_\rho(\tau_0)\}$ if and only if $\{\#\Phi^\eta_{\mathfrak{Q}_\rho}(\tau_0) > 0\}$, it follows that

$$\mathbb{P}\left(\#\Phi^\eta_{\mathfrak{Q}_\rho}(\tau_0) > 0 \right) \underset{\rho \to \infty}{\sim} \frac{\lambda_d(\mathfrak{Q}_\rho)}{\rho} \cdot \theta\tau_0.$$

This together with Proposition 3 implies that $\pi_k = \lim_{\rho \to \infty} \pi_{k,\mathfrak{Q}_\rho}(\tau_0)$ exists and $\pi_k = p_k/(k \cdot \theta)$ for any $k \geq 1$. Since $\pi_{k,\mathfrak{Q}_\rho}(\tau_0) \leq a_k$ with $\sum_{k=1}^\infty a_k < \infty$, it follows from the dominated convergence theorem that $\pi := (\pi_k)_{k \geq 1}$ is a probability measure on $\mathbf{N}_+$. Applying Proposition 6, we deduce that $\Phi^\eta_{\mathbf{W}_\rho}(\tau_0)$ converges to a homogeneous compound Poisson point process with intensity $\nu(\tau_0) := \theta\tau_0 > 0$ and cluster size distribution π. This together with Proposition 7 proves Assertion (B).

Secondly, we show that (B)$\Rightarrow$(A). The fact that the extremal index exists and is positive is a consequence of the fact that

$$\lim_{\rho \to \infty} \mathbb{P}\left(M^\eta_{\mathbf{W}_\rho}(1) \leq v_\rho(\tau_0) \right) = \lim_{\rho \to \infty} \mathbb{P}\left(\#\Phi^\eta_{\mathbf{W}_\rho}(\tau_0) = 0 \right) = e^{-\theta\tau_0},$$

where $\theta := \nu(\tau_0)/\tau_0 \in (0,1]$. By applying Proposition 3, we show that the limit of $p_k := p_{k,\mathfrak{Q}_\rho}(\tau_0)$ exists and $p_k = k\theta\pi_k$. This proves Assertion (A).

Proof of (ii). The fact that $p_k = k\theta\pi_k$ is established above. Moreover, we have $\sum_{k=1}^\infty k^{-1}p_k = \theta \sum_{k=1}^\infty \pi_k = \theta$ since $\pi = (\pi_k)_{k \geq 1}$ is a probability measure. $\square$

4   Numerical illustrations

Layout   In this section, we illustrate our main theorem throughout simulations for three geometric characteristics which satisfy Condition (C). Each geometric characteristic is chosen in such a way that the value of the extremal index is known or can be conjectured. For sake of simplicity, we only do our simulations in the particular setting $d = 2$. We provide approximations of $p_1, \ldots, p_9$ and of the extremal index by using the fact that $\theta = \sum_{k=1}^\infty k^{-1}p_k$ (see Theorem 4, (ii)) and we compare this approximation to the theoretical value of θ.

For each geometric characteristic g, we proceed as follows. We take $\tau = 1$ and $\rho = \exp(100)$. In particular, the cube $\mathfrak{Q}_\rho$, as defined in (2.3), is approximatively

$$\mathfrak{Q}_\rho \simeq [-173, 173]^2,$$

by taking $q_\rho = (\log\log\rho)^{\log\log\rho} \simeq 1134$ and $\varepsilon = 0.01$. Then, we compute theoretically $v_\rho(1)$ so that $\rho \cdot \mathbb{P}(g(\mathcal{C}) > v_\rho(1)) \xrightarrow[\rho\to\infty]{} 1$. We simulate 10000 realizations of independent Poisson-Voronoi tessellations given that the typical cell is an exceedance, i.e. $g^{\eta\cup\{0\}}(0) > v_\rho(1)$ (see Lemma 2). This sample of size 10000 is divided into 100 sub-samples of size 100. For each $1 \leq i \leq 100$ and for each $1 \leq k \leq 9$, we denote by $\hat{p}_k^{(i)}$ the empirical mean of p_k, i.e. the mean number of realizations in which there exist exactly k Voronoi cells with nucleus in $\mathfrak{Q}_\rho \simeq [-173, 173]$ and such that the geometric characteristic is larger than $v_\rho(1)$.

We summarize our empirical results by box plots associated with the empirical values $(\hat{p}_k^{(i)})_{1 \leq i \leq 100}$. For each geometric characteristic, we explain how we simulate a Poisson-Voronoi tessellation conditional on the fact that $g^{\eta\cup\{0\}}(0) > v_\rho(1)$.

