Construction of ergodic IDLA forests in $\mathbb{Z}^d$

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Abstract

We prove the existence of infinite-volume IDLA forests in $\mathbb{Z}^d$, with $d \geq 2$, based on a multi-source IDLA protocol. Unlike IDLA aggregates, the laws of the IDLA forests studied here depend on the trajectories of particles, and then do not satisfy the famous *Abelian property*. Their existence is due to a stabilization result (Theorem 1.1, our main result) that we establish using percolation tools. Although the sources are infinitely many, we also prove that each of them play the same role in the building procedure, which results in an ergodicity property for the IDLA forests (Theorem 1.2).

Keywords: Random walks, growth model, percolation, stabilization, multiscale argument.

Mathematics Subject Classification: 60K35, 82C24, 82B41.

1 Introduction

The Internal Diffusion Limited Aggregation (IDLA) model gives a protocol to build random aggregates $(A_n)_{n\geq 0}$ recursively in $\mathbb{Z}^d$. Initially, we assume that $A_0 = \emptyset$. Then, at some step $n \geq 1$, given A_{n-1}, the first site visited outside of A_{n-1} by a random walk started at the origin is added to A_{n-1} in order to obtain A_n. In this context, such random walks are called *particles.* IDLA was initially introduced by Meakin and Deutch in [22] to model an industrial chemical technique known as electropolishing. The goal of such a process is to eliminate a small coat of material off of a metallic surface in order to make it smoother. It became pertinent to quantify how smooth the surface of a polished metal could get through such a process.

A first shape theorem was established by Lawler, Bramson and Griffeath in [19] to describe the asymptotic shape of A_n, as n tends to infinity, as a Euclidean ball. This result was later made sharper with the works of Jerison, Levine and Sheffield [15, 16, 17] and Asselah and Gaudillière [1, 2, 3]. In their works, the bounds for fluctuations around the limit shape are improved from linear to logarithmic in dimension $d = 2$ and sublogarithmic in dimensions $d \geq 3$. See also [12] for a continuous time version. From then on, variants of the IDLA model have been studied on many other graphs, such as on cylinder graphs in [18, 21, 25], on supercritical percolation clusters in [10, 24], on comb lattices in [4, 14] or on non-amenable graphs in [13, 23].

While the previously cited IDLA models are single-source, multi-source IDLA models have also been considered in [7, 8, 20] and by the authors in [5]. This question was originally investigated by Diaconis and Fulton [9] in the context of the *smash sum* of two domains in which they discover the famous *Abelian Property* of IDLA aggregates, meaning that modifying the order in which particles are launched does not change the distribution of the final aggregate. This beautiful property will be a powerful tool for the study of IDLA models.

In the present paper, we introduce new random graphs on $\mathbb{Z}^d$ with $d \geq 2$, called *IDLA forests*, whose construction is based on a multi-source IDLA protocol. We consider an infinite set of sources, namely the hyperplane $\mathcal{H} := \{0\} \times \mathbb{Z}^{d-1}$. Basically, in addition to the site at which the current particle exits the aggregate and stops, we also retain the edge by which the particle reaches that site. This procedure leads to a random forest on the lattice $\mathbb{Z}^d$, i.e. a collection of disjoint random trees rooted at sources of $\mathcal{H}$. Unlike IDLA aggregates, trajectories of particles really matter for the IDLA forests, meaning the Abelian property is no longer true for this new model, making it more difficult to show the existence of these forests.

However, we prove in Theorem 1.1 (our main result) the existence of the infinite-volume IDLA forests, generated by the infinite set of sources $\mathcal{H}$. See Figure 1 for a simulation in dimension $d = 2$. Moreover, our construction does not favor any source in the sense that (roughly speaking), at any time, the next source to emit a particle is selected 'uniformly' among all the sources of $\mathcal{H}$. This remarkable feature is stated in Theorem 1.2 in which we prove that the IDLA forests are ergodic w.r.t. the translations of $\mathcal{H}$.

Let us notice that the existence of bi-dimensional IDLA forests has been already explored by the authors in [6] but their proof strongly used the one-dimensional aspect of the set of sources–which is $\{0\} \times \mathbb{Z}$ when $d = 2$–and then completely collapses in higher dimensions. More than a generalization of [6], we think that the method developed here and based on percolation tools is original in the context of IDLA and is certainly promising to deal with graphs built from IDLA protocols with infinitely many sources.

Construction of the finite-volume IDLA forests

Let us start by describing our random inputs. Let us first consider a family of i.i.d. Poisson Point Processes (PPP) on $\mathbb{R}_+$, with intensity 1 and denoted by $\{\mathcal{N}_z\}_{z\in\mathcal{H}}$. Each PPP $\mathcal{N}_z$ provides a sequence $(\tau_{z,j})_{j\geq 1}$ of successive tops which act as random clocks for the emission of particles from the source z. Thus, to the sequence $(\tau_{z,j})_{j\geq 1}$, is associated a sequence of i.i.d. simple random walks $(S_{z,j})_{j\geq 1}$ on $\mathbb{Z}^d$ starting at z. Note that the $S_{z,j}$'s, for $z \in \mathcal{H}$ and $j \geq 1$, are independent from each other and also independent from the PPP $\mathcal{N}_z$'s. Hence, at time $\tau_{z,j}$, a particle is emitted from the source z (precisely, the j-th one coming from z), and follows the trajectory given by the random walk $S_{z,j}$ until exiting the current aggregate. To avoid having multiple particles alive at the same time, we assume that particles realize their trajectories instantaneously (w.r.t. the Poisson clocks).

Let $M, n \geq 0$ be integers. Set $\mathcal{H}_M := \{0\} \times [\![-M, M]\!]^{d-1}$. Let us consider the random set $A_n^\dagger[M] \subset \mathbb{Z}^d$ defined as the IDLA aggregate generated by particles emitted from the sources of $\mathcal{H}_M$ and during the time interval $[0, n]$. Since the number of particles involved in $A_n^\dagger[M]$ is a.s. finite ($n(2M + 1)^{d-1}$ in mean), this aggregate is a.s. well defined. See the left hand side of Figure 1 for a simulation of $A_{20}^\dagger[50]$ in dimension $d = 3$.

Now, we can build quite naturally from $A_n^\dagger[M]$ a finite-volume IDLA forest $\mathcal{F}_n[M]$, simply by considering the edges of $\mathbb{Z}^d$ from which the particles involved in $A_n^\dagger[M]$ exit the current aggregate. Let $\kappa := \sum_{z\in\mathcal{H}_M} \#\mathcal{N}_z([0, n])$ be the total number of such particles. Let us enumerate them according to their starting times, say $0 < \tau_1 < \tau_2 < ... < \tau_\kappa < n$ (they are a.s. different). For $j \in [\![1, \kappa]\!]$, we denote by $A[j]$ the aggregate obtained until time τ_j, including the site added by the particle sent at time τ_j. We then have $A[0] = \emptyset$ (with $\tau_0 = 0$) and $A[\kappa] = A_n^\dagger[M]$. We proceed by induction to build the associated forest $\mathcal{F}_n[M]$. Let us first set $\mathcal{F}_n[M, 0] = (\emptyset, \emptyset)$. Now, for $j \in [\![1, \kappa]\!]$, given the random graph $\mathcal{F}_n[M, j-1] = (V_{j-1}, E_{j-1})$, we define $\mathcal{F}_n[M, j] = (V_j,\ E_j)$ as follows. Let x be the new site added to $A[j - 1]$ by the j-th particle.

- If x is the source from which the j-th particle is emitted, then this particle actually is the first one emitted from x, and x will be the root of a new tree in the graph. We set $V_j = V_{j-1} \cup \{x\}$ and $E_j = E_{j-1}$.
- Otherwise, let x' be the last site of $A[j-1]$ visited by the j-th particle before reaching x. Then we set $V_j = V_{j-1} \cup \{x\}$ and $E_j = E_{j-1} \cup \{(x', x)\}$.

Finally, we define $\mathcal{F}_n[M] := \mathcal{F}_n[M, \kappa]$. See the right hand side of Figure 1 for a simulation of $\mathcal{F}_{30}[20]$ in dimension $d = 2$. This construction ensures that $\mathcal{F}_n[M]$ is a finite union of trees with roots in $\mathcal{H}_M$ and whose vertex set is equal to $A_n^\dagger[M]$.

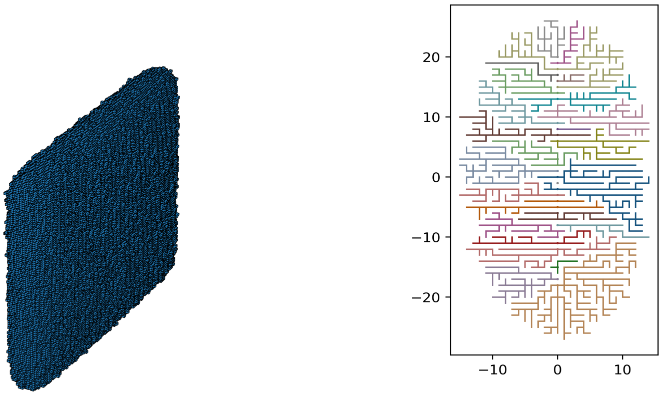

Figure 1: To the left: A realization in dimension $d = 3$ of the multi-source aggregate $A_{20}^\dagger[50]$ with particles emitted from $\mathcal{H}_{50}$ and on the time interval $[0, 20]$, looking like a 'bar of soap'. Each settled particle is represented by a blue cube. To the right: A realization in dimension $d = 2$ of the finite-volume IDLA forest $\mathcal{F}_{30}[20]$ associated to the aggregate $A_{30}^\dagger[20]$. Each tree is represented in a different color. Unfortunately, for visual reasons, we will only represent IDLA forests in dimension $d = 2$.

No monotonicity because of chains of changes

Thanks to the *natural coupling* defined in Section 2.2, one can construct on the same probability space the aggregates $A_n^\dagger[M]$, for all $M \geq 0$, in such a way that a.s. $A_n^\dagger[M] \subset A_n^\dagger[M+1]$ for any M. This monotonicity property allows us to a.s. define a limiting aggregate as

$$A_n^\dagger[\infty] := \bigcup_{M \geq 0} \uparrow A_n^\dagger[M] \ . \tag{1.1}$$

However the same monotonicity property does not hold for the sequence $(\mathcal{F}_n[M])_{M\geq 0}$ of associated IDLA forests, as depicted in Figure 2. Consequently, we cannot define an infinite-volume forest as the increasing union of finite-volume forests, in the same spirit as (1.1). Let $M' \geq M \geq 0$. Although their vertex sets satisfy the inclusion $V(\mathcal{F}_n[M]) = A_n^\dagger[M] \subset V(\mathcal{F}_n[M']) = A_n^\dagger[M']$ (thanks to the natural coupling), this is no longer true for their edge sets. Indeed, some vertices present in both forests $\mathcal{F}_n[M]$ and $\mathcal{F}_n[M']$ are reached using different particles and possibly through different edges. This contributes to an edge in $\mathcal{F}_n[M]$ which is not present in $\mathcal{F}_n[M']$ (and conversely). These discrepancies between $\mathcal{F}_n[M]$ and $\mathcal{F}_n[M']$ can occur through a tricky phenomenon called *chains of changes* that we detail now. In a first time, the reader may skip this part and go directly to the result section below.

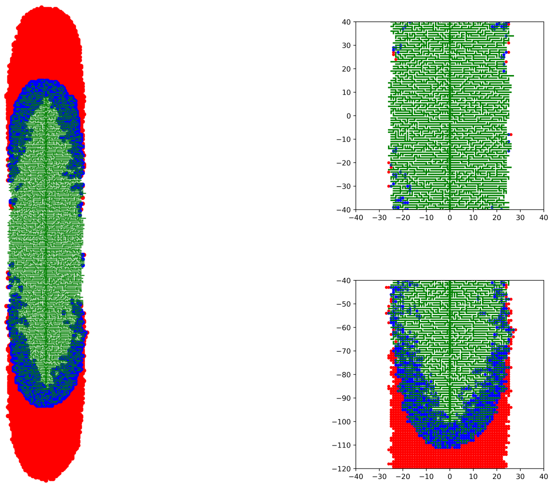

Figure 2: A realization of $\mathcal{F}_{50}[100]$ and $\mathcal{F}_{50}[150]$ using the natural coupling, i.e. their vertex sets are such that $A^{\dagger}_{50}[100] \subset A^{\dagger}_{50}[150]$. The green edges are common to both forests. The blue points are vertices common to both aggregates, but reached by different particles, and may lead to discrepancies between $\mathcal{F}_{50}[100]$ and $\mathcal{F}_{50}[150]$. The red points are vertices of $A^{\dagger}_{50}[150] \setminus A^{\dagger}_{50}[100]$. Both pictures on the right are zooms of the one on the left. One can see the presence of many blue points, especially on both extremities of $A^{\dagger}_{50}[100]$. Remark also that some of them appear in the vicinity of the e_1-axis (e_1 denoting the first vector of the canonical basis): see the top left picture. These possible discrepancies are produced by chains of changes.

Let $M' \geq M \geq 0$ and $n \geq 0$. To explain what a chain of changes is, we need to slightly describe the natural coupling we use. The basic idea of that coupling consists in using the same random clocks $(\mathcal{N}_z)$ and the same random walks $(S_{z,j})$ for both aggregates $A^{\dagger}_n[M]$ and $A^{\dagger}_n[M']$. Hence a particle starting from a source in $\mathcal{H}_M$ will work for both aggregates, i.e. it will add a new site to both aggregates (but not necessarily the same site). However, a particle starting from a source in $\mathcal{H}_{M'} \setminus \mathcal{H}_M$ will only work for the larger aggregate $A^{\dagger}_n[M']$. Now, consider such a particle starting at time $t_1 \in (0, n)$ from a source in $\mathcal{H}_{M'} \setminus \mathcal{H}_M$, it adds a site x_1 to the larger aggregate. Precisely, if we write $A^{\dagger}_{t_1^-}[M']$ the current aggregate produced right before sending that particle, then we get $A^{\dagger}_{t_1}[M'] = A^{\dagger}_{t_1^-}[M'] \cup \{x_1\}$, while $A^{\dagger}_{t_1}[M] = A^{\dagger}_{t_1^-}[M]$ remains unchanged. At this time, x_1 belongs to the larger aggregate $A^{\dagger}_{t_1}[M']$ but not to the smaller one $A^{\dagger}_{t_1}[M]$. If no other future particles starting from $\mathcal{H}_M$ visit the site x_1, then x_1 will remain a discrepancy until time n between both aggregates. Conversely, assume x_1 is visited at time $t_2 \in (t_1, n)$ (and for the first time) by a particle coming from $\mathcal{H}_M$, then both aggregates are updated as follows:

- The site x_1 is added to the smaller aggregate: $A^{\dagger}_{t_2}[M] = A^{\dagger}_{t_2^-}[M] \cup \{x_1\}$.
- Since x_1 already belongs to $A^{\dagger}_{t_2^-}[M']$, the particle continues its trajectory until exiting the larger aggregate, say through a new site x_2, that it is then added: $A^{\dagger}_{t_2}[M'] = A^{\dagger}_{t_2^-}[M'] \cup \{x_2\}$.

At time t_2, the site x_1 is no longer a discrepancy between both aggregates but it has been reached by two different particles: x_1 actually is a blue point using the color code of Figure 2, i.e. both edges leading to x_1 in $\mathcal{F}_n[M]$ and $\mathcal{F}_n[M']$ could be different. Moreover, the site x_2

has become a discrepancy between both aggregates $A^{\dagger}_{t_2}[M]$ and $A^{\dagger}_{t_2}[M']$. In other words, the discrepancy has been *relayed* from x_1 to x_2 by the particle emitted at time t_2.

From then on, one can imagine a scenario where, at a random time $t_3 \in (t_2, n)$, the discrepancy at x_2 is relayed (by a third particle from $\mathcal{H}_M$) to a new site x_3 and possibly becomes another difference between both forests (i.e. a blue point), and so on. Such a phenomenon is referred to as a *chain of changes*. We point out that, even if it is initiated by a particle from $\mathcal{H}_{M'} \setminus \mathcal{H}_M$–i.e. quite far away from the e_1-axis when M is large–a chain of changes can spread up to the e_1-axis thanks to several relays. See the blue points in the top left picture of Figure 2.

Results

Our main task consists in controlling the chains of changes described in the previous section and proving that they cannot spread up too much inside the aggregate. This leads to the following stabilization result for the sequence of IDLA forests $(\mathcal{F}_n[M])_{M\geq 0}$. For that purpose, we need to define the strip $\mathbb{Z}_K := \mathbb{Z} \times [\![-K, K]\!]^{d-1}$ for any integer $K \geq 0$.

Theorem 1.1 (Forest stabilization result)**.** *Let $d \geq 2$. For all $n \geq 1$ and $K \geq 1$, the following holds with probability one:*

$$\exists N_0 = N_0(n,K) \geq 0, \forall N \geq N_0, \mathcal{F}_n[N] \cap \mathbb{Z}_K = \mathcal{F}_n[N_0] \cap \mathbb{Z}_K \ , \tag{1.2}$$

where the above identity means that all vertices and edges of $\mathcal{F}_n[N]$ and $\mathcal{F}_n[N_0]$ inside the strip $\mathbb{Z}_K$ coincide.

From then on, Theorem 1.1 allows us to take the limit $M \to \infty$ in the sequence $(\mathcal{F}_n[M])_{M\geq 0}$ in order to obtain an infinite-volume forest $\mathcal{F}_n$. First remark that the sequence $(N_0(n,K))_{n,K}$ in (1.1) can be chosen increasing in K, which implies for any $K' \geq K$,

$$\mathcal{F}_n[N_0(n,K)] \cap \mathbb{Z}_K = \mathcal{F}_n[N_0(n,K')] \cap \mathbb{Z}_K \subset \mathcal{F}_n[N_0(n,K')] \cap \mathbb{Z}_{K'} \ . \tag{1.3}$$

Inclusion (1.3) compensates for the lack of monotonicity of the sequence $(\mathcal{F}_n[M])_{M\geq 0}$ and allows us to define the *infinite-volume* IDLA *forest up to time n* denoted by $\mathcal{F}_n$ as

$$\mathcal{F}_n := \bigcup_{K\geq 1} \uparrow \mathcal{F}_n[N_0(n,K)] \cap \mathbb{Z}_K \ , \tag{1.4}$$

a.s. and for any $n \geq 1$. A realization of $\mathcal{F}_{100}[200]$ is given in Figure 3 seen through the strip $\mathbb{Z}_{40}$.