4.1 Inradius

For any $x \in \eta \subset \mathbf{R}^d$, we define the so-called inradius of the Voronoi cell $C_\eta(x)$ as

$$r^\eta(x) := r(C_\eta(x)) := \sup\{r \geq 0 : B(x,r) \subset C_\eta(x)\},$$

where $B(x,r)$ is the ball centered at x with radius r. The distribution of $r(\mathcal{C})$, where $r(\mathcal{C}) = r(C_{\eta\cup\{0\}}(0))$ is the typical inradius, is given by $\mathbb{P}(r(\mathcal{C}) > v) = \mathbb{P}(\eta \cap B(0, 2v) \neq \emptyset) = e^{-2^d \kappa_d v^d}$ for each $v \geq 0$. Hence, for any $\tau > 0$, we have $\rho \cdot \mathbb{P}(r(\mathcal{C}) > v_\rho(\tau)) = \tau$, when

$$v_\rho(\tau) := 2^{-1}\kappa_d^{-1/d}\left(\log(\rho\tau^{-1})\right)^{1/d}.$$

Moreover it is proved in [7] that

$$\mathbb{P}\left(\max_{x\in\eta\cap\mathbf{W}_\rho} r^\eta(x) \leq v_\rho(\tau)\right) \xrightarrow[\rho\to\infty]{} e^{-\tau}.$$

Actually, the convergence was established for a fixed window and for a Poisson point process such that the intensity goes to infinity. By scaling property of the Poisson point process, the result can be re-written as above for a fixed intensity and for a window $\mathbf{W}_\rho$ as ρ goes to infinity. Therefore, we deduce that the extremal index of the inradius of a Poisson-Voronoi tessellation exists and is equal to $\theta = 1$. Actually, according to Theorem 2 in [10], the point process of exceedances $\Phi^\eta_{\mathbf{W}_\rho}(\tau)$ converges to a simple Poisson point process of intensity τ in W. In particular, the distributions π and p are equal to the dirac measure at 1.

Now, we explain how we evaluate by simulation the value of the extremal index and the distribution p when $d = 2$. It is known (see e.g. [19]) that for each $v \geq 0$, we have

$$\left(\eta \cup \{0\} | r^{\eta\cup\{0\}}(0) = v\right) \overset{\mathcal{D}}{=} \eta_{B(0,2v)^c} \cup \{(2v)X_0\} \cup \{0\},$$

where $\eta_{B(0,2v)^c}$ is a Poisson point process of intensity measure $\mathbb{I}_{\{x\in B(0,2v)^c\}}\mathrm{d}x$ and where X_0 is a random point uniformly distributed on the boundary of $B(0,1)$. Hence, to simulate a Poisson-Voronoi tessellation provided that $r^{\eta\cup\{0\}}(0) > v_{\exp(100)}(1) \simeq 2.82$, we first simulate a random variable r with distribution given by $\mathbb{P}(r > v) = e^{-4\pi v^2}$, conditional on the fact that $r > 2.82$. Then we generate a Poisson-Voronoi tessellation associated with the point process $\eta_{B(0,2r)^c} \cup \{(2r)X_0\} \cup \{0\}$.

On the left part of Figure 2, we provide a simulation of a Poisson-Voronoi tessellation given that $r^{\eta\cup\{0\}}(0) > 2.82$. We notice that the typical cell has a shape which tends to be circular. Actually, such an observation is related to the D. G. Kendall's conjecture which claims that the shape of the
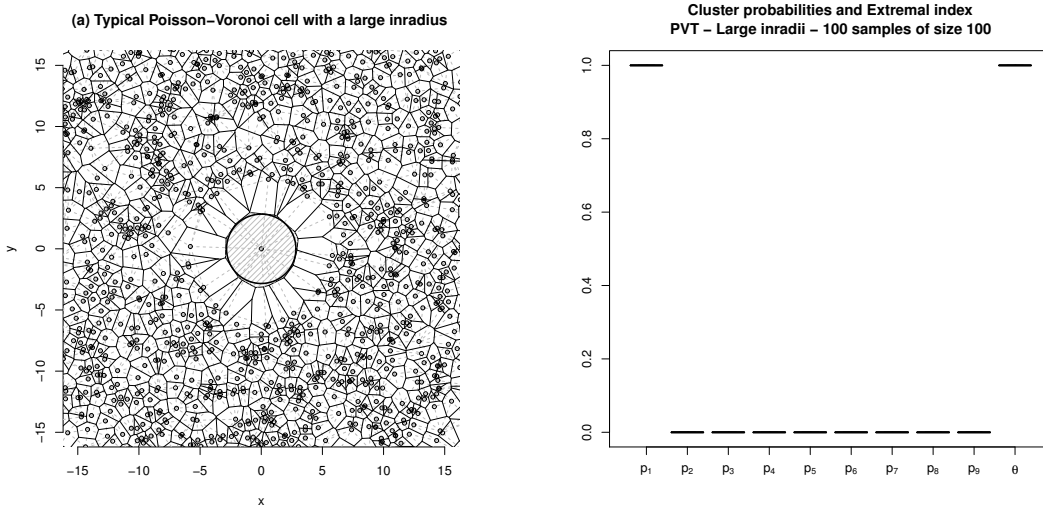
Figure 2: Large inradius for a Poisson-Voronoi tessellation

typical Poisson-Voronoi cell in $\mathbf{R}^d$, given that the volume of the cell goes to infinity, tends a.s. to a ball in $\mathbf{R}^d$. Many results concerning typical cells with a large geometric characteristic can be found in [8] and [16]. On the left part of Figure 2, we also notice that there is no cell with a large inradius, excepted the typical cell. This confirms that the cluster of exceedances are of size 1, i.e. $p_1 = 1$ and $\theta = 1$. The right part of Figure 2 provides the box plots of the empirical distributions. In particular, for all simulations, we notice that there is always exactly one cell with a large inradius.

4.2 Reciprocal of the inradius

In this example, we consider the large values of the reciprocal of the inradii for a Poisson-Voronoi tessellation in $\mathbf{R}^d$. Equivalently, this consists of the small values of the inradii. Since $\mathbb{P}\left(r(\mathcal{C}) < v \right) = 1 - e^{-2^d \kappa_d v^d}$, we have $\rho \cdot \mathbb{P}\left(r(\mathcal{C}) < v_\rho(\tau) \right) \underset{\rho \to \infty}{\longrightarrow} \tau$, when

$$v_\rho(\tau) := 2^{-1}(\kappa_d \rho)^{-1/d} \tau^{1/d}.$$

Here, we have written "$r(\mathcal{C}) < v_\rho(\tau)$" instead of "$r(\mathcal{C}) > v_\rho(\tau)$" in the probability because we consider the smallest inradii. Moreover, according to [7], we know that

$$\mathbb{P}\left(\min_{x \in \eta \cap \mathbf{W}_\rho} r(C_\eta(x)) \geq v_\rho(\tau) \right) \underset{\rho \to \infty}{\longrightarrow} e^{-\tau/2}.$$

We deduce that the extremal index of the reciprocal of the inradius of a Poisson-Voronoi tessellation exists and equals $\theta = 1/2$.

As in Section 4.1, we can easily simulate a Poisson-Voronoi tessellation in $\mathbf{R}^2$, conditional on the fact that $r^{\eta \cup \{0\}}(0) < v_{\exp(100)}(1) \simeq 5.44 \cdot 10^{-23}$. The left part of Figure 3 provides a realization of a Poisson-Voronoi tessellation when $r^{\eta \cup \{0\}}(0) < v_{\exp(4)}(1) \simeq 0.0381$ (here, we have taken the threshold $v_{\exp(4)}(1)$ instead of $v_{\exp(100)}(1)$ for convenience). The fact that $\theta = 1/2$ can be explained by a trivial heuristic argument: if a cell minimizes the inradius, one of its neighbors has to do the same (see also the left part of Figure 3). Moreover, we can easily prove that the probability that there is more than one such a cell is negligible. Therefore clusters are necessarily of size
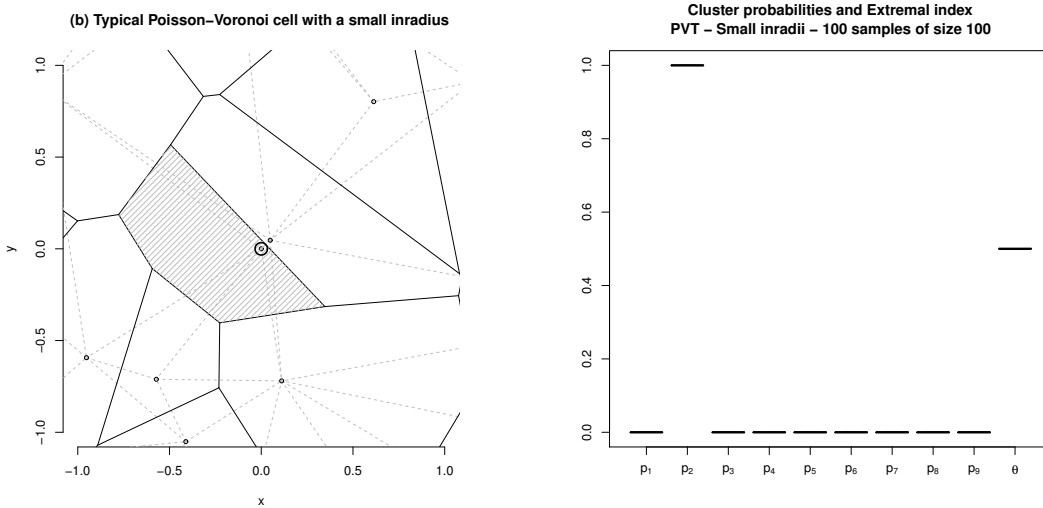
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Figure 3: Small inradius for a Poisson-Voronoi tessellation

2, i.e. $p_2 = 1$. The right part of Figure 3 provides the box plots of the empirical distributions. In particular, for all simulations, we notice that there are always exactly two cells with a small inradius.