After taking the limit $M \to \infty$ in space, let us take the limit $n \to \infty$ in time. The sequence $(N_0(n,K))_{n,K}$ can also be chosen increasing in n. So using once again the stabilization result of Theorem 1.1, we can write:

$$\begin{aligned} \mathcal{F}_n \cap \mathbb{Z}_K = \mathcal{F}_n[N_0(n,K)] \cap \mathbb{Z}_K &= \mathcal{F}_n[N_0(n+1,K)] \cap \mathbb{Z}_K \\ &\subset \mathcal{F}_{n+1}[N_0(n+1,K)] \cap \mathbb{Z}_K = \mathcal{F}_{n+1} \cap \mathbb{Z}_K, \end{aligned}$$

where the inclusion $\mathcal{F}_n[M] \subset \mathcal{F}_{n+1}[M]$ is merely due to extra particles emitted during the time interval $(n, n+1]$ (from $\mathcal{H}_M$). Hence, the sequence of random graphs $(\mathcal{F}_n)_{n\geq 1}$ is increasing in the sense that a.s. for any $n \geq 1$, $V(\mathcal{F}_n) \subset V(\mathcal{F}_{n+1})$ and $E(\mathcal{F}_n) \subset E(\mathcal{F}_{n+1})$. We then define the *infinite-volume* IDLA *forest $\mathcal{F}_\infty$* by

$$\mathcal{F}_\infty := \bigcup_{n\geq 1} \uparrow \mathcal{F}_n \quad \text{a.s.} \tag{1.5}$$

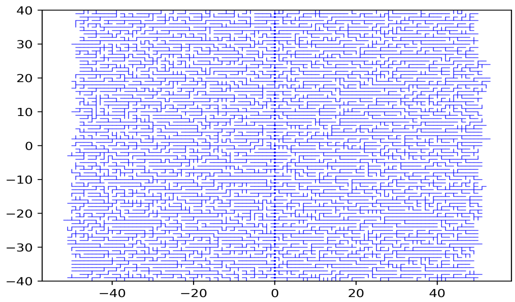

Figure 3: Here is a realization of $\mathcal{F}_{100}[200]$ intersected with the strip $\mathbb{Z}_{40}$. Taking $M = 200$ large enough, one can expect the infinite-volume forest $\mathcal{F}_{100}$ and the finite volume forest $\mathcal{F}_{100}[200]$ to coincide on $\mathbb{Z}_{40}$ thanks to Theorem 1.1.

The infinite-volume IDLA forests $\mathcal{F}_\infty$ and $(\mathcal{F}_n)_{n\geq 0}$ previously defined are built from an infinite set of sources, namely the hyperplane $\mathcal{H} = \{0\} \times \mathbb{Z}^{d-1}$. Theorem 1.2 below asserts that they are invariant in distribution w.r.t. translations of $\mathcal{H}$, meaning that all sources in $\mathcal{H}$ play the same role in their constructions. This gives a mathematical sense to what was announced at the beginning about $\mathcal{F}_\infty$ and $(\mathcal{F}_n)_{n\geq 0}$; at each time, the next source to emit a particle is chosen 'uniformly' among $\mathcal{H}$.

For $k \in \mathcal{H}$, let us denote by T_k the translation operator on $\mathbb{Z}^d$ defined by $\forall x \in \mathbb{Z}^d$, $T_k(x) = x + k$.

Theorem 1.2. *Let $d \geq 2$. The infinite-volume IDLA forests $\mathcal{F}_\infty$ and $(\mathcal{F}_n)_{n\geq 0}$ satisfy the following properties:*

1. *Almost surely, the set of vertices of $\mathcal{F}_\infty$ satisfies $V(\mathcal{F}_\infty) = \mathbb{Z}^d$.*
2. *The distributions of $(\mathcal{F}_n)_{n\geq 0}$ and $\mathcal{F}_\infty$ are invariant w.r.t. translations T_k, $k \in \mathcal{H}$.*
3. *The distributions of $(\mathcal{F}_n)_{n\geq 0}$ and $\mathcal{F}_\infty$ are α-mixing, and then ergodic, w.r.t. translations T_k, $k \in \mathcal{H}$.*

Strategy for proving Theorem 1.1

Let us call *level* M the set of sources in $\mathcal{H}$ at distance M from the origin (w.r.t. the infinite norm $\|(z_1, \dots, z_d)\| := \max_i |z_i|$). Our main task consists in proving the stabilization result Theorem 1.1, i.e. given $n \geq 1$ and $K \geq 1$, that (1.2) recalled below occurs with probability 1:

$$\exists N_0 = N_0(n, K) \geq 0, \forall N \geq N_0, \mathcal{F}_n[N] \cap \mathbb{Z}_K = \mathcal{F}_n[N_0] \cap \mathbb{Z}_K \ .$$

Since the event $\{\mathcal{F}_n[N] \cap \mathbb{Z}_K \neq \mathcal{F}_n[N_0] \cap \mathbb{Z}_K\}$ implies the existence of a chain of changes between $A_n^\dagger[N]$ and $A_n^\dagger[N_0]$, a natural approach would be, using the Borel-Cantelli Lemma combined with a union bound, to show that

$$\sum_{N_0} \sum_{N \geq N_0} \mathbb{P}(\mathcal{F}_n[N] \cap \mathbb{Z}_K \neq \mathcal{F}_n[N_0] \cap \mathbb{Z}_K) < \infty \ .$$

However, it is difficult to obtain any upper bounds which decrease with respect to N *and* make the corresponding series summable. Indeed, this event provides no control on the level from which that chain of changes is initiated: it could be initiated anywhere from level $N_0 + 1$, independently of N. A different strategy is therefore required.

Our original approach is to interpret the chain of changes phenomenon with a percolation point of view. Consider $N \geq N_0 \geq K$ and a chain of changes between the forests $\mathcal{F}_n[N_0]$ and $\mathcal{F}_n[N]$ creating (at least) a discrepancy inside the strip $\mathbb{Z}_K$. Then, the sequence of successive relays will be interpreted as a sequence of successive overlapping balls, centered at the sources emitting the relay particles, whose cluster goes from level N_0 to the strip $\mathbb{Z}_K$. In particular, when $N_0 \gg K$, a very large cluster will correspond to this chain of changes.

Given $n, K \geq 1$, we proceed by contradiction and assume that, with positive probability,

$$\forall N_0,\, \exists N \geq N_0,\, \mathcal{F}_n[N] \cap \mathbb{Z}_K \neq \mathcal{F}_n[N_0] \cap \mathbb{Z}_K \ . \tag{1.6}$$

Throughout the whole paper, we will refer to (1.6) as the *Absurd hypothesis*. According to our percolation viewpoint, (1.6) leads to the existence of a Boolean model, say Σ, which percolates (i.e. admits an unbounded cluster) with positive probability. To get a contradiction, we will also state that Σ is actually subcritical with probability 1, concluding the proof of Theorem 1.1. To do it, we will act on three characteristics of the Boolean model Σ: its intensity (the density of its centers), its radii distribution and its long-range correlations.

First, the relay particles involved in a chain of changes are emitted by space-time points (z, t), i.e. from a source z and at time t. Denoting by (z_i, t_i)'s the sequence of emitting space-time points of a given chain of changes, the time sequence (t_i) is increasing by construction. Taking advantage of this monotonicity property, we prove that the intensity of the Boolean model Σ can be chosen as small as we want.

Thus, recall that the (infinite) aggregate $A_n^\dagger[\infty]$ defined in (1.1) is also the vertex set of the IDLA forest $\mathcal{F}_n$. So any information about it will help us to analyze the chain of changes phenomenon. In particular, in the Boolean model Σ, the radius of a ball associated to a relay particle is given by the maximal fluctuations performed by that particle from its source to exiting $A_n^\dagger[\infty]$. So stating a global upper bound (see Proposition 2.2) for this infinite aggregate–within a kind of cone–will allow us to control these fluctuations and prove that the radii distribution of Σ satisfies good integrability conditions.

Finally, in order to show that Σ is subcritical, we will apply a multiscale argument in the manner of [11]. A crucial ingredient making this strategy successful is to quantify how much Σ, when restricted to a finite window, depends on what happens far away. An important step towards such a local property satisfied by the Boolean model Σ lies in the stabilization result below. Theorem 1.3, interesting in itself, asserts that with high probability, the infinite aggregate $A_n^\dagger[\infty]$, when restricted to the strip $\mathbb{Z}_M$, does not depend on particles launched from levels larger than $2M$.

Theorem 1.3 (Aggregate stabilization result)**.** *Let $d \geq 2$ and $n \geq 1$. There exists a positive constant $C = C(n, d)$ such that for any $M, L \geq 1$,*

$$\mathbb{P}(A_n^\dagger[\infty] \cap \mathbb{Z}_M = A_n^\dagger[2M] \cap \mathbb{Z}_M) \geq 1 - \frac{C}{M^L} \ .$$

Proof of Theorem 1.3 is based on the global upper bound for $A_n^\dagger[\infty]$ mentioned above, on a donut argument already used in [6] and on a variant of the natural coupling between two aggregates, called the *special coupling.*

Why does the proof of [6] collapse in higher dimensions?

In [6], the authors proved the existence of IDLA forests $(\mathcal{F}_n)_{n\geq 1}$ and $\mathcal{F}_\infty$–see (1.4) and (1.5)–in the case of dimension $d = 2$, i.e. with the set of sources $\mathcal{H} = \{0\} \times \mathbb{Z}$. Their proof only works for the dimension $d = 2$ and cannot be generalized to higher dimensions: let us explain why.

For any integer n, let us define the *vacant set* $\mathcal{V}_n \subset \mathcal{H}$ by

$$\mathcal{V}_n := \{z \in \mathcal{H} : L(z) \cap A_n^\dagger[\infty] = \emptyset\}$$

where $L(z) := z + \{(k, 0) : k \in \mathbb{Z}\}$ is the horizontal line passing by z. It is proved in Corollary 5.2 of [6] that the random set $\mathcal{V}_n$ contains a.s. infinitely many sources. Due to the dimension $d = 2$, this implies that the aggregate $A_n^\dagger[\infty]$ is made up of (infinitely many) disjoint, finite connected components. The aggregate $A_n^\dagger[\infty]$ being the vertex set of the IDLA forest $\mathcal{F}_n$, it is then impossible for a chain of changes initiated from a very far away source to spread and create a discrepancy inside a given strip $\mathbb{Z}_K$ (the relay particles cannot jump from a connected component of $A_n^\dagger[\infty]$ to another one). Corollary 5.2 of [6] is certainly still true in dimension $d \geq 3$ (in the sense that $A_n^\dagger[\infty]$ contains an infinite number of empty lines $\mathbb{Z} \times \{y\}$, for $y \in \mathbb{Z}^{d-1}$) but its consequence about $A_n^\dagger[\infty]$, due to planarity, definitely collapses in higher dimensions.

However, one could ask whether the aggregate $A_n^\dagger[\infty]$ would still be an union of disjoint, finite connected components in dimension $d \geq 3$. Actually that is wrong whenever n is large enough. Indeed, let us call *rooted* a source z having emitted at least one particle during the time interval $[0, n]$. Hence, z is rooted if and only if the corresponding PPP $\mathcal{N}_z$ has generated at least one top in $[0, n]$, which occurs with probability $1 - e^{-n}$. The events $\{z \text{ is rooted}\}$, $z \in \mathcal{H}$, being independent from each other and their (common) probability tending to 1 as $n \to \infty$, we get that the set of rooted sources percolates in $\mathcal{H}$ for n large enough. Since a rooted source belongs to $A_n^\dagger[\infty]$, we conclude that this aggregate contains a (unique) infinite connected component.

Organization of the paper

Our paper is organized as follows. In Section 2, are gathered several properties about the aggregate $A_n^\dagger[\infty]$ that will be useful in the sequel. The natural and special couplings are detailed and the *Aggregate stabilization* result Theorem 1.3 is proved. In Section 3, we explain how chains of changes can be interpreted in terms of percolation. In particular, we define in Section 3.3 a discrete Boolean model $\hat{\Sigma}_\varepsilon$, with intensity $p_\varepsilon \to 0$ as $\varepsilon \to 0$, which is proved to be supercritical (i.e. percolating) for any $\varepsilon > 0$ under the *Absurd hypothesis* (1.6). Conversely, in Section 4, we adapt to our context a multiscale argument due to [11] which allows us to prove that $\hat{\Sigma}_\varepsilon$ does not percolate provided the intensity p_ε is small enough, leading to a contradiction with the conclusion of Section 3. Finally, Section 5 is devoted to the proof of Theorem 1.2. A detailed proof of Proposition 2.2 is given in the Appendix A.

2 The infinite aggregate $A_n^\dagger[\infty]$

Let $n \geq 1$. Recall that the infinite aggregate $A_n^\dagger[\infty]$ has been defined in (1.1) as the limit of the increasing sequence of (finite) aggregates $(A_n^\dagger[M])_{M\geq 1}$. In this section, are gathered several results about $A_n^\dagger[\infty]$ which will be helpful for the proof of our main result Theorem 1.1. Indeed, the infinite aggregate $A_n^\dagger[\infty]$ is intended to be the vertex set of the infinite-volume IDLA forest $\mathcal{F}_n$. Results about $A_n^\dagger[\infty]$ are stated in Section 2.1. Both natural and special couplings are given in Section 2.2. Finally, we prove the stabilization result for $A_n^\dagger[\infty]$ (Theorem 1.3) in Section 2.3.

2.1 Results

Let us start with an invariance property in distribution for the infinite aggregate $A_n^\dagger[\infty]$. This result is based on the fact that the random ingredients generating $A_n^\dagger[\infty]$, namely the collection of PPP $\{\mathcal{N}_z : z \in \mathcal{H}\}$ and the random walks $\{S_{z,j} : z \in \mathcal{H}, j \geq 1\}$, are iid. See Proposition 2.2 of [6] for the same result but in dimension $d = 2$. The same proof actually works for any $d \geq 2$.

Proposition 2.1. *The distribution of $A_n^\dagger[\infty]$ is invariant w.r.t. translations of the source set $\mathcal{H}$:*

$$T_k A_n^\dagger[\infty] \overset{\text{law}}{=} A_n^\dagger[\infty]$$

where T_k, for $k \in \mathcal{H}$, is defined on $\mathbb{Z}^d$ by $T_k(x) = x + k$.

The following result, given by Proposition 2.2, provides a global control of the shape of $A_n^\dagger[\infty]$. It is referred to as a *Global Upper Bound* and is necessary in the proof of Theorem 1.3 in Section 2.3. Let us begin by introducing some notations. For $0 < \alpha < 1$ and $\varepsilon > 0$, let us consider the cone $\mathscr{C}_\varepsilon^\alpha$ as

$$\mathscr{C}_\varepsilon^\alpha = \bigcup_{\ell \geq 0} \left\{ z \in \mathbb{Z}^d,\ \|p_{\mathcal{H}}(z)\| = \ell,\ |z_1| \leq \varepsilon \ell^\alpha \right\},$$

where $p_{\mathcal{H}}$ denotes the orthogonal projection onto $\mathcal{H}$. Then, for any integer $M \geq 0$, we define the event

$$\mathbf{Over}_\alpha^\dagger(M, n, \varepsilon) := \{A_n^\dagger[\infty] \cap \mathbb{Z}_M^c \not\subset \mathscr{C}_\varepsilon^\alpha\} , \tag{2.1}$$

meaning that the aggregate $A_n^\dagger[\infty]$ exceeds the cone $\mathscr{C}_\varepsilon^\alpha$ outside the strip $\mathbb{Z}_M$. Proposition 2.2 states an upper bound for the probability of $\mathbf{Over}_\alpha^\dagger(M, n, \varepsilon)$ which implies that above a certain level, the aggregate is almost surely contained inside the cone $\mathscr{C}_\varepsilon^\alpha$.

Proposition 2.2. *(Global upper bound) For any $\varepsilon > 0$ and $\alpha \in (1 - 1/d, 1)$, there exists a positive constant $C = C(d, n, \varepsilon, \alpha)$ such that for all integers $L, M > 1$,*

$$\mathbb{P}\left(\mathbf{Over}_\alpha^\dagger(M, n, \varepsilon)\right) \leq \frac{C}{M^L} .$$

In particular, with probability 1, there exists a (random) level from which $A_n^\dagger[\infty] \cap \mathbb{Z}_M^c$ is included in the cone $\mathscr{C}_\varepsilon^\alpha$.

Proposition 2.2 actually is a refined version of Theorem 4.1 of [5] which states the same result but for $\alpha = 1$ (i.e. for a wider cone). So only a short proof of Proposition 2.2 is given in the Appendix, focusing on the differences due to the use of the thinner cone $\mathscr{C}_\varepsilon^\alpha$, with $\alpha \in (1 - 1/d, 1)$. However this refinement (using $\mathscr{C}_\varepsilon^\alpha$ instead of $\mathscr{C}_\varepsilon^1$) is required here to get Theorem 1.3–and then our main result Theorem 1.1–as explained at the end of Section 2.3.

2.2 Two couplings

Let $n \geq 1$ and $M' > M$. In this section, we detail two different couplings allowing to construct both aggregates $A_n^\dagger[M]$ and $A_n^\dagger[M']$ on the same probability space in such a way that

$$A_n^\dagger[M] \subset A_n^\dagger[M'] \quad \text{a.s.} \tag{2.2}$$

The first one, called the *natural coupling*, will be used intensively in Section 3 to describe the chains of changes. It has been introduced in [6]. In this paper, we will require a variant of the natural coupling, called the *special coupling*, ensuring a special property ($\star$) (see below) in addition to 2.2. The special coupling will be used in Section 2.3 to get Theorem 1.3. Hence, we first recall the natural coupling in details and thus its variant.