4.3 Circumradius

For $x \in \eta \subset \mathbf{R}^2$, we define the so-called circumradius of $C_\eta(x)$ as

$$R^\eta(x) := R(C_\eta(x)) := \inf\{r \geq 0 : B(x, r) \supset C_\eta(x)\}.$$

According to [6], we know that

$$2ve^{-v} \leq \mathbb{P}\left(\pi R(\mathcal{C})^2 > v,\right) \leq 4ve^{-v},$$

for each $v \geq 0.337$. Actually, simulations suggest that the upper bound above is the order of $\mathbb{P}\left(\pi R(\mathcal{C})^2 > v\right)$ as v goes to infinity (see Table 1 in [6]). If we assume that $\mathbb{P}\left(\pi R(\mathcal{C})^2 > v\right) \underset{\rho \to \infty}{\sim} ave^{-v}$ for some $2 \leq a \leq 4$, we have $\rho \cdot \mathbb{P}\left(R(\mathcal{C}) > v_\rho(\tau)\right) \underset{\rho \to \infty}{\longrightarrow} \tau$, when

$$v_\rho(\tau) := \pi^{-1/2}\left(\log\left(a\rho \log \rho \tau^{-1}\right)\right)^{1/2}.$$

Thanks to (2.c) in [7], we know that

$$\mathbb{P}\left(\max_{x \in \eta \cap \mathbf{W}_\rho} R^\eta(x) \leq v_\rho(\tau)\right) \underset{\rho \to \infty}{\longrightarrow} e^{-\tau/a}.$$

Hence, provided that $\mathbb{P}\left(\pi R(\mathcal{C})^2 > v\right) \underset{\rho \to \infty}{\sim} ave^{-v}$, the extremal index of the maximum of circumradius of a planar Poisson-Voronoi tessellation exists and should be equal to $\theta = 1/a$.

Now, we explain how we evaluate by simulation the value of the extremal index and the distribution p. According to Lemma 1 in [13], we know that $R^{\eta \cup \{0\}}(0) > v$ if and only if there exists a disk of radius v containing the origin on its boundary and no particle inside. Without loss of generality,

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(c) Typical Poisson–Voronoi cell with a large circumradius

Cluster probabilities and Extremal index
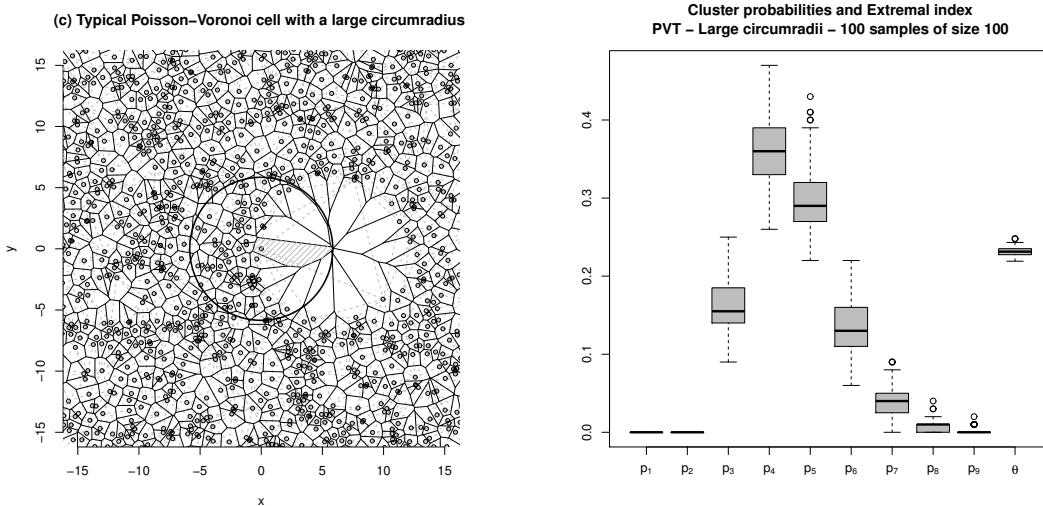**PVT – Large circumradii – 100 samples of size 100**

Figure 4: Large circumradius for a Poisson-Voronoi tessellation

we can assume that the disk, that contains the origin on its boundary and no particle inside, has its center on the x-axis, since the Poisson point process is isotropic. Hence, we proceed as follows. First, we simulate a random variable R_b, with distribution such that $\mathbb{P}\left(\pi R_b^2 > v\right) \underset{v\to\infty}{\sim} bve^{-v}$, with $b = 4$, given that $R_b > v_{\exp(100)}(1) \simeq 5.81$. We have taken $b = 4$ since we should have $\mathbb{P}\left(\pi R(\mathcal{C})^2 > v\right) \underset{v\to\infty}{\sim} 4ve^{-v}$ as suggested in the simulations in [6]. However, this choice is arbitrary and does not have influence on the final result since the conditional distribution of R_b does not depend on b for high thresholds. Then, we generate a Voronoi tessellation induced by the point process $\eta_{B((R,0),R)^c} \cup \{0\}$, where $\eta_{B((R,0),R)^c}$ is a Poisson point process of intensity measure $\mathbb{I}_{x \in B((R,0),R)^c}\,\mathrm{d}x$.

On the left part of Figure 4, we provide a simulation of the Palm version of the Poisson-Voronoi tessellation, given that $R(\mathcal{C}) > v_{\exp(100)}(1) \simeq 5.81$. We notice that the typical cell is very elongated and that the same fact holds for a large number of its connected cells. In particular, the size of a cluster of exceedances is random. On the right part of Figure 4, we provide the box plots of the empirical distributions. This time, the empirical distributions of the cluster size probabilities are not degenerated for $k = 3, \ldots, 9$, and their interquartile ranges are quite large for $k = 3, 4, 5$. We also notice that the empirical value of the extremal index is very concentrated around a value close to $1/4$. This confirms that if a exists, it should be close to 4.

5 The case of the Poisson-Delaunay tessellation

The Poisson-Delaunay tessellation Let $\chi \in \mathcal{F}_{lf}$ be a locally finite subset of $\mathbf{R}^d$ such that each subset of size $n < d+1$ of points are affinely independent and no $d+2$ points lie on a sphere. If two points $x, y \in \chi$ are Voronoi neighbors, i.e. $C_\chi(x) \cap C_\chi(y) \neq \varnothing$, we connect these two points by an edge. The family of these edges defines a partition of $\mathbf{R}^d$ into simplices which is the so-called Delaunay tessellation. Another useful characterization of the Delaunay tessellation is the following: a simplex associated with $d+1$ points of χ is a Delaunay simplex if and only if its circumball contains no point of χ in its interior. Delaunay tessellations are very popular structures in computational geometry [1] and are extensively used in many areas such as surface reconstruction [9] or mesh

generation [11].

For each cell C of the Delaunay tessellation, the nucleus $z(C)$ is defined as the center of the circumball of C. The set of this nuclei is denoted by $Z(\chi)$. Besides, for each $z \in Z(\chi)$, we denote by $C(z)$ the Delaunay cell whose the center of its circumball is z.

When $\chi = \eta$ is a homogeneous Poisson point process, the family of these cells is the so-called *Poisson-Delaunay tessellation*. If we denote by γ_η the intensity of η, then the intensity of the Poisson-Delaunay tessellation is $\gamma_{Z(\eta)} = \beta_d^{-1} \cdot \gamma_\eta$, where

$$\beta_d := \frac{(d^3 + d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^d\left(\frac{d+2}{2}\right)2^{d+1}\pi^{\frac{d-1}{2}}}.$$

In particular, if $d = 2$, we have $\beta_2 = 1/2$. In the rest of the paper, we assume that $\gamma_\eta = \beta_d$ to ensure that $\gamma_{Z(\eta)} = 1$.

The typical cell of a Poisson-Delaunay tessellation can be made explicit as follows. Let $S^{d-1} := \{x \in \mathbf{R}^d : |x| = 1\}$ be the unit sphere of $\mathbf{R}^d$ and, for $u_{1:d+1} \in (S^{d-1})^{d+1}$, let $\Delta(u_{1:d+1}) := conv(u_1, \ldots, u_{d+1})$. According to Miles (see e.g. Theorem 10.4.4 in [28]), for any bounded measurable function $f : \mathbf{R}_+ \times S^{d-1} \to \mathbf{R}$, we have

$$\mathbb{E}[f(\mathcal{C})] := a_d\gamma_\eta^d \int_{\mathbf{R}_+} \int_{S^{d-1}} \cdots \int_{S^{d-1}} a(u_{1:d+1})r^{d^2-1}e^{-\gamma_\eta\kappa_d r^d}f(\Delta(ru_{1:d+1}))\sigma(\mathrm{d}u_{1:d+1})\mathrm{d}r, \qquad (5.1)$$

where $a_d := \beta_d/(d+1)$ and $a(u_{1:d+1}) := \lambda_d(\Delta(u_{1:d+1}))$. The measure $\sigma(\mathrm{d}u)$ is the uniform distribution on S^{d-1} with normalization $\sigma(S^{d-1}) = \omega_{d-1}$, where $\omega_{d-1} := d\kappa_d$ is the area of the unit sphere and $\sigma(\mathrm{d}u_{1:d+1}) := \bigotimes_{i=1}^{d+1}\sigma(\mathrm{d}u_i)$. Hence, the typical satisifes the equality in distribution $\mathcal{C} \stackrel{\mathcal{D}}{=} \Delta(RU_{1:d+1})$, where $R > 0$ and $U_{1:d+1} \in (S^{d-1})^{d+1}$ are two independent random variables whose the distributions are provided in (1.1).