Let us begin by describing the natural coupling. Let $\kappa := \sum_{z\in\mathcal{H}_{M'}} \#\mathcal{N}_z([0,n])$ be the total number of particles sent from $\mathcal{H}_{M'}$ during the time interval $[0,n]$. Let us build two sequences of aggregates $(A_i)_{0\le i\le\kappa}$ and $(B_i)_{0\le i\le\kappa}$ such that

$$\text{for all } 0 \le i \le \kappa,\ \ A_i \subset B_i \ \text{ and } \ A_\kappa = A_n^\dagger[M],\ B_\kappa = A_n^\dagger[M'] \ .$$

We proceed by induction on $i \in [\![0,\kappa]\!]$ by sorting the κ particles according to their starting times (from time 0 to n). When $i = 0$ (no particles have been emitted), we have that $A_0 = B_0 = \emptyset$. Now, suppose $i \ge 0$ and $A_i \subset B_i$ and let us say that the $(i+1)$-th particle is sent from a source $z \in \mathcal{H}_{M'}$.

- If $z \in \mathcal{H}_{M'}\backslash\mathcal{H}_M$ then the $(i+1)$-th particle only contributes to B_i. It adds a random site x to B_i while A_i remains unchanged:

$$A_{i+1} := A_i \subset B_i \subset B_i \cup \{x\} =: B_{i+1}.$$

- If $z \in \mathcal{H}_M$, the $(i+1)$-th particle contributes to both aggregates. Since $A_i \subset B_i$, it exits A_i before B_i, and adds a random site x to A_i. Now, we must consider two cases.
  - If $x \notin B_i$, then x is added to B_i. Hence,

$$A_{i+1} := A_i \cup \{x\} \subset B_i \cup \{x\} =: B_{i+1} \ .$$

  - If $x \in B_i$, then the $(i+1)$-th particle does not exit B_i in x, and continues its trajectory until exiting B_i on some site $x' \neq x$. In this case,

$$A_{i+1} := A_i \cup \{x\} \subset B_i \subset B_i \cup \{x'\} =: B_{i+1} \ .$$

Let us now detail the special coupling. The total number of particles sent from $\mathcal{H}_{M'}$ during $[0,n]$ is still denoted by κ and, as before, we build two sequences of aggregates $(\tilde{A}_i)_{0\le i\le\kappa}$ and $(\tilde{B}_i)_{0\le i\le\kappa}$ by induction on $i \in [\![0,\kappa]\!]$. Our construction will ensure that

$$\text{for all } 0 \le i \le \kappa,\ \ \tilde{A}_i \subset \tilde{B}_i \ \text{ and } \ \tilde{A}_\kappa = A_n^\dagger[M],\ \tilde{B}_\kappa \overset{\text{law}}{=} A_n^\dagger[M'] \ ,$$

while also ensuring that the following condition holds, for any $0 \le i \le \kappa$:

($\star$) Any element $x \in \tilde{B}_i\backslash\tilde{A}_i$ is produced by a particle emitted from a source in $\mathcal{H}_{M'}\backslash\mathcal{H}_M$.

Let us build the $\tilde{A}_i$'s and $\tilde{B}_i$'s. When $i = 0$, we have that $\tilde{A}_0 = \tilde{B}_0 = \emptyset$. Let us assume for some $i \ge 0$ that $\tilde{A}_i \subset \tilde{B}_i$, and that they both satisfy condition ($\star$). Let us say that the $(i+1)$-th particle is sent from a source $z \in \mathcal{H}_{M'}$ (and moves according to a random walk S_z).

- If $z \in \mathcal{H}_{M'}\backslash\mathcal{H}_M$ then the $(i+1)$-th particle only contributes to B_i. As for the natural coupling, it adds a random site x to B_i while A_i remains unchanged:

$$\tilde{A}_{i+1} := \tilde{A}_i \subset \tilde{B}_i \subset \tilde{B}_i \cup \{x\} =: \tilde{B}_{i+1}$$

  and the couple $(\tilde{A}_{i+1}, \tilde{B}_{i+1})$ still satisfies ($\star$).

- If $z \in \mathcal{H}_M$, the $(i+1)$-th particle contributes to both aggregates. Since $\tilde{A}_i \subset \tilde{B}_i$, it exits $\tilde{A}_i$ before $\tilde{B}_i$, and adds a random site x to $\tilde{A}_i$. Once again, we consider two cases.

- If $x \notin \tilde{B}_i$ then we proceed as for the natural coupling: x is also added to $\tilde{B}_i$ which again implies
$$\tilde{A}_{i+1} := \tilde{A}_i \cup \{x\} \subset \tilde{B}_i \cup \{x\} =: \tilde{B}_{i+1} \ .$$
The couple $(\tilde{A}_{i+1}, \tilde{B}_{i+1})$ still satisfies $(\star)$ since the just added site x belongs to $\tilde{A}_{i+1}$ and $\tilde{B}_{i+1}$.
- If $x \in \tilde{B}_i \backslash \tilde{A}_i$, then thanks to condition $(\star)$, the site x was reached by a particle (say with index $i' < i$) originating from some source $z' \in \mathcal{H}_{M'} \backslash \mathcal{H}_M$ (and moving according to a random walk $S_{z'}$). Then, the $(i+1)$-th particle settles at x, wakes up the i'-th particle which continues its trajectory (according to $S_{z'}$) until exiting $\tilde{B}_i$ on some site y. In this case,
$$\tilde{A}_{i+1} := \tilde{A}_i \cup \{x\} \subset \tilde{B}_i \subset \tilde{B}_i \cup \{y\} =: \tilde{B}_{i+1} \ .$$
Note that the couple $(\tilde{A}_{i+1}, \tilde{B}_{i+1})$ satisfies $(\star)$ since y, which is a discrepancy between $\tilde{A}_{i+1}$ and $\tilde{B}_{i+1}$, has been produced by a particle sent from $\mathcal{H}_{M'} \backslash \mathcal{H}_M$.

To conclude, let us remark that in both couplings, the aggregates A_i and $\tilde{A}_i$ are built in the same way. They are a.s. equal and then $\tilde{A}_\kappa = A_\kappa = A_n^\dagger[M]$. This is not the case for the $\tilde{B}_i$'s: even if $\tilde{B}_i = B_i$, the site y added following the random walk $S_{z'}$ could be different from the site x' added following S_z. However, in distribution, they are equal:
$$\tilde{B}_\kappa \overset{\text{law}}{=} B_\kappa = A_n^\dagger[M']$$
since the trajectory used to add the site y to $\tilde{B}_i$ is the concatenation of S_z from the source z to x, thus $S_{z'}$ after x. This trajectory is a random walk.

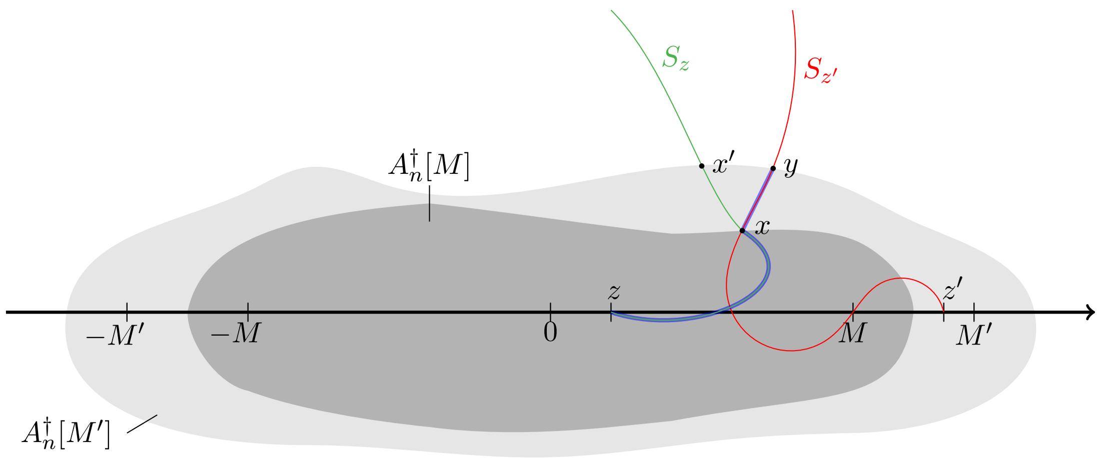

Figure 4: The aggregates $A_n^\dagger[M]$ and $A_n^\dagger[M']$ are represented in dark and light gray. The trajectories of random walks S_z and $S_{z'}$ are respectively depicted in green and red. In the natural coupling, the site x' is added to $\tilde{B}_i$ using solely S_z whereas in the special coupling, the site y is added to $\tilde{B}_i$ using first S_z until exiting $\tilde{A}_i = A_i$ and then $S_{z'}$. We highlight in blue the actual path that is realized when doing so.

2.3 Proof of Theorem 1.3

Let $n \geq 1$. Our goal is to prove that there exists a positive constant $C = C(n,d)$ such that for any $M, L \geq 1$,
$$\mathbb{P}(A_n^\dagger[\infty] \cap \mathbb{Z}_M = A_n^\dagger[2M] \cap \mathbb{Z}_M) \geq 1 - \frac{C}{M^L} \ .$$

To do it, let us introduce the event

$$D_M := \left\{ \begin{array}{c} \text{The trajectory of any random walk associated with a particle of } A_n^\dagger[\infty] \text{ and starting} \\ \text{from a level greater than } 2M \text{ does not visit the strip } \mathbb{Z}_M \text{ before exiting } \mathscr{C}_\varepsilon^\alpha \end{array} \right\} .$$

The event D_M has a probability tending to 1 as $M \to \infty$:

Lemma 2.3. *For $\varepsilon > 0$ and $\alpha \in (0,1)$, there exists a positive constant $C = C(d,n,\varepsilon,\alpha)$ such that for all $M, L \geq 1$,*

$$\mathbb{P}\left(D_M\right) \geq 1 - \frac{C}{M^L}.$$

The proof of Theorem 1.3 works in two steps. We first explain how to conclude from Lemma 2.3 and then we prove this auxiliary result.

Let us pick $\varepsilon > 0$ and $\alpha \in (1 - 1/d, 1)$. Let $M, L \geq 1$. Let us consider the event $\mathcal{G}_M$ defined by

$$\mathcal{G}_M := \left\{ A_n^\dagger[\infty] \cap \mathbb{Z}_M^c \subset \mathscr{C}_\varepsilon^\alpha \right\} \cap D_M \ .$$

Thanks to Proposition 2.2 and Lemma 2.3, there exists a positive constant $C = C(d,n,\varepsilon,\alpha)$ such that $\mathbb{P}(\mathcal{G}_M) \geq 1 - CM^{-L}$. Given $M' > 2M$, we consider the aggregates $A_n^\dagger[2M]$ and $A_n^\dagger[M']$ under the special coupling. Hence, a.s. $A_n^\dagger[2M]$ is included in $A_n^\dagger[M']$ and any element x in $A_n^\dagger[M'] \backslash A_n^\dagger[2M]$ is produced by a particle emitted from a source in $\mathcal{H}_{M'} \backslash \mathcal{H}_{2M}$ thanks to the condition ($\star$). On the event $\mathcal{G}_M$, the random walk associated to this particle necessarily exited $\mathscr{C}_\varepsilon^\alpha$ before reaching $\mathbb{Z}_M$, and hence necessarily exited $A_n^\dagger[M']$ before reaching $\mathbb{Z}_M$. This means that, on the event $\mathcal{G}_M$, both aggregates $A_n^\dagger[2M]$ and $A_n^\dagger[M']$ coincide on $\mathbb{Z}_M$ and leads to

$$\mathbb{P}\left(A_n^\dagger[M'] \cap \mathbb{Z}_M = A_n^\dagger[2M] \cap \mathbb{Z}_M\right) \geq 1 - \frac{C}{M^L}, \ \forall M' > 2M \ . \tag{2.3}$$

The key argument here is the special coupling which ensures that any discrepancy between both aggregates is due to a single particle and not to a chain of changes.

We can now conclude. On the one hand, thanks to (2.3),

$$\liminf_{M' \to \infty} \mathbb{P}\left(A_n^\dagger[M'] \cap \mathbb{Z}_M \subset A_n^\dagger[2M] \cap \mathbb{Z}_M\right) \geq 1 - \frac{C}{M^L} \ .$$

On the other hand, the infinite aggregate $A_n^\dagger[\infty]$ being the limit of the increasing sequence $(A_n^\dagger[M'])_{M'}$ (thanks to the natural coupling), we have

$$\lim_{M' \to \infty} \mathbb{P}\left(A_n^\dagger[M'] \cap \mathbb{Z}_M \subset A_n^\dagger[2M] \cap \mathbb{Z}_M\right) = \mathbb{P}\left(A_n^\dagger[\infty] \cap \mathbb{Z}_M \subset A_n^\dagger[2M] \cap \mathbb{Z}_M\right) \ .$$

As a consequence, we can write

$$\mathbb{P}\left(A_n^\dagger[\infty] \cap \mathbb{Z}_M \subset A_n^\dagger[2M] \cap \mathbb{Z}_M\right) \geq 1 - \frac{C}{M^L}$$

and the same holds for $\mathbb{P}(A_n^\dagger[\infty] \cap \mathbb{Z}_M = A_n^\dagger[2M] \cap \mathbb{Z}_M)$ since $A_n^\dagger[2M] \subset A_n^\dagger[\infty]$ with probability 1. This concludes the proof of Theorem 1.3.

Proof of Lemma 2.3. Let $\varepsilon > 0$, $\alpha \in (0,1)$ and $M, j \geq 1$. Let us set

$$E_{M,j} := \left\{ \begin{array}{c} \text{At least one random walk starting from } \mathrm{Ann}(M,j) \\ \text{visits the strip } \mathbb{Z}_M \text{ before exiting } \mathscr{C}_\varepsilon^\alpha \end{array} \right\}$$

where $\mathrm{Ann}(M,j) := \mathcal{H}_{(j+2)M} \setminus \mathcal{H}_{(j+1)M}$. Hence, the event D_M^c is equal to $\cup_{j\geq 1} E_{M,j}$ and we focus on bounding each term $\mathbb{P}(E_{M,j})$.

As in the proof of Theorem 1.2 of [5], our strategy consists in building donuts from level $(j+1)M$ down to level M, symmetric w.r.t. the hyperplane $\mathcal{H}$ and containing the cone $\mathscr{C}_\varepsilon^\alpha$ (restricted to $\mathbb{Z}_{(j+1)M} \setminus \mathbb{Z}_M$). The largest donut is the one built at level $(j+1)M$. Its width is equal to $2\varepsilon((j+1)M)^\alpha$ and all the following donuts have smaller sizes. Therefore, the number of donuts $k^o = k^o(j, M, \varepsilon, \alpha)$ we can build from level $(j+1)M$ to level M verifies

$$k^o \geq \frac{(j+1)M - M}{2\varepsilon(j+1)^\alpha M^\alpha} \geq \frac{(jM)^{1-\alpha}}{4\varepsilon}, \tag{2.4}$$

where the last inequality is due to $(j+1)^\alpha \leq 2j$.

We know that if a random walk sent from a level greater than $(j+1)M$ reaches the strip $\mathbb{Z}_M$ while staying inside the cone $\mathscr{C}_\varepsilon^\alpha$, it necessarily crossed over the k^o donuts previously built. The probabilistic cost to cross any given donut while staying inside the cone is at most $1-c$ with $c := (2d)^{-2}$. So the probability for such random walk to cross the k^o donuts before exiting $\mathscr{C}_\varepsilon^\alpha$ is at most $(1-c)^{k^o}$. See Proposition 3.1 of [5] for details.

Besides, let us denote by $N_{tot}^{(j)} = N_{tot}^{(j)}(d,n,M,j)$ the total number of particles sent from $\mathrm{Ann}(M,j)$ during the time interval $[0,n]$:

$$N_{tot}^{(j)} := \sum_{z \in \mathrm{Ann}(M,j)} \mathcal{N}_z([0,n]) \ .$$

The next result allows us to bound from above $N_{tot}^{(j)}$ with high probability.

Lemma 2.4. *Let $M, j \geq 1$ and set $C_{M,j} := \#\mathrm{Ann}(M,j)$. Then,*

$$\mathbb{P}(N_{tot}^{(j)} > 2nC_{M,j}) \leq \exp\left(-c_0 j^{d-2} M^{d-1}\right),$$

where $c_0 = c_0(d,n)$ denotes a positive constant.

Proof of Lemma 2.4. The searched inequality is a direct consequence of the concentration inequality for Poisson variables (2.5) stated below, applied to $N_{tot}^{(j)}$ which is distributed as a Poisson random variable with parameter $nC_{M,j}$ and to the fact that $C_{M,j}$ is of order $j^{d-2}M^{d-1}$.

If X is a Poisson random variable of parameter $\lambda > 0$, then for any $t \geq 0$, the following holds:

$$\mathbb{P}\left(X - \mathbb{E}\left[X\right] \geq t\right) \leq \exp\left(-\frac{t^2}{2\left(\lambda + \frac{t}{3}\right)}\right). \tag{2.5}$$

□

All the ingredients are gathered, we can compute:

$$\begin{aligned}
\mathbb{P}(E_{M,j}) &\leq \mathbb{P}(E_{M,j} \cap \{N_{tot}^{(j)} \leq 2nC_{M,j}\}) + \mathbb{P}(N_{tot}^{(j)} > 2nC_{M,j}) \\
&\leq \sum_{i=1}^{2nC_{M,j}} \mathbb{P}(\{\text{walk } i \text{ visits } \mathbb{Z}_M \text{ before exiting } \mathscr{C}_\varepsilon^\alpha\}) + \exp\left(-c_0 j^{d-2} M^{d-1}\right) \\
&\leq \sum_{i=1}^{2nC_{M,j}} \mathbb{P}(\{\text{walk } i \text{ crosses } k^o \text{ donuts before exiting } \mathscr{C}_\varepsilon^\alpha\}) + \exp\left(-c_0 j^{d-2} M^{d-1}\right) \\
&\leq 2nC_{M,j}(1-c)^{k^o} + \exp\left(-c_0 j^{d-2} M^{d-1}\right) \ .
\end{aligned}$$

Thus, (2.4) leads to

$$2nC_{M,j}(1-c)^{k^o} \leq C_1 M^{d-1}(j+1)^{d-2} \exp\big(-C_2(jM)^{1-\alpha}\big)$$

where C_1, C_2 are positive constants depending only on d, n, ε. Summing over $j \geq 1$, we get

$$\mathbb{P}\Big(\bigcup_{j\geq 1} E_{M,j}\Big) \leq \sum_{j\geq 1} \mathbb{P}(E_{M,j}) \leq \sum_{j\geq 1} C_1 M^{d-1}(j+1)^{d-2} \exp\big(-C_2(jM)^{1-\alpha}\big) + \sum_{j\geq 1} \exp\big(-c_0 j^{d-2} M^{d-1}\big) \ .$$

Since $1-\alpha > 0$, both terms of the upper bound above are summable and decrease faster than any power of M^{-1}, which concludes the proof.

Remark that the previous conclusion holds only if $d \geq 3$. When $d = 2$, we have to proceed slightly differently. Each $\mathrm{Ann}(M, j)$ has the same cardinality, equal to M. So, in the previous computation, it suffices to intersect the event $E_{M,j}$ with $\{N_{tot}^{(j)} \leq 2nMj^\beta\}$ (for some $\beta > 0$) since (2.5) allows to bound the probability of $\{N_{tot}^{(j)} > 2nMj^\beta\}$ by $\exp(-c'Mj^\beta)$–which is summable w.r.t. j and M. □

Theorem 4.1 of [5] is a weaker version of Proposition 2.2: it gives the same result but in the particular case of a linear cone $\mathscr{C}_\varepsilon^1$, i.e. with $\alpha = 1$. Taking $\alpha = 1$ in the computation (2.4) leads to a constant number of donuts k^o (no longer depending on j, M) which prevents us to conclude as above. Hence, in [5], in order to get sufficiently many donuts to make unlikely the crossing to $\mathbb{Z}_M$ for particles coming far away, we required more space. This argument led to a stabilization result (Theorem 1.2 of [5]) weaker than our Theorem 1.3 since it claimed that we need to go above levels M^γ, $\gamma > 1$, to stabilize $A_n^\dagger[\infty] \cap \mathbb{Z}_M$.

However, to apply with success the multiscale argument of Section 4, we need a stabilization result for $A_n^\dagger[\infty] \cap \mathbb{Z}_M$ requiring a linear number of levels of M (rather than M^γ). This is why we had to improve the stabilization result Theorem 1.2 of [5] into Theorem 1.3. This justifies the use of the thinner cone $\mathscr{C}_\varepsilon^\alpha$ leading to the refined global upper bound Proposition 2.2.

3 From chains of changes to percolation models

To get the stabilization result Theorem 1.1, we proceed by contradiction by assuming the *Absurd hypothesis* (1.6) that we recall now: there exist positive integers n_0, K_0 (fixed for the whole section) such that

$$\forall N_0,\ \exists N \geq N_0,\ \mathcal{F}_{n_0}[N] \cap \mathbb{Z}_{K_0} \neq \mathcal{F}_{n_0}[N_0] \cap \mathbb{Z}_{K_0}$$

occurs with positive probability.

In section 3.1, we use a space-time representation of a chain of changes between the forests $\mathcal{F}_{n_0}[N_0]$ and $\mathcal{F}_{n_0}[N]$ to describe the propagation of discrepancies between the corresponding aggregates $A_{n_0}^\dagger[N_0]$ and $A_{n_0}^\dagger[N]$. In Section 3.2, a percolation model Σ with good properties (Lemmas 3.1 and 3.2) is introduced. Under the *Absurd hypothesis*, we prove that Σ percolates in the sense that it contains an *infinite descending chain* with positive probability. Finally, in Section 3.3, we take advantage of the monotonicity in time of such infinite descending chain in order to state that it actually appears instantaneously. The final result of Section 3 is summarized in Proposition 3.4 saying that a discrete Boolean model $\hat{\Sigma}_\varepsilon$ percolates even if its intensity tends to 0 with $\varepsilon \to 0$.

For $z \in \mathcal{H}$ and $r \in \mathbb{N}$, let us set $\mathbb{B}(z, r) := z + \mathcal{H}_r$ the $(d-1)$-dimensional ball, included in $\mathcal{H}$, centered at z and with radius r (for the infinity norm). We also denote by $p_{\mathcal{H}} : \mathbb{Z}^d \to \mathcal{H}$ the orthogonal projection onto the hyperplane $\mathcal{H}$.

3.1 A space-time representation of chains of changes

Let (N_0, N), with $N \geq N_0$, a couple of integers such that the forests $\mathcal{F}_{n_0}[N]$ and $\mathcal{F}_{n_0}[N_0]$ do not coincide on the strip $\mathbb{Z}_{K_0}$. From now on, we consider their vertex sets, namely the aggregates $A^\dagger_{n_0}[N]$ and $A^\dagger_{n_0}[N_0]$, under the natural coupling defined in Section 2.2, i.e. satisfying a.s. the inclusion $A^\dagger_{n_0}[N_0] \subset A^\dagger_{n_0}[N]$. A (random) space-time couple (z, t) where z is a source with $\|z\| \leq N$ and $t \in [0, n_0]$ is a 'top' of the PPP $\mathcal{N}_z$, is called a *starting point*. It means that, at time t, a particle using the random walk $S_{z,t} = (S_{z,t}(k))_{k\geq 0}$ and launched from the source z, contributes to the construction of $A^\dagger_{n_0}[N]$–and also of $A^\dagger_{n_0}[N_0]$ if $\|z\| \leq N_0$. As explained in the introduction, the difference between both forests $\mathcal{F}_{n_0}[N]$ and $\mathcal{F}_{n_0}[N_0]$ inside $\mathbb{Z}_{K_0}$ is due to a chain of changes, i.e. a sequence of $\kappa \geq 1$ particles coming from starting points $(z_i, t_i)_{1\leq i\leq\kappa}$ satisfying the following conditions:

$$\begin{cases} N_0 < \|z_1\| \leq N \text{ and } \|z_i\| \leq N_0, \text{ for } 2 \leq i \leq \kappa \\ 0 < t_1 < t_2 < \cdots < t_\kappa < n_0 \\ \text{for } 1 \leq i \leq \kappa - 1, \text{ the } i\text{-th particle is relayed by the } (i+1)\text{-th one} \\ \text{the } \kappa\text{-th particle exists } A^\dagger_{t_\kappa^-}[N] \text{ through } \mathbb{Z}_{K_0} \end{cases} \tag{3.1}$$

Recall that 'the i-th particle is relayed by the $(i + 1)$-th one' means that the discrepancy $x_i \in A^\dagger_{t_i}[N] \setminus A^\dagger_{t_i}[N_0]$ created by the i-th particle at time t_i, is visited by the $(i+1)$-th particle at time t_{i+1}, which contributes to both aggregates. So, at time t_{i+1}, x_i is now longer a discrepancy and is replaced with a new one $x_{i+1} \in A^\dagger_{t_{i+1}}[N] \setminus A^\dagger_{t_{i+1}}[N_0]$ which actually is the site at which the $(i + 1)$-th particle settles when it exits the current aggregate $A^\dagger_{t^-_{i+1}}[N]$.

Associated with a given starting point (z, t) and with the corresponding particle, we define the radius $R_N(z, t)$ as follows:

$$R_N(z,t) := \min\left\{r \in \mathbb{N} : \mathbb{B}(z,r) \text{ contains } p_{\mathcal{H}}(S_{z,t}(0)), p_{\mathcal{H}}(S_{z,t}(1)), \ldots, p_{\mathcal{H}}(S_{z,t}(\tau))\right\}$$

where

$$\tau = \tau(z,t,N) := \min\{k : S_{z,t}(k) \notin A^\dagger_{t-}[N]\}$$

denotes the time at which the particle moving according to $S_{z,t}$ exits the current aggregate $A^\dagger_{t-}[N]$. In other words, the ball $\mathbb{B}(z, R_N(z,t))$ contains the part of the projected trajectory $p_{\mathcal{H}}(S_{z,t})$ until $S_{z,t}$ exits $A^\dagger_{t-}[N]$. It is worth pointing out here that $R_N(z, t)$ only depends on the random walk $S_{z,t}$ and the current aggregate $A^\dagger_{t-}[N]$.

Now, let us come back to the sequence of $\kappa \geq 1$ particles satisfying (3.1). The fact that the i-th particle is relayed by the $(i + 1)$-th one means that

$$\mathbb{B}(z_i, R_N(z_i,t_i)) \cap \mathbb{B}(z_{i+1}, R_N(z_{i+1}, t_{i+1})) \neq \emptyset \ .$$

Figure 5 properly illustrates our argument. On the left hand side, aggregates $A^\dagger_n[N_0]$ and $A^\dagger_n[N]$ are respectively in dark and light gray–the first one being included in the second one according to the natural coupling. The trajectory of the first particle, starting at (z_1, t_1), is depicted in red. Since $\|z_1\| > N_0$, it works only for the larger aggregate and creates a first discrepancy x_1. The trajectory of the second particle, starting at (z_2, t_2), is depicted in black and green. This particle works for both aggregates until it visits x_1 for the first time (the black path): x_1 is then added to the smaller aggregate and is no longer a discrepancy. Thus, the second particle continues its trajectory (the green path) but now working only for the larger aggregate: it creates a new discrepancy x_2 between both aggregates. Note that the radii $R_N(z_1, t_1)$ and

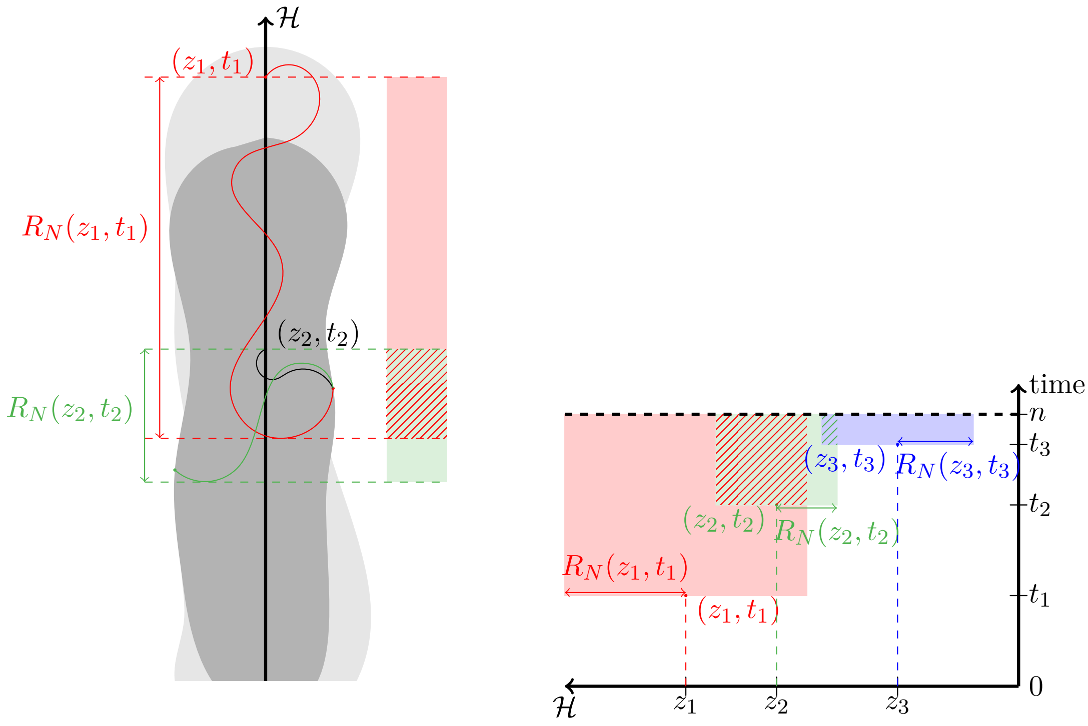

Figure 5: Representations of a chain of changes

$R_N(z_2,t_2)$ are represented on the left hand side of Figure 5. The hatched area emphasizes the fact that the balls $\mathbb{B}(z_1, R_N(z_1,t_1))$ and $\mathbb{B}(z_2, R_N(z_2,t_2))$ overlap.

This phenomenon is perhaps better visualized in the right hand side of Figure 5 where the extra time parameter is taken into account. Remark that not only the colored rectangles have to intersect, but they need to do so with respect to increasing times. This additional constraint will turn out to be crucial in our proof.

Conclusion. Given (N_0, N) such that the forests $\mathcal{F}_{n_0}[N]$ and $\mathcal{F}_{n_0}[N_0]$ do not coincide on $\mathbb{Z}_{K_0}$, we can assert that there exists a sequence of $\kappa \geq 1$ particles coming from starting points $(z_i, t_i)_{1\leq i\leq\kappa}$ and satisfying the following conditions:

$$\begin{cases} N_0 < \|z_1\| \leq N \text{ and } \|z_i\| \leq N_0, \text{ for } 2 \leq i \leq \kappa \\ 0 < t_1 < t_2 < \cdots < t_\kappa < n_0 \\ \text{for } 1 \leq i \leq \kappa-1,\ \mathbb{B}(z_i, R_N(z_i,t_i)) \cap \mathbb{B}(z_{i+1}, R_N(z_{i+1},t_{i+1})) \neq \emptyset \\ \mathbb{B}(z_\kappa, R_N(z_\kappa,t_\kappa)) \cap \mathbb{Z}_{K_0} \neq \emptyset \end{cases} \tag{3.2}$$

At this stage, the radii $R_N(z_i,t_i)$'s we consider are far from easy to handle, as the law of $R_N(z_i,t_i)$ strongly depends on the shape of $A^\dagger_{t_i}[N]$ as well as the starting point (z_i,t_i), which is random. Consequently, the radii $R_N(z_i,t_i)$'s are neither independent, nor identically distributed. To overcome this latter obstacle, we will replace in the next section the aggregate $A^\dagger_{t_i^-}[N]$ involved in the definition of $R_N(z_i,t_i)$ with the larger aggregate $A^\dagger_T[\infty]$ (with $T \geq n_0$) whose distribution is translation invariant.

3.2 Existence of infinite descending chains under the *Absurd hypothesis*

Let $T \geq n_0$. Associated to the starting point (z,t), with $z \in \mathcal{H}$ and $t \in [0,n_0]$, let us introduce the random radius $R((z,t),T)$ defined by

$$R((z,t),T) := \min\left\{r \in \mathbb{N} : \mathbb{B}(z,r) \text{ contains } p_{\mathcal{H}}(S_{z,t}(0)), p_{\mathcal{H}}(S_{z,t}(1)), \dots, p_{\mathcal{H}}(S_{z,t}(\tau'))\right\}$$

where

$$\tau' = \tau'(z,t,N) := \min\{k : S_{z,t}(k) \notin A_T^{\dagger}[\infty]\}$$

denotes the time at which the particle moving according to $S_{z,t}$ exits the infinite aggregate $A_T^{\dagger}[\infty]$. The new radius $R((z,t),T)$ is defined similarly as $R_N(z,t)$, but we are now considering the trajectory of $S_{z,t}$ until it exits $A_T^{\dagger}[\infty]$ rather than $A_{t^-}^{\dagger}[N]$. Growing the aggregate both in time and space, from $A_{t^-}^{\dagger}[N]$ to $A_T^{\dagger}[\infty]$, we then a.s have

$$R_N(z,t) \leq R((z,t),T) \ .$$

Hence, given (N_0,N) such that the forests $\mathcal{F}_{n_0}[N]$ and $\mathcal{F}_{n_0}[N_0]$ do not coincide on $\mathbb{Z}_{K_0}$, we get the existence of a sequence of $\kappa \geq 1$ particles coming from starting points $(z_i,t_i)_{1\leq i\leq\kappa}$ and satisfying the following conditions:

$$\begin{cases} N_0 < \|z_1\| \\ 0 < t_1 < t_2 < \cdots < t_\kappa < n_0 \\ \text{for } 1 \leq i \leq \kappa-1,\ \mathbb{B}(z_i, R((z_i,t_i),T)) \cap \mathbb{B}(z_{i+1}, R((z_{i+1},t_{i+1}),T)) \neq \emptyset \\ \mathbb{B}(z_\kappa, R((z_\kappa,t_\kappa),T)) \cap \mathbb{Z}_{K_0} \neq \emptyset \end{cases} \tag{3.3}$$

So, we now consider the space-time percolation model $\Sigma = \Sigma(n_0,T)$ defined as the collection of balls $\mathbb{B}(z,R((z,t),T))$, for any starting point (z,t) with $z \in \mathcal{H}$ and $t \in [0,n_0]$. Note also that considering larger radii, i.e. replacing $R_N(z,t)$ with $R((z,t),T)$, all dependency on the parameter N disappears in (3.3) and this allows us to take advantage of the stationarity property of $A_T^{\dagger}[\infty]$, which will greatly facilitate the study of Σ.

In the rest of this section, we state two properties about the percolation model Σ; a finite moment property for its radii (Lemma 3.1) and a finite degree property (Lemma 3.2).