The extremes of the Poisson-Delaunay tessellation   Let g be a geometric characteristic such that (1.6) holds. As for a Poisson-Voronoi tessellation, we consider the point process of normalized exceedances, say

$$\Psi^\eta(\tau) := \rho^{-1/d} \cdot \{z \in Z(\eta) : g(C(z)) > v_\rho(\tau)\}.$$

For any Borel subset $B \subset \mathbf{R}^d$, we write $\Psi_B^\eta(\tau) := \Psi^\eta(\tau) \cap (\rho^{-1/d}B)$. We also let $\Psi^{\eta,0}(\tau)$ be the Palm version of $\Psi^\eta(\tau)$ and $\Psi_B^{\eta,0}(\tau) = \Psi^{\eta,0}(\tau) \cap B$. In the rest of the paper, the quantity $p_{k,B}(\tau)$ refers to as the probability that there exist k exceedance cells in B conditional on the fact that the typical cell is an exceedance, i.e. $p_{k,B}(\tau) := \mathbb{P}\left(\#\psi_B^{\eta,0}(\tau) = k\right)$. In the same spirit as Lemma 2, we provide below an explicit characterization of this probability.

Proposition 8 *Let $\mathcal{A}$ be a Borel subset in $\mathcal{F}_{lf}$. Then*

$$\mathbb{P}\left(\Psi^{\eta,0}(\tau) \in \mathcal{A}\right) = \mathbb{P}\left(\Psi^{\eta_{\mathbf{R}^d\backslash B(0,R)}\cup\{RU_{1:d+1}\}}(\tau) \in \mathcal{A}\,\middle|\,g(\Delta(RU_{1:d+1})) > v_\rho(\tau)\right).$$

Therefore, for any $B \subset \mathbf{R}^d$,

$$p_{k,B}(\tau) = \mathbb{P}\left(\#\Psi_B^{\eta_{\mathbf{R}^d\backslash B(0,R)}\cup\{RU_{1:d+1}\}}(\tau) = k\,\middle|\,g(\Delta(RU_{1:d+1})) > v_\rho(\tau)\right).$$

Proof Let $A \subset \mathbf{R}^d$ be such that $\lambda_d(A) = 1$. It follows from the definition of the Palm distribution of the point process $\Psi^\eta(\tau)$, that

$$
\mathbb{P}\left(\Psi^{\eta,0}(\tau) \in \mathcal{A}\right) := \frac{1}{\gamma_{\Psi^\eta(\tau)}} \mathbb{E}\left[\sum_{z \in \Psi^\eta(\tau) \cap A} \mathbb{I}_{(\Psi^\eta(\tau)-z) \in \mathcal{A}} \right]
$$

$$
= \frac{1}{\gamma_{\Psi^\eta(\tau)}} \mathbb{E}\left[\sum_{\{x_{1:d+1}\} \subset \eta} \mathbb{I}_{(\Psi^\eta(\tau)-\rho^{1/d}z(x_{1:d+1})) \in \mathcal{A}} \, \mathbb{I}_{z(x_{1:d+1}) \in Z(\eta) \cap \mathbf{A}_\rho} \right],
$$

where $z(x_{1:d+1})$ is the center of the circumball of the simplex $\Delta(x_{1:d+1})$. According to the Slivnyak-Mecke formula and the Blaschke-Petkantschin formula (e.g. Theorem 7.3.1 in [28]) that

$$
\mathbb{P}\left(\Psi^{\eta,0}(\tau) \in \mathcal{A}\right) = \frac{\gamma_\eta^{d+1} d!}{\gamma_{\Psi^\eta(\tau)}(d+1)!} \int_{\mathbf{A}_\rho} \int_{\mathbf{R}_+} \int_{S^{d+1}} r^{d^2-1} a(u_{1:d+1})
$$

$$
\times \mathbb{P}\left((\Psi^\eta(\tau) - \rho^{1/d}z) \in \mathcal{A}, \eta \cap B(z,r) = \varnothing\right) \mathbb{I}_{g(z+\Delta(ru_{1:d+1}))>v_\rho(\tau)} \, \sigma(du_{1:d+1}) dr dz.
$$

Since η is stationary and since g is translation-invariant, the integrand does not depend on z. Integrating over $z \in \mathbf{A}_\rho$, and using the fact that $\lambda_d(\mathbf{A}_\rho) = \rho$, we get

$$
\mathbb{P}\left(\Psi^{\eta,0}(\tau) \in \mathcal{A}\right) = \frac{\gamma_\eta^{d+1} \rho}{\gamma_{\Psi^\eta(\tau)}(d+1)} \int_{\mathbf{R}_+} \int_{S^{d+1}} r^{d^2-1} a(u_{1:d+1})
$$

$$
\times \mathbb{P}\left(\Psi^\eta(\tau) \in \mathcal{A}, \eta \cap B(0,r) = \varnothing\right) \mathbb{I}_{g(\Delta(ru_{1:d+1}))>v_\rho(\tau)} \, \sigma(du_{1:d+1}) dr.
$$

We give below an explicit representation for the integrand. Let $\eta_{B(0,r)}$ and $\eta_{\mathbf{R}^d \backslash B(0,r)}$ be two independent Poisson point processes with intensity measures $\gamma_\eta \mathbb{I}_{x \in B(0,R)} \, dx$ and $\gamma_\eta \mathbb{I}_{x \in \mathbf{R}^d \backslash B(0,r)} \, dx$ respectively. We know that

$$
\eta \overset{\mathcal{D}}{=} \eta_{B(0,r)} \cup \eta_{\mathbf{R}^d \backslash B(0,r)}.
$$

This gives

$$
\mathbb{P}\left(\Psi^\eta(\tau) \in \mathcal{A}, \eta \cap B(0,r) = \varnothing\right) = \mathbb{P}\left(\Psi^{\eta_{\mathbf{R}^d \backslash B(0,r)} \cup \{ru_{1:d+1}\}}(\tau) \in \mathcal{A}, \eta_{B(0,r)} \cap B(0,r) = \varnothing\right)
$$

$$
= e^{-\gamma_\eta \kappa_d r^d} \mathbb{P}\left(\Psi^{\eta_{\mathbf{R}^d \backslash B(0,r)} \cup \{ru_{1:d+1}\}}(\tau) \in \mathcal{A}\right).
$$

Hence,

$$
\mathbb{P}\left(\Psi^{\eta,0}(\tau) \in \mathcal{A}\right) = \frac{\gamma_\eta^{d+1} \rho}{\gamma_{\Psi^\eta(\tau)}(d+1)} \int_{\mathbf{R}_+} \int_{S^{d+1}} r^{d^2-1} a(u_{1:d+1})
$$

$$
\times e^{-\gamma_\eta \kappa_d r^d} \mathbb{P}\left(\Psi^{\eta_{\mathbf{R}^d \backslash B(0,r)} \cup \{ru_{1:d+1}\}}(\tau) \in \mathcal{A}\right) \sigma(du_{1:d+1}) dr.
$$

Moreover, we know that $\gamma_{\Psi^\eta(\tau)} = \rho \mathbb{P}\left(g(\mathcal{C}) > v_\rho(\tau)\right)$ and that $\gamma_\eta = (d+1)a_d$. Then, we get

$$
\mathbb{P}\left(\Psi^{\eta,0}(\tau) \in \mathcal{A}\right) = \frac{a_d \gamma_\eta^d}{\mathbb{P}\left(g(\mathcal{C}) > v_\rho(\tau)\right)} \int_{\mathbf{R}_+} \int_{S^{d+1}} r^{d^2-1} a(u_{1:d+1})
$$

$$
\times e^{-\gamma_\eta \kappa_d r^d} \mathbb{P}\left(\Psi^{\eta_{\mathbf{R}^d \backslash B(0,r)} \cup \{ru_{1:d+1}\}} \in \mathcal{A}\right) \sigma(du_{1:d+1}) dr.
$$

This proves the first equality in Proposition 8 since $\mathcal{C} \overset{\mathcal{D}}{=} \Delta(RU_{1:d+1})$. The second equality is a direct consequence of the first one. □

We think that Theorem 4 can be adapted in the context of a Poisson-Delaunay tessellation. To do it, we have to replace the point process $\Phi^\eta(\tau)$ by the point process $\Psi^\eta(\tau)$ and we have to use the characterization of the probability $p_{k,\mathfrak{Q}_\rho}(\tau_0)$ as described in the above proposition. We can easily extend Lemma 1 and adapt Condition (C) in the particular setting of a Poisson-Delaunay tessellation. However, the main difficulty to adapt Theorem 4 focuses on an analogous version of Proposition 3 since its proof seems very technical. We give below a numerical illustration which confirms that Theorem 4 should be true for a Poisson-Delaunay tessellation.

A numerical illustration   Let m_{PDT} be a Poisson-Delaunay tessellation generated by a Poisson point process η in $\mathbf{R}^d$ with intensity $\gamma_\eta = \beta_d$. For each cell $C \in m_{PDT}$, we consider the so-called circumcenter of C defined as

$$R(C) := \inf\{R \geq 0 : C \subset B(z,R), z \in \mathbb{R}^d\}.$$

According to (5.1), the random variable $\kappa_d R(\mathcal{C})^d$ is Gamma distributed with parameters (d, β_d). A Taylor expansion of $\mathbb{P}(R(\mathcal{C}) > v)$ as v goes to infinity (e.g. Equation (3.14) in [10]), shows that $\rho \cdot \mathbb{P}(R(\mathcal{C}) > v_\rho(\tau)) \underset{\rho \to \infty}{\longrightarrow} \tau$, when

$$v_\rho(\tau) := (\kappa_d \beta_d)^{-1/d} \cdot \left(\log \left([(d-1)!]^{-1} \rho \log(\beta_d \rho)^{d-1} \tau^{-1} \right) \right)^{1/d}.$$