Let us start by introducing the random radius $R_T(z)$, for $z \in \mathcal{H}$ and $T \geq n_0$, defined as

$$R_T(z) := \max_{t \in \mathcal{N}_z([0,T])} R((z,t),T) \tag{3.4}$$

where $\mathcal{N}_z$ denotes the PPP with intensity 1 associated to the source z. Also, $t \in \mathcal{N}_z([0,T])$ means that t is a 'top' of the PPP $\mathcal{N}_z$ occurring in $[0,T]$. Since the aggregate $A_T^{\dagger}[\infty]$ is translation invariant in distribution (w.r.t. translations of $\mathcal{H}$, see Proposition 2.1) then all the radii $R_T(z)$, $z \in \mathcal{H}$, have the same distribution. Setting $R_T := R_T(0)$, the next result states that the $R_T(z)$'s admit finite moments.

Lemma 3.1. *Let $T \geq n_0$ and $L > 0$ be real numbers. There exists a constant $C > 0$ such that for any integer M,*

$$\mathbb{P}(R_T \geq M) \leq CM^{-L} \ .$$

In particular $\mathbb{E}[(R_T)^k]$ is finite for any integer k.

Proof. Let $M \geq T$ be an a positive integer, fix $\alpha \in (1-1/d,\ 1)$. Let us first write:

$$\mathbb{P}(R_T \geq 2M) \leq \mathbb{P}\left(R_T \geq 2M,\ \#\mathcal{N}_0([0,T]) \leq M,\ \mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c\right) + \mathbb{P}\left(\#\mathcal{N}_0([0,T]) > M\right) + \mathbb{P}\left(\mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)\right)\ , \tag{3.5}$$

where $\mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon) := \{A^{\dagger}_T[\infty] \cap \mathbb{Z}^c_M \not\subset \mathscr{C}^{\alpha}_{\varepsilon}\}$. We know from Proposition 2.2 that the probability of $\mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)$ is smaller than any power of M^{-1} provided M is sufficiently large. Additionally, we also know that $\mathbb{P}\left(\#\mathcal{N}_0([0,T]) > M\right)$ decreases faster than any power of M^{-1}, given M is large enough. It then remains to focus on the first term of (3.5). Hence,

$$\begin{aligned}
\mathbb{P}&\left(R_T \geq 2M,\ \#\mathcal{N}_0([0,T]) \leq M,\ \mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c\right) \\
&= \mathbb{P}\left(\exists t \in \mathcal{N}_0([0,T]),\ R((0,t),T) \geq 2M,\ \#\mathcal{N}_0([0,T]) \leq M,\ \mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c\right) \\
&\leq \mathbb{E}\left[\sum_{t\in\mathcal{N}_0([0,T])} \mathbb{1}_{R((0,t),T)>2M}\, \mathbb{1}_{\#\mathcal{N}_0([0,T])\leq M}\, \mathbb{1}_{\mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{\#\mathcal{N}_0([0,T])\leq M} \sum_{t\in\mathcal{N}_0([0,T])} \mathbb{P}\left(R((0,t),T) > 2M,\ \mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c \mid \mathcal{N}_0([0,T])\right)\right].
\end{aligned}$$

Now, the event $\{R((0,t),T) > 2M\}$ implies that the random walk traveled a distance at least $2M$, and in particular, that it traveled from levels M to $2M$, while staying contained inside $A^{\dagger}_T[\infty]$. Now, working on $\mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c$ implies that $A_T[\infty]$ is contained inside $\mathscr{C}^{\alpha}_{\varepsilon}$ above levels M. Hence working on both events implies that the random walk traveled from level M to level $2M$, all while staying contained inside $\mathscr{C}^{\alpha}_{\varepsilon}$. By taking $j = 1$ in (2.4), we know that the number of boxes between levels M and $2M$, denoted by k^o, is such that

$$k^o \geq \frac{(2M)^{1-\alpha}}{4\varepsilon} = \frac{M^{1-\alpha}}{2^{1+\alpha}\varepsilon}.$$

Hence, using a box argument similar to the one used in the proof of Theorem 1.3, we can show that:

$$\begin{aligned}
\sum_{t\in\mathcal{N}_0([0,T])} \mathbb{P}\left(R((0,t),T) > 2M,\ \mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c \mid \mathcal{N}_0([0,T])\right) &\leq \sum_{t\in\mathcal{N}_0([0,T])} \left(1 - \frac{1}{4d^2}\right)^{k^o} \\
&= \#\mathcal{N}_0([0,T]) \exp\left(-c_0 M^{1-\alpha}\right),
\end{aligned}$$

where $c_0 = -\frac{1}{2^{1+\alpha}\varepsilon} \ln\left(1 - \frac{1}{4d^2}\right) > 0$. Hence,

$$\begin{aligned}
\mathbb{P}\left(R_T \geq 2M,\ \#\mathcal{N}_0([0,T]) \leq M,\ \mathbf{Over}^{\dagger}_{\alpha}(M,T,\varepsilon)^c\right) &\leq \mathbb{E}\left[\#\mathcal{N}_0([0,T]) e^{-c_0 M^{1-\alpha}} \mathbb{1}_{\#\mathcal{N}_0([0,T])\leq M}\right] \\
&\leq M \exp(-c_0 M^{1-\alpha})\ ,
\end{aligned}$$

which decreases faster than any power of M^{-1}. Finally, we have proved that $\mathbb{P}(R_T \geq 2M)$ decreases faster than any power of M^{-1} from which one easily concludes. □

Lemma 3.1 paves the way to a finite degree property for the percolation model $\Sigma = \Sigma(n_0, T)$. Precisely, any given ball $\mathbb{B}(z, R((z,t),T))$ of Σ overlaps only a finite number of other balls $\mathbb{B}(z', R((z',t'),T))$ of Σ, with probability 1.

Lemma 3.2. *Let $T \geq n_0$ and $z \in \mathcal{H}$. Then, with probability one, for any starting point (z,t) with $t \leq n_0$, the number of starting points (z',t'), with $z' \in \mathcal{H}$ and $t' \leq n_0$, such that $\mathbb{B}(z, R((z,t),T)) \cap \mathbb{B}(z', R((z',t'),T)) \neq \emptyset$ is finite.*

Proof. Since the radius $R_T(z)$ is by definition larger than any $R((z,t),T)$–see (3.4)–it is enough to prove that a.s.

$$\#\{z' \in \mathcal{H} :\, \mathbb{B}(z, R_T(z)) \cap \mathbb{B}(z', R_T(z')) \neq \emptyset\} < \infty \ .$$

Thus using the translation invariance property of the $R_T(z)$'s, it is also enough to state that the expectation

$$\mathbb{E}\Big[\#\{z \in \mathcal{H} :\, \mathbb{B}(0, R_T(0)) \cap \mathbb{B}(z, R_T(z)) \neq \emptyset\}\Big]$$

is finite. This follows from the next computation:

$$\begin{aligned}
\mathbb{E}\Big[\#\{z \in \mathcal{H} :\, \mathbb{B}(0, R_T(0)) \cap \mathbb{B}(z, R_T(z)) \neq \emptyset\}\Big] &= \mathbb{E}\Big[\sum_{z \in \mathcal{H}} \mathbb{1}_{R_T(0)+R_T(z) \geq \|z\|}\Big] \\
&= \sum_{z \in \mathcal{H}} \mathbb{P}\big(R_T(0) + R_T(z) \geq \|z\|\big) \\
&\leq \sum_{z \in \mathcal{H}} \mathbb{P}\big(\{R_T(0) \geq \|z\|/2\} \cup \{R_T(z) \geq \|z\|/2\}\big) \\
&\leq 2 \sum_{\ell=0}^{\infty} \sum_{\|z\|=\ell} \mathbb{P}\big(R_T(0) \geq \|z\|/2\big) \\
&\leq C_d \sum_{\ell=0}^{\infty} \ell^{d-2}\, \mathbb{P}\big(R_T(0) \geq \ell\big) \ .
\end{aligned}$$

This latter sum is finite thanks to Lemma 3.1. □

Conclusion. Let us interpret the percolation model Σ as the (undirected) graph $\mathcal{G}$ whose vertex set is given by the starting points (z,t), with $z \in \mathcal{H}$ and $t \leq n_0$ and whose edge set is made of pairs $\{(z,t),(z',t')\}$ such that the corresponding balls $\mathbb{B}(z, R((z,t),T))$ and $\mathbb{B}(z', R((z',t'),T))$ overlap. Lemma 3.2 asserts that each vertex of the graph $\mathcal{G}$ almost surely has a finite degree.

Now, combining this finite degree property with the *Absurd hypothesis*, we get the existence of an *infinite descending chain*. Let us explain why. Recall that $T \geq n_0$ and K_0 are fixed parameters. The *Absurd hypothesis* says that, with positive probability, for any integer N_0, there exists a sequence of $\kappa = \kappa(N_0) \geq 1$ particles satisfying (3.3). With each of these sequences can be associated a cluster in the percolation model $\Sigma = \Sigma(n_0, T)$ joining the outside of $\mathbb{Z}_{N_0}$ to $\mathbb{Z}_{K_0}$ whose centers have increasing times. Roughly speaking, these sequences connect the strip $\mathbb{Z}_{K_0}$ to regions as far as desired through the percolation model Σ. Then, using the finite degree property and proceeding step by step, we can extract an infinite sequence of starting points $((z_i,t_i))_{i\geq 1}$ such that

$$\begin{cases}
\|z_i\| \to \infty \\
\text{for any index } i,\ 0 < t_{i+1} < t_i < n_0 \\
\text{for any index } i,\ \mathbb{B}(z_i, R((z_i,t_i),T)) \cap \mathbb{B}(z_{i+1}, R((z_{i+1},t_{i+1}),T)) \neq \emptyset \\
\mathbb{B}(z_1, R((z_1,t_1),T)) \cap \mathbb{Z}_{K_0} \neq \emptyset
\end{cases} \tag{3.6}$$

Note that, w.r.t. (3.3), we have reversed the indices of the time sequence $(t_i)_{i\geq 1}$. On the one hand, this sequence of starting points $((z_i,t_i))_{i\geq 1}$ is *infinite* and means that Σ percolates since it contains an unbounded cluster. On the other hand, it is also *descending* since the sequence of starting times $(t_i)_{i\geq 1}$ is decreasing.

Let $\mathcal{K}_0 := \mathcal{H}_{K_0}$. The previous analysis allows us to say that, under the *Absurd hypothesis*, the following holds

$$\mathbb{P}(\mathbf{Perco}(n_0, \mathcal{K}_0, n_0)) > 0 \ , \tag{3.7}$$

where, for any $0 \leq t \leq T$ and any compact set $\mathcal{K} \subset \mathcal{H}$,

$$\mathbf{Perco}(t, \mathcal{K}, T) := \left\{ \begin{array}{c} \exists \text{ a sequence } ((z_i, t_i))_{i\geq 1} \text{ of starting points s.t. } \|z_i\| \to \infty, \\ \mathbb{B}(z_1, R((z_1, t_1), T)) \cap \mathcal{K} \neq \emptyset, \text{ and for any } i \geq 1, \ t_{i+1} < t_i < t \\ \text{and } \mathbb{B}(z_i, R((z_i, t_i), T)) \cap \mathbb{B}(z_{i+1}, R((z_{i+1}, t_{i+1}), T)) \neq \emptyset \end{array} \right\} .$$

In the next section, we will take advantage of the monotonicity of the time sequence $(t_i)_{i\geq 1}$ to establish that the infinite descending chain mentioned above appears instantaneously.

3.3 Instantaneous percolation

For $T \geq t \geq 0$, let us set:

$$\mathbf{Perco}(t, T) := \bigcup_{\mathcal{K}} \mathbf{Perco}(t, \mathcal{K}, T) \ , \tag{3.8}$$

where the union above is taken over all compact subsets of $\mathcal{H}$. The event $\mathbf{Perco}(t, T)$ ensures the existence of an infinite descending chain made up with balls coming from the percolation model Σ, anchored at some (random) $\mathcal{K} \subset \mathcal{H}$. The event $\mathbf{Perco}(t, T)$ is monotone w.r.t. parameters t and T:

Lemma 3.3. *Let $0 \leq t \leq t' \leq T \leq T'$. Then,*

$$\mathbf{Perco}(t, T) \subset \mathbf{Perco}(t, T') \ \text{ and } \ \mathbf{Perco}(t, T) \subset \mathbf{Perco}(t', T) \ .$$

Proof. Let $0 \leq t \leq T \leq T'$. Recall that the infinite aggregate $A^{\dagger}_T[\infty]$ corresponds to particles launched from the whole set of sources $\mathcal{H}$ and during the time interval $[0, T]$. Hence, $A^{\dagger}_T[\infty]$ and $A^{\dagger}_{T'}[\infty]$ can be naturally coupled so that a.s. $A^{\dagger}_T[\infty] \subset A^{\dagger}_{T'}[\infty]$. This leads to $R((z_i, t_i), T) \leq R((z_i, t_i), T')$ whatever the starting point (z_i, t_i) with $t_i \leq t$. So, $\mathbf{Perco}(t, \mathcal{K}, T)$ is included in $\mathbf{Perco}(t, \mathcal{K}, T')$, for any compact set $\mathcal{K}$, and the same holds for $\mathbf{Perco}(t, T)$ and $\mathbf{Perco}(t, T')$.

The second inclusion is easy to prove. Indeed, replacing t with $t' \geq t$ amounts to relax the upper bound on the decreasing sequence $(t_i)_{i\geq 1}$ appearing in the event $\mathbf{Perco}(t, \mathcal{K}, T)$. So $\mathbf{Perco}(t, \mathcal{K}, T)$ is included in $\mathbf{Perco}(t', \mathcal{K}, T)$, for any $\mathcal{K}$, and the same holds for $\mathbf{Perco}(t, T)$ and $\mathbf{Perco}(t', T)$. □

From now on, we fix $T := n_0 + 1$. Let us introduce the critical percolation time as

$$t_c = t_c(T) := \inf \left\{ 0 \leq t \leq T : \mathbb{P}\big(\mathbf{Perco}(t, T)\big) > 0 \right\} \in [0, T] \ .$$

Combining to the monotone property given by Lemma 3.3, we can then write:

$$\left\{ \begin{array}{l} \mathbb{P}\left(\mathbf{Perco}(t, T)\right) = 0 \text{ for any } 0 \leq t < t_c, \\ \mathbb{P}\left(\mathbf{Perco}(t, T)\right) > 0 \text{ for any } t_c < t \leq T. \end{array} \right. \tag{3.9}$$

Both statements of (3.9) become meaningful provided the critical percolation time t_c is non-trivial, i.e. different from 0 and T. This is where the *Absurd hypothesis* steps in since (3.7) and Lemma 3.3 imply that

$$\mathbb{P}(\,\mathbf{Perco}(n_0,T)) \geq \mathbb{P}(\,\mathbf{Perco}(n_0,n_0)) \geq \mathbb{P}(\,\mathbf{Perco}(n_0,\mathcal{K}_0,n_0)) > 0$$

that is to say

$$t_c \leq n_0 < T \ . \tag{3.10}$$

Actually, the condition $t_c < T$–ensured by the *Absurd hypothesis*–leads to a phenomenon of *instantaneous percolation* for the percolation model Σ that we describe below.

Let us first assume that the critical percolation time t_c is positive. The case $t_c = 0$ is similar and will be treated after. Hence for any $\varepsilon > 0$ small enough (i.e. such that $t_c - \varepsilon \geq 0$ and $t_c + \varepsilon \leq T$), we have $\mathbb{P}(\mathbf{Perco}(t_c+\varepsilon,T)) > 0$ while $\mathbb{P}(\mathbf{Perco}(t_c-\varepsilon,T)) = 0$ by (3.9), meaning that

$$\mathbb{P}(\,\mathbf{Perco}(t_c+\varepsilon,T)\setminus\mathbf{Perco}(t_c-\varepsilon,T)) > 0 \ .$$

Let us analyze what happens on the event $\mathbf{Perco}(t_c+\varepsilon,T)\setminus\mathbf{Perco}(t_c-\varepsilon,T)$. First of all, the event $\mathbf{Perco}(t_c+\varepsilon,T)$ asserts the existence of an infinite descending chain in Σ associated to a sequence of starting points $((z_i,t_i))_{i\geq 1}$ with, for any $i \geq 1$, $t_{i+1} < t_i < t_c + \varepsilon$. Besides, the event $\mathbf{Perco}(t_c-\varepsilon,T)^c$ forces all the t_i's to be larger than $t_c - \varepsilon$. Otherwise there would exist some index i_0 such that $t_{i_0} < t_c - \varepsilon$. In that case, one would have an infinite descending chain associated to the sequence of starting points $((z_i,t_i))_{i\geq i_0}$ anchored at $\mathbb{B}(z_{i_0},R((z_{i_0},t_{i_0}),T))$ and such that for any i, $t_{i+1} < t_i \leq t_{i_0} < t_c - \varepsilon$, meaning that $\mathbf{Perco}(t_c-\varepsilon,T)$ actually occurs. In conclusion, the sequence of starting points $((z_i,t_i))_{i\geq 1}$ satisfies $t_c - \varepsilon < t_{i+1} < t_i < t_c + \varepsilon$ for any index i. This means that this infinite descending chain appears entirely during the time interval $[t_c-\varepsilon,t_c+\varepsilon]$, with length 2ε, for $\varepsilon > 0$ arbitrarily small. This is why we talk about instantaneous percolation. Let us emphasize that the previous argument works because the radii $R((z_i,t_i),T)$'s remain the same in both events $\mathbf{Perco}(t_c-\varepsilon,T)$ and $\mathbf{Perco}(t_c+\varepsilon,T)$ since they have the same second parameter T.

In the particular case $t_c = 0$, we know that there is no percolation at the critical time t_c since no particles have been launched yet. So we apply the previous analysis with $t_c + \varepsilon = \varepsilon$ and $t_c - \varepsilon$ replaced with 0. As before, an infinite descending chain appears entirely during the time interval $[0,\varepsilon]$, for any small $\varepsilon > 0$.