Moreover, with standard arguments, we can easily show that the maximum of circumradii of Delaunay cells $\max_{C \in m_{PDT}, z(C) \in \mathbf{W}_\rho} R(C)$ has the same asymptotic behavior as the maximum of circumradii of the associated Voronoi cells $\max_{x \in \eta \cap \mathbf{W}_\rho} R(C_\eta(x))$. Besides, according to (2c) in [10], we know that

$$\mathbb{P}\left(\max_{x \in \eta \cap \mathbf{W}_\rho} R(C_\eta(x)) \leq (\kappa_d \beta_d)^{-1/d} \left(\log \left(\alpha_d \beta_d \rho \log(\beta_d \rho)^{d-1} \tau^{-1} \right) \right)^{1/d} \right) \underset{\rho \to \infty}{\longrightarrow} e^{-\tau},$$

where $\alpha_d := \frac{1}{d!} \left(\frac{\pi^{1/2} \Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)} \right)^{d-1}$. It follows that

$$\mathbb{P}\left(\max_{\substack{C \in m_{PDT}, \\ z(C) \in \mathbf{W}_\rho}} R(C) \leq v_\rho(\tau) \right) \underset{\rho \to \infty}{\longrightarrow} e^{-\theta_d \tau},$$

where

$$\theta_d := \alpha_d \beta_d (d-1)! = \frac{(d^3 + d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{d\Gamma\left(\frac{d^2+1}{2}\right)\Gamma\left(\frac{d+2}{2}\right) 2^{d+1}}.$$

In particular, when $d = 1, 2, 3$, the extremal index equals $\theta_1 = 1$, $\theta_2 = 1/2$ and $\theta_3 = 35/128$ respectively.

Now, we explain how we evaluate by simulation the value of the extremal index and the distribution p when $d = 2$. First, we simulate a random variable R such that πR^2 is Gamma distributed with parameters $(2, 1/2)$, given that $R > v_{\exp(100)}(1) \simeq 8.16$. Then we simulate a typical cell $\mathcal{C}$, with circumradius R, by using the method described in [17]. The Poisson-Delaunay tessellation
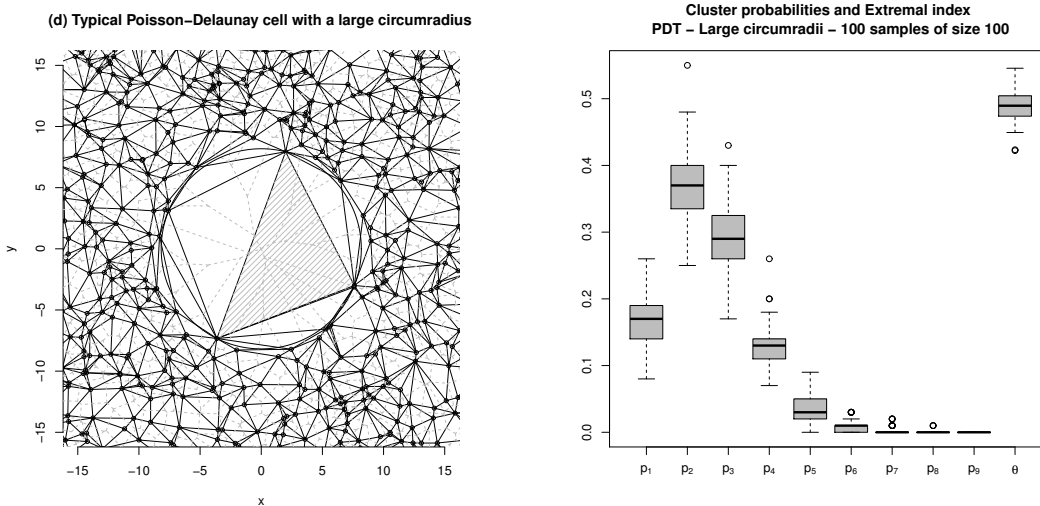
Figure 5: Large circumradius for a Poisson-Delaunay tessellation

which is generated is induced by the point process $\eta_{B(0,2R)^c} \cup \{0\}$, where $\eta_{B(0,2R)^c}$ is a Poisson point process with intensity measure $\mathbb{I}_{x \in B(0,2R)^c} \, \mathrm{d}x$ (see Proposition 8).

On the left part of Figure 5, we provide a simulation of the Palm version of the Poisson-Delaunay tessellation given that the typical cell has a circumradius larger than 8.16. The number of neighbors of the typical cells which are exceedances is random. The right part of Figure 5 provides the box plots of the empirical probabilities. Notice that these empirical distributions are not degenerated for $k = 1, \ldots, 8$. Their interquartile ranges are not so important as for the circumradii of the Poisson-Voronoi tessellation, but the spread of the empirical distribution of the extremal index is larger. Besides, the empirical value of the extremal index is very concentrated around a value close to $1/2$, which is the theoretical value of θ.

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