In order to simplify the argumentation and lighten notations, let us reduce the first case $t_c > 0$ to the second one $t_c = 0$. Indeed, because the radii $R((z_i,t_i),T)$'s remain unchanged in the events $\mathbf{Perco}(\cdot,T)$, the time interval during which the infinite descending chain entirely occurs matters in distribution only through its length (the PPP $\mathcal{N}_z$'s have stationary increments). Henceforth,

$$\Big(\forall\varepsilon > 0,\ \mathbb{P}(\,\mathbf{Perco}(t_c+\varepsilon,T)\setminus\mathbf{Perco}(t_c-\varepsilon,T)) > 0\Big) \implies \Big(\forall\varepsilon > 0,\ \mathbb{P}(\,\mathbf{Perco}(\varepsilon,T)) > 0\Big) \ .$$

Conclusion. We have taken advantage of the monotonicity in time of the sequence of starting points $((z_i,t_i))_{i\geq 1}$ to state that the corresponding infinite descending chain in the percolation model Σ appears instantaneously. Precisely, we have proven under the *Absurd hypothesis* that

$$\forall\varepsilon > 0,\ \mathbb{P}(\,\mathbf{Perco}(\varepsilon,T)) > 0 \ . \tag{3.11}$$

We can now forget the time dimension of our model: (3.11) allows us to exhibit a *supercritical* discrete Boolean model, denoted by $\hat{\Sigma}_\varepsilon$, whose intensity tends to 0 as $\varepsilon \to 0$.

Let us set, for any source $z \in \mathcal{H}$,

$$Y_z := \mathbb{1}_{\#\mathcal{N}_z([0,\varepsilon])>0} \ .$$

So $Y_z = 1$ means that at least one particle has been emitted during the time interval $[0, \varepsilon]$. The Y_z's are Bernoulli random variables with common parameter

$$p_\varepsilon := \mathbb{P}\left(Y_z = 1\right) = 1 - e^{-\varepsilon} \tag{3.12}$$

which tends to 0 as $\varepsilon \to 0$. By hypothesis on the PPP $\mathcal{N}_z$'s, the random variables Y_z, $z \in \mathcal{H}$, are also independent from each other. Let us denote by

$$\chi_\varepsilon := \{z \in \mathcal{H} : Y_z = 1\}$$

the random set of emitting sources during the time interval $[0, \varepsilon]$. The set χ_ε can be interpreted as a discrete PPP on $\mathcal{H}$ with intensity p_ε: it will play the role of the center set for the Boolean model $\hat{\Sigma}_\varepsilon$. In all that follows, let us keep in mind that when $\varepsilon \to 0$ there are very few centers, and so very few balls, in the Boolean model $\hat{\Sigma}_\varepsilon$.

Thus, for $z \in \mathcal{H}$, let us define in the same spirit of (3.4) the radius $R_T(z; \varepsilon)$ as

$$R_T(z; \varepsilon) := \max_{t \in \mathcal{N}_z([0,\varepsilon])} R((z, t), T) \ .$$

The only difference between radii $R_T(z; \varepsilon)$ and $R_T(z)$ defined in (3.4) is that $R_T(z; \varepsilon)$ involves particles launched during $[0, \varepsilon]$ while $R_T(z)$ involves particles of $[0, T]$, so that $R_T(z; \varepsilon) \leq R_T(z)$ (and then $R_T(z; \varepsilon)$ also satisfies the finite moment property of Lemma 3.1). They both refer to radii $R(\cdot, T)$ defined from the aggregate $A_T^\dagger[\infty]$ (with $T = n_0 + 1$).

We are now ready to define the (discrete) Boolean model $\hat{\Sigma}_\varepsilon$ by

$$\hat{\Sigma}_\varepsilon := \bigcup_{z \in \chi_\varepsilon} \mathbb{B}(z, R_T(z; \varepsilon)) \ .$$

Statement (3.11) says that, for any $\varepsilon > 0$ and with positive probability, there exists a sequence of starting points $((z_i, t_i))_{i \geq 1}$ such that $\|z_i\| \to \infty$ and, for any $i \geq 1$, $t_{i+1} < t_i < \varepsilon$ and the balls $\mathbb{B}(z_i, R((z_i, t_i), T))$ and $\mathbb{B}(z_{i+1}, R((z_{i+1}, t_{i+1}), T))$ overlap. So, the z_i's all belong to χ_ε and the larger balls $\mathbb{B}(z_i, R_T(z_i; \varepsilon))$ and $\mathbb{B}(z_{i+1}, R_T(z_{i+1}; \varepsilon))$ overlap too. In other words, the Boolean model $\hat{\Sigma}_\varepsilon$ contains an unbounded cluster.

Finally, the argumentation of the whole Section 3 provides the following result:

Proposition 3.4. *Under the Absurd hypothesis (1.6), the following holds:*

$$\forall \varepsilon > 0, \ \mathbb{P}\Big(\hat{\Sigma}_\varepsilon \text{ percolates}\Big) > 0 \ . \tag{3.13}$$

4 A multiscale argument

In this section, we study the $(d-1)$-dimensional, discrete Boolean model

$$\hat{\Sigma}_\varepsilon = \bigcup_{z \in \chi_\varepsilon} \mathbb{B}(z, R_T(z; \varepsilon)) \ ,$$

defined in the previous section, and whose intensity p_ε tends to 0 as $\varepsilon \to 0$. We adapt the strategy of [11] to our context to state :

Proposition 4.1. *For any $\varepsilon > 0$ small enough, the Boolean model $\hat{\Sigma}_\varepsilon$ does not percolate with probability* 1.

To implement the strategy of [11], we need to make our model more local. This is the role of $\hat{\Sigma}^{\text{loc}}_\varepsilon$ defined below. For that purpose, the stabilization result for the infinite aggregate $A^\dagger_n[\infty]$ (Theorem 1.3) is an important ingredient to ensure that the localized model $\hat{\Sigma}^{\text{loc}}_\varepsilon$ is a good approximation of $\hat{\Sigma}_\varepsilon$.

Proof of Theorem 1.1. Propositions 3.4 and 4.1 together say that the *Absurd hypothesis* (1.6) leads to a contradiction. We then conclude that, with probability 1, there exists N_0 such that, for any $N \geq N_0$, the forests $\mathcal{F}_{n_0}[N]$ and $\mathcal{F}_{n_0}[N_0]$ coincide on the strip $\mathbb{Z}_{K_0}$. This concludes the proof of Theorem 1.1. □

4.1 The localized Boolean models $\hat{\Sigma}^{\text{loc}}_\varepsilon$

Recall that $T = n_0 + 1$ and $\varepsilon > 0$ is thought to be small. Given a source $x \in \mathcal{H}$, we denote by $A^\dagger_T[\mathbb{B}(x, 20M)]$ the aggregate $A^\dagger_T[\cdot]$ using only sources of $\mathbb{B}(x, 20M)$. Associated to the starting point (z, t), with $z \in \mathcal{H}$ and $t \in [0, \varepsilon]$, let us introduce the local radius $R^{\text{loc}}_{x,M}((z, t), T)$ defined by

$$R^{\text{loc}}_{x,M}((z,t),T) := \min\left\{r \in \mathbb{N} : \mathbb{B}(z,r) \text{ contains } p_{\mathcal{H}}(S_{z,t}(0)), p_{\mathcal{H}}(S_{z,t}(1)), \ldots, p_{\mathcal{H}}(S_{z,t}(\tau''))\right\} \tag{4.1}$$

where

$$\tau'' := \min\{k : S_{z,t}(k) \notin A^\dagger_T[\mathbb{B}(x, 20M)]\}$$

denotes the time at which the particle moving according to $S_{z,t}$ exits $A^\dagger_T[\mathbb{B}(x, 20M)]$. Taking the maximum over starting points (z, t) with $t \in \mathcal{N}_z([0, \varepsilon])$, we get

$$R^{\text{loc}}_T(z;\varepsilon) = R^{\text{loc}}_{T,x,M}(z;\varepsilon) := \max_{t \in \mathcal{N}_z([0,\varepsilon])} R^{\text{loc}}_{x,M}((z,t),T) \ .$$

Let us now define the *localized Boolean model* $\hat{\Sigma}^{\text{loc}}_\varepsilon(x, M)$ which is a version of $\hat{\Sigma}_\varepsilon$ localized to the neighborhood of x:

$$\hat{\Sigma}^{\text{loc}}_\varepsilon = \hat{\Sigma}^{\text{loc}}_\varepsilon(x,M) := \bigcup_{z \in \chi_\varepsilon \cap \mathbb{B}(x,10M)} \mathbb{B}\Big(z, R^{\text{loc}}_{T,x,M}(z;\varepsilon)\Big) \ .$$

Unlike $\hat{\Sigma}_\varepsilon$, the localized Boolean model $\hat{\Sigma}^{\text{loc}}_\varepsilon$ is local in the following sense. It only uses sources of χ_ε contained inside $\mathbb{B}(x, 10M)$. The radii that it considers depend only on $A^\dagger_T[\mathbb{B}(x, 20M)]$.

As in [11], we consider the event

$$G_\varepsilon(x,M) := \left\{ \begin{array}{c} \text{The connected component of } x \text{ in } \hat{\Sigma}^{\text{loc}}_\varepsilon(x,M) \cup \mathbb{B}(x,M) \\ \text{is not included in } \mathbb{B}(x,8M) \end{array} \right\} .$$

Since our model is translation invariant, we have for any x:

$$\pi_\varepsilon(M) := \mathbb{P}\big(G_\varepsilon(0,M)\big) = \mathbb{P}\big(G_\varepsilon(x,M)\big) \ .$$

Let us denote by $\hat{C}_\varepsilon(0)$ the cluster of the source 0 in the discrete Boolean model $\hat{\Sigma}_\varepsilon$ whose diameter $\operatorname{diam} \hat{C}_\varepsilon(0)$ is defined as the minimal integer r such that $\hat{C}_\varepsilon(0) \subset \mathbb{B}(0, r)$. The next two results allow us to conclude.

Proposition 4.2. *There exists a constant $C = C_{T,d} > 0$ such that for all $M, L \geq 1$,*

$$\mathbb{P}\big(\mathrm{diam}\, \hat{C}_\varepsilon(0) \geq 8M\big) \leq \pi_\varepsilon(M) + \frac{C}{M^L} \ .$$

Proposition 4.3. *For any $\varepsilon > 0$ small enough we have*

$$\liminf_{M\to\infty} \pi_\varepsilon(M) = 0 \ .$$

Proof of Proposition 4.1. Taking $\varepsilon > 0$ small enough according to Proposition 4.3, the two previous results imply that

$$\begin{aligned} \lim_{M\to\infty} \mathbb{P}\big(\mathrm{diam}\, \hat{C}_\varepsilon(0) \geq 8M\big) &= \liminf_{M\to\infty} \mathbb{P}\big(\mathrm{diam}\, \hat{C}_\varepsilon(0) \geq 8M\big) \\ &\leq \liminf_{M\to\infty} \pi_\varepsilon(M) \ = \ 0 \ , \end{aligned}$$

meaning that the source 0 almost surely belongs to a finite connected component in $\hat{\Sigma}_\varepsilon$. This discrete Boolean model being translation invariant in distribution, we conclude that it a.s. admits only finite connected components. I.e. $\hat{\Sigma}_\varepsilon$ does not percolate with probability 1. □

The next two sections are devoted to the proofs of Propositions 4.2 and 4.3.

4.2 Proof of Proposition 4.2

Let us introduce for any $M \geq 1$ the event $H(M)$ defined by

$$H(M) := \Big\{\text{each ball of } \hat{\Sigma}_\varepsilon \text{ intersecting } \mathbb{B}(0, 10M) \text{ has a radius smaller than } M\Big\} \ .$$

The following lemma gives a control for the probability of $H(M)$.

Lemma 4.4. *There exists a positive constant $C = C(d, \varepsilon)$ such that for any $M, L \geq 1$,*

$$\mathbb{P}\big(H(M)^c\big) \leq \frac{C}{M^L} \ .$$

Let us first prove Proposition 4.2 from Lemma 4.4 and in a second step, Lemma 4.4 will be proven.

Proof of Proposition 4.2. We define the event

$$\mathcal{G}(M) := H(M) \cap \Big\{A_T^\dagger[\infty] \cap \mathbb{Z}_{10M} = A_T^\dagger[\mathbb{B}(0, 20M)] \cap \mathbb{Z}_{10M}\Big\} .$$

Now, we use the stabilization result Theorem 1.3 which, combined to Lemma 4.4, provides the following control on the probability of $\mathcal{G}(M)$:

$$\mathbb{P}\left(\mathcal{G}(M)\right) \geq 1 - \frac{C}{M^L},$$

for some positive constant C and for any $M, L \geq 1$. Recalling that $\pi_\varepsilon(M)$ denotes the probability of $G_\varepsilon(0, M)$, it is then sufficient to prove the inclusion:

$$\Big\{\mathrm{diam}\, \hat{C}_\varepsilon(0) \geq 8M\Big\} \cap \mathcal{G}(M) \subset G_\varepsilon(0, M) \ .$$

Let us assume that the event $\{\mathrm{diam}\, \hat{C}_\varepsilon(0) \geq 8M\} \cap \mathcal{G}(M)$ holds. This implies the existence of a cluster $\mathcal{C}$ of $\hat{\Sigma}_\varepsilon$ containing 0 and going beyond $\mathbb{B}(0, 8M)$. Each ball of $\mathcal{C}$ overlaps $\mathbb{B}(0, 8M)$

and then has a radius smaller than M thanks to $H(M)$. So they are included in $\mathbb{B}(0,10M)$ and their centers belong to $\chi_\varepsilon \cap \mathbb{B}(0,10M)$.

Given a vertex z, the radii $R_T^{\mathrm{loc}}(z;\varepsilon)$ and $R_T(z;\varepsilon)$ used resp. for $\hat{\Sigma}_\varepsilon^{\mathrm{loc}}$ and $\hat{\Sigma}_\varepsilon$ are possibly different since they resp. refer to $A_T^\dagger[\mathbb{B}(0,20M)]$ and $A_T^\dagger[\infty]$. However, we prove that on the event $\{\operatorname{diam}\hat{C}_\varepsilon(0) \geq 8M\} \cap \mathcal{G}(M)$, these radii are equal for the balls involved in the cluster $\mathcal{C}$. Indeed, these balls are included in $\mathbb{B}(0,10M)$ and, on $\mathcal{G}(M)$, the aggregates $A_T^\dagger[\infty]$ and $A_T^\dagger[\mathbb{B}(0,20M)]$ coincide on $\mathbb{Z}_{10M}$. So the radii $R_T^{\mathrm{loc}}(z;\varepsilon)$ and $R_T(z;\varepsilon)$ coincide for each center z of balls involved in $\mathcal{C}$. Therefore, on $\{\operatorname{diam}\hat{C}_\varepsilon(0) \geq 8M\} \cap \mathcal{G}(M)$, there exists a cluster of balls of $\hat{\Sigma}_\varepsilon$ such that:

- all these balls have their centers in $\chi_\varepsilon \cap \mathbb{B}(0,10M)$;
- all these balls have their radii given by $R_T^{\mathrm{loc}}(z;\varepsilon)$;
- the cluster contains 0 and goes beyond $\mathbb{B}(0,8M)$.

Then, this cluster of balls is also a cluster of $\hat{\Sigma}_\varepsilon^{\mathrm{loc}}$ and the event $G_\varepsilon(0,M)$ occurs. □

Proof of Lemma 4.4. Fix $M, L \geq 1$. Let us introduce the following events:

$$\mathbf{Inside}(M) := \left\{ \begin{array}{c} \text{There exists a ball of } \hat{\Sigma}_\varepsilon \text{ centered } \textbf{inside } \mathbb{B}(0,20M) \\ \text{that intersects } \mathbb{B}(0,10M) \text{ with a radius greater than } M \end{array} \right\},$$

$$\mathbf{Outside}(M) := \left\{ \begin{array}{c} \text{There exists a ball of } \hat{\Sigma}_\varepsilon \text{ centered } \textbf{outside } \mathbb{B}(0,20M) \\ \text{that intersects } \mathbb{B}(0,10M) \end{array} \right\}.$$

Hence, the event $H(M)^c$ can be written as the union $\mathbf{Inside}(M) \cup \mathbf{Outside}(M)$ and we have to work with the probability of these two events. We begin by handling the event $\mathbf{Inside}(M)$. Let us write:

$$\begin{aligned}
\mathbb{P}(\mathbf{Inside}(M)) &= \mathbb{P}\left(\bigcup_{z \in \chi_\varepsilon \cap \mathbb{B}(0,20M)} \{R_T(z;\varepsilon) \geq M\} \right) \\
&\leq \sum_{z \in \mathbb{B}(0,20M)} \mathbb{P}(R_T(z;\varepsilon) \geq M) \\
&\leq C_d M^{d-1} \mathbb{P}(R_T(0;\varepsilon) \geq M) \ .
\end{aligned}$$

Using Lemma 3.1, $\mathbb{P}(R_T(0;\varepsilon) \geq M)$ decreases faster than any power of M^{-1}, which handles this case.

We switch our focus to the event $\mathbf{Outside}(M)$ which is trickier to handle since it deals with an infinite number of particles outside of $\mathbb{B}(0,20M)$. Let us recall the event

$$\mathbf{Over}_\alpha^\dagger(10M,T,\delta) = \{A_T^\dagger[\infty] \cap \mathbb{Z}_{10M}^c \not\subset \mathscr{C}_\delta^\alpha\}$$

introduced in Section 2.1, where

$$\mathscr{C}_\delta^\alpha = \bigcup_{\ell \geq 0} \left\{ z \in \mathbb{Z}^d,\ \|p_{\mathcal{H}}(z)\| = \ell,\ |z_1| \leq \delta \ell^\alpha \right\} .$$

For $\delta > 0$ and $\alpha \in (1-1/d, 1)$, we know thanks to the global upper bound Proposition 2.2 that the probability of $\mathbf{Over}_\alpha^\dagger(10M,T,\delta)$ decreases faster than any power of M^{-1}. So we can restrict our attention to the event $\mathbf{Outside}(M) \cap \mathbf{Over}^{\dagger}_{\alpha}(10M, T, \delta)^c$. The event $\mathbf{Outside}(M)$ provides the existence of a ball of the Boolean model $\hat{\Sigma}_\varepsilon$ that intersects $\mathbb{B}(0, 10M)$ and whose center is beyond level $20M$. This ball is due to a particle starting (during the time interval $[0, \varepsilon]$) from a source beyond level $20M$ and visiting the strip $\mathbb{Z}_{10M}$ before exiting the aggregate $A^{\dagger}_T[\infty]$. Thanks to $\mathbf{Over}^{\dagger}_{\alpha}(10M, T, \delta)^c$, we can assert that the random walk associated to that particle starts from a level greater than $20M$ and visits $\mathbb{Z}_{10M}$ before exiting the cone $\mathscr{C}^{\alpha}_{\delta}$. This implies the event D^c_{10M} introduced in Section 2.3 whose probability is smaller than any power of M^{-1} thanks to Lemma 2.3. This concludes the proof. □

4.3 Proof of Proposition 4.3

This section is an adaptation of [11]. Lemmas 4.5 and 4.6 together imply, by induction, that $\pi_\varepsilon(10^n M) \to 0$ as $n \to \infty$ for some M and ε small enough. Lemma 4.5 is the induction step, allowing to go from scale $10^n M$ to $10^{n+1} M$, while Lemma 4.6 is the base step.

Lemma 4.5. *There exist positive constants $c = c(d)$ and $C = C(T, d)$ such that for any $M, L \geq 1$,*

$$\pi_\varepsilon(10M) \leq c\pi_\varepsilon(M)^2 + \frac{C}{M^L} \ .$$

Lemma 4.6. *There exists a positive constant $C' = C'(d)$ such that for all $M \geq 1$ and all $\varepsilon > 0$:*

$$\pi_\varepsilon(M) \leq \varepsilon C' M^{d-1} \ .$$

Let us prove Proposition 4.3 from Lemmas 4.5 and 4.6.

Proof of Proposition 4.3. This is an adaptation of Lemma 3.7 of [11]. Setting $f(M) := c\pi_\varepsilon(M)$ and $g(M) := 10cC/M$, Lemma 4.5 provides

$$f(10M) \leq f(M)^2 + g(10M) \ . \tag{4.2}$$

Since $g(M) \to 0$, we can pick M_0 such that, for any $M \geq M_0$, $g(M) \leq 1/4$. Thus, using Lemma 4.6, there exists $\varepsilon_0 = \varepsilon_0(M_0) > 0$ small enough such that, for $\varepsilon \leq \varepsilon_0$ and $M \leq M_0$,

$$f(M) = c\pi_\varepsilon(M) \leq c\varepsilon_0 C' M_0^{d-1} \leq 1/2 \ .$$

So, the function f is bounded by $1/2$ on the interval $(0, M_0]$. Let us first extend this bound on $(M_0, 10M_0]$. To do it, let us fix $\varepsilon \in (0, \varepsilon_0)$. Thanks to (4.2), we can write for any $M \in (M_0, 10M_0]$:

$$f(M) \leq f(M/10)^2 + g(M) \leq \Big(\frac{1}{2}\Big)^2 + \frac{1}{4} = \frac{1}{2} \ .$$

Iterating this argument, we prove by induction that $f(M) \leq 1/2$ for any $M > 0$.

As a consequence, we deduce from (4.2) that, for any $M > 0$,

$$f(10M) \leq \frac{1}{2} f(M) + g(10M) \leq \frac{1}{2} + g(10M) \ ,$$

from which is not difficult to get, once again by induction, that the following holds for any integer n,

$$f(10^n M) \leq \frac{1}{2^n} + g(10^n M) + \frac{g(10^{n-1}M)}{2} + \ldots + \frac{g(10M)}{2^{n-1}} \ . \tag{4.3}$$

Henceforth, using (4.3) and $g(10^n M) \to 0$ as $n \to \infty$, we prove that $f(10^n M)$ tends to 0 as $n \to \infty$, which is the searched result. □

Proof of Lemma 4.4. Let us first consider the event $F(M)$ defined by

$$F(M) := \{\forall z \in \chi_\varepsilon \cap \mathbb{B}(0, 100M),\ R_T(z; \varepsilon) \leq M\}$$

for which, any ball of $\hat{\Sigma}_\varepsilon$ centered at some $z \in \chi_\varepsilon \cap \mathbb{B}(0, 100M)$ has a radius $R_T(z; \varepsilon)$ smaller than M. Since these radii are identically distributed and satisfy Lemma 3.1), the event $F(M)$ is very likely. There exists a constant $C > 0$ such that for any $M, L \geq 1$,

$$\mathbb{P}\big(F(M)^c\big) \leq \frac{C}{M^L} \ . \tag{4.4}$$

Besides, for $M \geq 1$, let us consider the event

$$\mathbf{Stab}(0, 100M) := \bigcap_{z \in \mathbb{B}(0,100M)} \mathbf{Stab}^M(z)$$

where, for any $z \in \mathcal{H}$,

$$\mathbf{Stab}^M(z) := \left\{ \begin{array}{c} A_T^\dagger[\mathbb{B}(z, 20M)] \cap (\mathbb{Z} \times \mathbb{B}(z, 10M)) = A_T^\dagger[S] \cap (\mathbb{Z} \times \mathbb{B}(z, 10M)) \,, \\ \text{for any } S \subset \mathcal{H} \text{ such that } \mathbb{B}(z, 20M) \subset S. \end{array} \right\} .$$

In particular, $\mathbf{Stab}^M(0)$ means that $A_T^\dagger[\mathbb{B}(0, 20M)]$ (also denoted by $A_T^\dagger[20M]$ for short) and $A_T^\dagger[S]$ coincide on $\mathbb{Z}_{10M}$, for any $\mathcal{H}_{20M} \subset S \subset \mathcal{H}$. Let us prove that there exists a constant $C > 0$ such that for any $M, L \geq 1$,

$$\mathbb{P}\big(\,\mathbf{Stab}(0, 100M)^c\big) \leq \frac{C}{M^L} \ . \tag{4.5}$$

Let us start by writing, using translation invariance of the aggregates $A_T^\dagger[\cdot]$,

$$\mathbb{P}\big(\,\mathbf{Stab}(0, 100M)^c\big) \leq \sum_{z \in \mathbb{B}(0,100M)} \mathbb{P}\big(\,\mathbf{Stab}^M(z)^c\big) \leq c(100M)^{d-1} \mathbb{P}\big(\,\mathbf{Stab}^M(0)^c\big) \ .$$

So it is sufficient to show that $\mathbf{Stab}^M(0)^c$ has a probability decreasing faster than any power of M^{-1}. The *Aggregate stabilization* result Theorem 1.3 asserts that, with probability larger than $1 - CM^{-L}$, the aggregates $A_T^\dagger[20M]$ and $A_T^\dagger[\infty]$ coincide on the strip $\mathbb{Z}_{10M}$. Then the same holds for $A_T^\dagger[20M]$ and $A_T^\dagger[S]$, for any $\mathcal{H}_{20M} \subset S \subset \mathcal{H}$ since $A_T^\dagger[20M] \subset A_T^\dagger[S] \subset A_T^\dagger[\infty]$ by the natural coupling. So $\mathbf{Stab}^M(0)$ occurs with probability larger than $1 - CM^{-L}$, and then (4.5) is proven.

Let $\mathbb{S}_r$ be the $(d-2)$-dimensional sphere centered at the origin, with radius r and included in the source set $\mathcal{H}$: $\mathbb{S}_r := \{z \in \mathcal{H} :\ \|z\| = r\}$. We claim that the following key inclusion holds for any M,

$$G_\varepsilon(0, 10M) \cap F(M) \cap \mathbf{Stab}(0, 100M) \subset \Big(\bigcup_{c \in \mathbb{S}_{10}} G_\varepsilon(Mc, M) \Big) \cap \Big(\bigcup_{c' \in \mathbb{S}_{80}} G_\varepsilon(Mc', M) \Big) \ . \tag{4.6}$$

Lemma 4.4 actually appears as a straight consequence of the inclusion (4.6) combined with (4.4) and (4.5). Let us first explain why and, in a second step, we will establish (4.6).

Inequalities (4.4) and (4.5) allow to write

$$\begin{aligned} \pi_\varepsilon(10M) = \mathbb{P}(G_\varepsilon(0, 10M)) \quad &\leq \quad \mathbb{P}(G_\varepsilon(0, 10M) \cap F(M) \cap \mathbf{Stab}(0, 100M)) + \frac{C}{M^L} \\ &\leq \quad \mathbb{P}\Big(\Big(\bigcup_{c \in \mathbb{S}_{10}} G_\varepsilon(Mc, M) \Big) \cap \Big(\bigcup_{c' \in \mathbb{S}_{80}} G_\varepsilon(Mc', M) \Big)\Big) + \frac{C}{M^L} \ . \end{aligned}$$

Recall that the event $G_\varepsilon(Mc, M)$ involves balls of $\hat{\Sigma}_\varepsilon^{\text{loc}}$ whose centers are included in $\mathbb{B}(Mc, 10M)$ and radii are defined w.r.t. the aggregate $A_T^\dagger[\mathbb{B}(Mc, 20M)]$. So the event $\bigcup_{c\in\mathbb{S}_{10}} G_\varepsilon(Mc, M)$ only concerns random inputs (i.e. Poisson clocks and random walks) associated to sources of $\mathcal{H}_{30M}$. In the same way, $\bigcup_{c'\in\mathbb{S}_{80}} G_\varepsilon(Mc', M)$ only concerns random inputs associated to sources outside of $\mathcal{H}_{59M}$. So they are independent from each other. It is then easy to conclude:

$$\begin{aligned}\pi_\varepsilon(10M) &\leq \mathbb{P}\Big(\bigcup_{c\in\mathbb{S}_{10}} G_\varepsilon(Mc, M)\Big)\,\mathbb{P}\Big(\bigcup_{c'\in\mathbb{S}_{80}} G_\varepsilon(Mc', M)\Big) + \frac{C}{M^L} \\ &\leq |\mathbb{S}_{10}||\mathbb{S}_{80}|\pi_\varepsilon(M)^2 + \frac{C}{M^L}\ .\end{aligned}$$

As a consequence, it only remains to establish the inclusion (4.6) and, to do it, let us assume that $G_\varepsilon(0, 10M) \cap F(M) \cap \mathbf{Stab}(0, 100M)$ occurs. On the event $G_\varepsilon(0, 10M)$, the localized Boolean model

$$\hat{\Sigma}_\varepsilon^{\text{loc}}(0, 10M) = \bigcup_{z\in\chi_\varepsilon\cap\mathbb{B}(0,100M)} \mathbb{B}\Big(z, R^{\text{loc}}_{T,0,10M}(z;\varepsilon)\Big)$$

contains a cluster $\mathcal{C}$ joining $\mathbb{B}(0, 10M)$ to $\mathbb{B}(0, 80M)^c$. The balls of $\hat{\Sigma}_\varepsilon^{\text{loc}}(0, 10M)$ are centered at vertices z in $\mathbb{B}(0, 100M)$ and their radii $R^{\text{loc}}_{T,0,10M}(z;\varepsilon)$ are by definition relative to the aggregate $A_T^\dagger[\mathbb{B}(0, 200M)]$. Since $A_T^\dagger[\mathbb{B}(0, 200M)] \subset A_T^\dagger[\infty]$, we have that, for any $z \in \chi_\varepsilon \cap \mathbb{B}(0, 100M)$, $R^{\text{loc}}_{T,0,10M}(z;\varepsilon) \leq R_T(z;\varepsilon) \leq M$ on the event $F(M)$. This means that all the balls of $\mathcal{C}$ have radii smaller than M. We can then extract from $\mathcal{C}$ a sub-cluster, say $\mathcal{C}'$, joining $\mathbb{B}(Mc, M)$ for some (random) $c \in \mathbb{S}_{10}$ to $\mathbb{B}(Mc, 8M)^c$. Indeed, the sphere $\mathbb{S}_{10M}$ is covered by the union of balls $\mathbb{B}(Mc, M)$, with $c \in \mathbb{S}_{10}$. Now, we have to prove that $\mathcal{C}'$ is also a cluster of $\hat{\Sigma}_\varepsilon^{\text{loc}}(Mc, M)$, ensuring the occurrence of $G_\varepsilon(Mc, M)$. Hence, we have to prove that each ball $\mathbb{B}(z, R^{\text{loc}}_{T,0,10M}(z;\varepsilon))$ involved in $\mathcal{C}'$ satisfies the two following properties:

- Its center z belongs to $\chi_\varepsilon \cap \mathbb{B}(Mc, 10M)$. This is clear since, by construction, each ball of $\mathcal{C}'$ overlaps $\mathbb{B}(Mc, 8M)$ and has a radius smaller than M.
- Its radius $R^{\text{loc}}_{T,0,10M}(z;\varepsilon)$ is actually equal to $R^{\text{loc}}_{T,Mc,M}(z;\varepsilon)$. This is where the event $\mathbf{Stab}(0, 100M)$ comes into play. The previous item implies that the ball $\mathbb{B}(z, R^{\text{loc}}_{T,0,10M}(z;\varepsilon))$ is completely included in $\mathbb{B}(Mc, 10M)$. Its radius is defined w.r.t. the aggregate $A_T^\dagger[\mathbb{B}(0, 200M)]$ (see (4.1)), but only through

  $$A_T^\dagger[\mathbb{B}(0, 200M)] \cap \Big(\mathbb{Z} \times \mathbb{B}(Mc, 10M)\Big)$$

  since $\mathbb{B}(z, R^{\text{loc}}_{T,0,10M}(z;\varepsilon)) \subset \mathbb{B}(Mc, 10M)$. Thanks to $\mathbf{Stab}(0, 100M)$, in particular $\mathbf{Stab}^M(cM)$ applied with $S := \mathbb{B}(0, 200M) \supset \mathbb{B}(Mc, 20M)$, we have

  $$A_T^\dagger[\mathbb{B}(0, 200M)] \cap \Big(\mathbb{Z} \times \mathbb{B}(Mc, 10M)\Big) = A_T^\dagger[\mathbb{B}(Mc, 20M)] \cap \Big(\mathbb{Z} \times \mathbb{B}(Mc, 10M)\Big)\ . \quad (4.7)$$

  Since the radius $R^{\text{loc}}_{T,Mc,M}(z;\varepsilon)$ is defined w.r.t. the aggregate $A_T^\dagger[\mathbb{B}(Mc, 20M)]$, the identity (4.7) implies that $R^{\text{loc}}_{T,Mc,M}(z;\varepsilon)$ and $R^{\text{loc}}_{T,0,10M}(z;\varepsilon)$ are equal.

Thus, we have proven that the event $G_\varepsilon(Mc, M)$ holds for some $c \in \mathbb{S}_{10}$. We can show in a similar fashion that $G_\varepsilon(Mc', M)$ also occurs for some $c' \in \mathbb{S}_{80}$. Inclusion (4.6) is established. □

Proof of Lemma 4.6. Let $M \geq 1$. Note that the occurrence of the event $G_\varepsilon(0, M)$ forces the random set $\chi_\varepsilon \cap \mathbb{B}(0, 10M)$ to be non-empty. Therefore,

$$\begin{aligned}
\mathbb{P}\left(G_\varepsilon(0, M)\right) &\leq \mathbb{P}\left(\#\left(\mathbb{B}(0, 10M) \cap \chi_\varepsilon\right) \geq 1\right) \\
&\leq \mathbb{E}\left[\#\left(\mathbb{B}(0, 10M) \cap \chi_\varepsilon\right)\right] \\
&= \sum_{z \in \mathbb{B}(0,10M)} \mathbb{E}\left[\mathbb{1}_{z \in \chi_\varepsilon}\right] \\
&= \#\mathbb{B}(0, 10M) p_\varepsilon \ .
\end{aligned}$$

Using $p_\varepsilon = 1 - e^{-\varepsilon} \leq \varepsilon$ and $\#\mathbb{B}(0, 10M) \leq C_d M^{d-1}$, we get the desired result. □

5 Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we give the following lemma:

Lemma 5.1. *For any finite subset $S \subset \mathbb{Z}^d$,*

$$\mathbb{P}\left(S \subset A_n^\dagger[\infty]\right) \underset{n \to \infty}{\longrightarrow} 1.$$

Proof of Lemma 5.1. To do so, we will be using the Shape Theorem for standard IDLA, in the case where exactly n particles are sent from the origin. Let us denote this aggregate by $A(n)$. We know from Theorem 1 of [19] that

$$\mathbb{P}\left(S \subset A(n)\right) \underset{n \to \infty}{\longrightarrow} 1. \tag{5.1}$$

Now, recall that particles of $A_n^\dagger[\infty]$ are given according to a family of PPP's in $\mathbb{R}_+$, denoted by $(\mathcal{N}_z)_{z \in \mathcal{H}}$. Let $N_0 = \mathcal{N}_0([0, n])$ denote the number of particles sent from the origin. Since N_0 is a Poisson random variable of parameter n, we know from concentration inequality theory that for all $0 < \varepsilon < 1/2$,

$$\mathbb{P}\left(N_0 - n \leq n^{1/2+\varepsilon}\right) \leq \exp\left(-\frac{n^{2\varepsilon}}{2}\right).$$

Hence, define $\mathscr{E}_n := \left\{N_0 \leq n - n^{1/2+\varepsilon}\right\}$. Let S denote a finite subset of $\mathbb{Z}^d$. We write:

$$\begin{aligned}
\mathbb{P}\left(S \subset A(n - n^{1/2+\varepsilon})\right) &\leq \mathbb{P}\left(\left\{S \subset A(n - n^{1/2+\varepsilon})\right\} \cap \mathscr{E}_n{}^c\right) + \mathbb{P}\left(\mathscr{E}_n\right) \\
&\leq \mathbb{P}\left(S \subset A_n^\dagger[0]\right) + \exp\left(-\frac{n^{2\varepsilon}}{2}\right) \\
&\leq \mathbb{P}\left(S \subset A_n^\dagger[\infty]\right) + \exp\left(-\frac{n^{2\varepsilon}}{2}\right).
\end{aligned}$$

Hence, for any finite subset $S \subset \mathbb{Z}^d$, we have that

$$\mathbb{P}\left(S \subset A_n^\dagger[\infty]\right) \geq \mathbb{P}\left(S \subset A(n - n^{1/2+\varepsilon})\right) - \exp\left(-\frac{n^{2\varepsilon}}{2}\right).$$

Using (5.1), we have the desired result. □

Proof of Theorem 1.2. We begin by showing 1. For any $n \geq 0$, $\mathcal{F}_n \subset \mathcal{F}_\infty$, so $V(\mathcal{F}_n) \subset V(\mathcal{F}_\infty)$. Now, consider S a finite subset of $\mathbb{Z}^d$. Since the vertex set of $\mathcal{F}_n$ is $A_n^\dagger[\infty]$, we have that for any $n \geq 0$,

$$\mathbb{P}(S \subset V(\mathcal{F}_\infty)) \geq \mathbb{P}(S \subset V(\mathcal{F}_n)) = \mathbb{P}\left(S \subset A_n^\dagger[\infty]\right).$$

This result is immediate using Lemma 5.1.

We move to the proof of 2. Consider a compact K of $\mathbb{R}^d$. Fix $k \in \mathcal{H}$ and $n \geq 0$. It is sufficient to show that $\mathcal{F}_n$ and $\mathcal{F}_\infty$ have the same probability of intersecting K. Assume that this holds for $\mathcal{F}_n$, that is:

$$\mathbb{P}(\mathcal{F}_n \cap K \neq \emptyset) = \mathbb{P}(T_k\mathcal{F}_n \cap K \neq \emptyset). \tag{5.2}$$

Note that for any subset C such that $(C + \mathbb{B}(0,1)) \cap \mathbb{Z}^d \subset A_n^\dagger[\infty]$, then $\mathcal{F}_n \cap C = \mathcal{F}_\infty \cap C$. Again using Lemma 5.1, we have that

$$\mathbb{P}\left((K + \mathbb{B}(0,1)) \cap \mathbb{Z}^d \subset A_n^\dagger[\infty]\right) \underset{n\to\infty}{\longrightarrow} 1.$$

Thus, there exists an integer n_0 such that $\mathbb{P}\left((K + \mathbb{B}(0,1)) \cap \mathbb{Z}^d \subset A_{n_0}^\dagger[\infty]\right) \geq 1 - \varepsilon$. We have:

$$\begin{aligned}|\mathbb{P}(\mathcal{F}_\infty \cap K \neq \emptyset) - \mathbb{P}(\mathcal{F}_{n_0} \cap K \neq \emptyset)| &\leq \mathbb{P}(\mathcal{F}_\infty \cap K \neq \mathcal{F}_{n_0} \cap K) \\ &\leq \mathbb{P}\left((K + \mathbb{B}(0,1)) \cap \mathbb{Z}^d \not\subset A_{n_0}^\dagger[\infty]\right) \leq \varepsilon. \end{aligned} \tag{5.3}$$

Similarly, we can show that

$$|\mathbb{P}(T_k\mathcal{F}_\infty \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_{n_0} \cap K \neq \emptyset)| \leq \varepsilon. \tag{5.4}$$

We can now conclude for $\mathcal{F}_\infty$, since

$$\begin{aligned}&|\mathbb{P}(\mathcal{F}_\infty \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_\infty \cap K \neq \emptyset)| \leq \\ &\qquad |\mathbb{P}(\mathcal{F}_\infty \cap K \neq \emptyset) - \mathbb{P}(\mathcal{F}_{n_0} \cap K \neq \emptyset)| + |\mathbb{P}(\mathcal{F}_{n_0} \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_{n_0} \cap K \neq \emptyset)| \\ &\qquad + |\mathbb{P}(T_k\mathcal{F}_{n_0} \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_\infty \cap K \neq \emptyset)|.\end{aligned}$$

Now, from (5.2), we know that the middle term is equal to 0, and from (5.3) and (5.4), we get that the first and third term are bounded by ε. Thus,

$$|\mathbb{P}(\mathcal{F}_\infty \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_\infty \cap K \neq \emptyset)| \leq 2\varepsilon.$$

It now remains to show (5.2). Fix $n \geq 0$. Take $M \geq 1$ sufficiently large such that $K \cap \mathbb{Z}^d \subset \mathbb{Z}_M$. From Theorem 1.1, there exists a random integer N_0 such that for any $N' \geq N_0$, with probability greater than $1 - \varepsilon$, we have

$$\mathcal{F}_n[N'] \cap \mathbb{Z}_M = \mathcal{F}_n \cap \mathbb{Z}_M, \qquad T_k\mathcal{F}_n[N'] \cap \mathbb{Z}_M = T_k\mathcal{F}_n \cap \mathbb{Z}_M. \tag{5.5}$$

Then, we have:

$$\begin{aligned}&|\mathbb{P}(\mathcal{F}_n \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_n \cap K \neq \emptyset)| \leq \\ &\ |\mathbb{P}(\mathcal{F}_n \cap K \neq \emptyset) - \mathbb{P}(\mathcal{F}_n[N_0] \cap K \neq \emptyset)| + |\mathbb{P}(\mathcal{F}_n[N_0] \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_n[N_0] \cap K \neq \emptyset)| + \\ &\ |\mathbb{P}(T_k\mathcal{F}_n[N_0] \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_n \cap K \neq \emptyset)| \\ &\leq |\mathbb{P}(\mathcal{F}_n[N_0] \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_n[N_0] \cap K \neq \emptyset)| + 2\varepsilon.\end{aligned}$$

Now, we grow the forests $\mathcal{F}_n[N_0]$ and $T_k\mathcal{F}_n[N_0]$ to obtain two forests $\mathfrak{F}_1$ and $\mathfrak{F}_2$. We obtain $\mathfrak{F}_1$ by sending the particles used to build $\mathcal{F}_n[N_0]$, and add the additional particles from $(T_k\mathcal{H}_{N_0}) \cap$

${\mathcal{H}_{N_0}}^c$. To build $\mathfrak{F}_2$, consider the forest induced by particles used to build $\mathcal{F}_n[N_0]$ and additional particles from $(T_{-k}\mathcal{H}_{N_0}) \cap {\mathcal{H}_{N_0}}^c$. Let $\mathfrak{F}$ denote this forest. We define $\mathfrak{F}_2$ as the translation of vector k of this forest, that is $\mathfrak{F}_2 := T_k\mathfrak{F}$. Now, from (5.5), we know that $\mathfrak{F}_1$ and $\mathcal{F}_n[N_0]$ coincide on the strip $\mathbb{Z}_M$ (and hence on K) with probability greater than $1-\varepsilon$. The same is true for $\mathfrak{F}_2$ and $T_k\mathcal{F}_n[N_0]$. Therefore, we have

$$|\mathbb{P}(\mathcal{F}_n \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_n \cap K \neq \emptyset)| \leq |\mathbb{P}(\mathfrak{F}_1 \cap K \neq \emptyset) - \mathbb{P}(\mathfrak{F}_2 \cap K \neq \emptyset)| + 4\varepsilon.$$

Lemma 5.2. *The set of sources used to build $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are identical.*

Now, the forest $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are built using the same IDLA protocol, using the same set of sources, with i.i.d Poisson clocks over the time interval $[0,n]$. They therefore have same distribution, which implies that

$$|\mathbb{P}(\mathcal{F}_n \cap K \neq \emptyset) - \mathbb{P}(T_k\mathcal{F}_n \cap K \neq \emptyset)| \leq 4\varepsilon,$$

which concludes the proof.
We now prove 3. We show that for any compacts $C_1,\ C_2$ of $\mathbb{R}^2$

$$\lim_{k\in\mathcal{H},\ \|k\|\to\infty} \mathbb{P}(\mathcal{F}_\infty \cap (C_1 \cup T_kC_2) = \emptyset) = \mathbb{P}(\mathcal{F}_\infty \cap C_1 = \emptyset)\,\mathbb{P}(\mathcal{F}_\infty \cap C_2 = \emptyset). \tag{5.6}$$

Fix $\varepsilon > 0$ and let $C_1,\ C_2$ be two compact sets of $\mathbb{R}^2$. Let $r > 0$ be such that $C_1 \cup C_2$ is included in $\mathbb{B}(0, r-1)$. From Lemma 5.1, we can pick n large enough such that

$$\mathbb{P}\left(\mathbb{B}(0,r) \cap \mathbb{Z}^2 \subset A_n^\dagger[\infty]\right) \geq 1-\varepsilon.$$

On the event $\left\{\mathbb{B}(0,r) \cap \mathbb{Z}^2 \subset A_n^\dagger[\infty]\right\}$, $\mathcal{F}_\infty \cap C_i$ and $\mathcal{F}_n \cap C_i$ are equal for any $i \in \{1,2\}$. Since the distribution of $A_n^\dagger[\infty]$ is invariant with respect to T_k, we have that for any $k \in \mathcal{H}$ (independent of ε),

$$\mathbb{P}\left(T_k(C_1 \cup C_2) \subset A_n^\dagger[\infty]\right) \geq 1-\varepsilon.$$

This implies:

$$\mathbb{P}(\mathcal{F}_\infty \cap (C_1 \cup T_kC_2) = \mathcal{F}_n \cap (C_1 \cup T_kC_2)) \geq 1-2\varepsilon.$$

Therefore,

$$\begin{aligned}&|\mathbb{P}(\mathcal{F}_\infty \cap (C_1 \cup T_kC_2) = \emptyset) - \mathbb{P}(\mathcal{F}_\infty \cap C_1 = \emptyset)\,\mathbb{P}(\mathcal{F}_\infty \cap C_2 = \emptyset)| \\ &\qquad \leq |\mathbb{P}(\mathcal{F}_n \cap (C_1 \cup T_kC_2) = \emptyset) - \mathbb{P}(\mathcal{F}_n \cap C_1 = \emptyset)\,\mathbb{P}(\mathcal{F}_n \cap C_2 = \emptyset)| + 4\varepsilon.\end{aligned}$$

It remains to show that $\mathcal{F}_n$ is mixing with respect to T_k. From Theorem 1.1, there exists N_0 such that

$$\mathbb{P}(\mathcal{F}_n \cap C_1 \neq \mathcal{F}_n[N_0] \cap C_1) \leq \varepsilon \ \text{ and } \ \mathbb{P}(\mathcal{F}_n \cap C_2 \neq \mathcal{F}_n[N_0] \cap C_2) \leq \varepsilon. \tag{5.7}$$

We have, by shifting all the random clocks by a vector $-k \in \mathcal{H}$:

$$\begin{aligned}\mathbb{P}(\mathcal{F}_n \cap C_2 \neq \mathcal{F}_n[N_0] \cap C_2) &= \mathbb{P}(\mathcal{F}_n(\omega) \cap C_2 \neq \mathcal{F}_n[N_0](\omega) \cap C_2) \\ &= \mathbb{P}(\mathcal{F}_n(\omega - k) \cap C_2 \neq \mathcal{F}_n[N_0](\omega - k) \cap C_2) \\ &= \mathbb{P}(\mathcal{F}_n(\omega) \cap T_kC_2 \neq \mathcal{F}_n[\mathbb{B}(k, N_0)](\omega) \cap T_kC_2).\end{aligned}$$

Hence, from (5.7), for any $k \in \mathcal{H}$ (independent of ε), there exists N_0 such that

$$\mathbb{P}(\mathcal{F}_n(\omega) \cap T_kC_2 \neq \mathcal{F}_n[\mathbb{B}(k, N_0)](\omega) \cap T_kC_2) \leq \varepsilon. \tag{5.8}$$

Now, we write:

$$
\begin{aligned}
&|\mathbb{P}\left(\mathcal{F}_n \cap (C_1 \cup T_k C_2) = \emptyset\right) - \mathbb{P}\left(\mathcal{F}_n \cap C_1 = \emptyset\right) \mathbb{P}\left(\mathcal{F}_n \cap C_2 = \emptyset\right)| \\
&= |\mathbb{P}\left(\{\mathcal{F}_n \cap C_1 = \emptyset\} \cap \{\mathcal{F}_n \cap T_k C_2 = \emptyset\}\right) - \mathbb{P}\left(\mathcal{F}_n \cap C_1 = \emptyset\right) \mathbb{P}\left(\mathcal{F}_n \cap C_2 = \emptyset\right)| \\
&\leq |\mathbb{P}\left(\{\mathcal{F}_n \cap C_1 = \emptyset\} \cap \{\mathcal{F}_n \cap T_k C_2 = \emptyset\}\right) - \mathbb{P}\left(\{\mathcal{F}_n[N_0] \cap C_1 = \emptyset\} \cap \{\mathcal{F}_n[\mathbb{B}(k, N_0)] \cap T_k C_2 = \emptyset\}\right)| \\
&+ |\mathbb{P}\left(\{\mathcal{F}_n[N_0] \cap C_1 = \emptyset\} \cap \{\mathcal{F}_n[\mathbb{B}(k, N_0)] \cap T_k C_2 = \emptyset\}\right) - \mathbb{P}\left(\mathcal{F}_n \cap C_1 = \emptyset\right) \mathbb{P}\left(\mathcal{F}_n \cap C_2 = \emptyset\right)|.
\end{aligned}
$$

Lemma 5.3. *Let A, A', B, and B' denote 4 events. Suppose $\mathbb{P}(A \neq A') \leq \varepsilon$ and $\mathbb{P}(B \neq B') \leq \varepsilon$. Then*

$$
|\mathbb{P}\left(\{A = \emptyset\} \cap \{B = \emptyset\}\right) - \mathbb{P}\left(\{A' = \emptyset\} \cap \{B' = \emptyset\}\right)| \leq 2\varepsilon.
$$

To alleviate notation, we define the following sets:

$$
\begin{cases} A = \mathcal{F}_n \cap C_1, \\ B = \mathcal{F}_n \cap T_k C_2, \end{cases} \quad \text{and} \quad \begin{cases} A' = \mathcal{F}_n[N_0] \cap C_1, \\ B' = \mathcal{F}_n[\mathbb{B}(k, N_0)] \cap T_k C_2. \end{cases}
$$

We can rewrite the previous inequality as:

$$
\begin{aligned}
|\mathbb{P}\left(\mathcal{F}_n \cap (C_1 \cup T_k C_2) = \emptyset\right) - \mathbb{P}\left(\mathcal{F}_n \cap C_1 = \emptyset\right) \mathbb{P}\left(\mathcal{F}_n \cap C_2 = \emptyset\right)| \\
\leq |\mathbb{P}\left(\{A = \emptyset\} \cap \{B = \emptyset\}\right) - \mathbb{P}\left(\{A' = \emptyset\} \cap \{B' = \emptyset\}\right)| \\
+ |\mathbb{P}\left(\{A' = \emptyset\} \cap \{B' = \emptyset\}\right) - \mathbb{P}\left(A = \emptyset\right) \mathbb{P}\left(B = \emptyset\right)|.
\end{aligned}
$$

Note that for the last term, we replaced $\mathbb{P}(\mathcal{F}_n \cap C_2 = \emptyset)$ by $\mathbb{P}(B = \emptyset)$. We can do so since the distribution of $\mathcal{F}_n$ is invariant with respect to T_k, so $\mathcal{F}_n$ has same probability of intersecting C_2 or $T_k C_2$.
Notice that for any k such that $\|k\| > 2N_0$, the events A' and B' are independent, since the two forests considered are built using disjoint sets of sources. Additionally, from (5.3) and (5.4), we have that $\mathbb{P}(A \neq A') \leq \varepsilon$ and $\mathbb{P}(B \neq B') \leq \varepsilon$. Therefore, using the result of Lemma 5.3, we get, for any $\|k\| > 2N_0$:

$$
\begin{aligned}
|\mathbb{P}\left(\mathcal{F}_n \cap (C_1 \cup T_k C_2) = \emptyset\right) - \mathbb{P}\left(\mathcal{F}_n \cap C_1 = \emptyset\right) \mathbb{P}\left(\mathcal{F}_n \cap C_2 = \emptyset\right)| \\
\leq |\mathbb{P}\left(A' = \emptyset\right) \mathbb{P}\left(B' = \emptyset\right) - \mathbb{P}\left(A = \emptyset\right) \mathbb{P}\left(B = \emptyset\right)| + 2\varepsilon.
\end{aligned}
$$

Now, since $\mathbb{P}(A \neq A') \leq \varepsilon$ and $\mathbb{P}(B \neq B') \leq \varepsilon$, one can show (in the same spirit of Lemma 5.3) that

$$
|\mathbb{P}\left(A' = \emptyset\right) \mathbb{P}\left(B' = \emptyset\right) - \mathbb{P}\left(A = \emptyset\right) \mathbb{P}\left(B = \emptyset\right)| \leq 2\varepsilon.
$$

Therefore,

$$
|\mathbb{P}\left(\mathcal{F}_\infty \cap (C_1 \cup T_k C_2) = \emptyset\right) - \mathbb{P}\left(\mathcal{F}_\infty \cap C_1 = \emptyset\right) \mathbb{P}\left(\mathcal{F}_\infty \cap C_2 = \emptyset\right)| \leq 8\varepsilon,
$$

which concludes the proof. □

A Appendix: Proof of Proposition 2.2

Before proving Proposition 2.2, we must first introduce a new family of random aggregates. Let $n \geq 1$. Just like for $A_n^\dagger[\infty]$, we begin by building a family of *finite* random aggregates $(A_n^*[M])_{M \geq 0}$. Like $A_n^\dagger[\infty]$, the number of particles sent from each source z is random, given this time by a Poisson random variable N_z of parameter n, but unlike $A_n^\dagger[\infty]$, these are sent in a predetermined order. Let $(N_z)_{z\in\mathcal{H}}$ denote a family of i.i.d Poisson random variables of parameter n. When $M=0$, $A_n^*[0]$ is the aggregate obtained after launching N_0 particles from the origin. Given a realization of $A_n[M-1]$, we throw N_z particles from each source z of level M according to the lexicographical order. So $A_n^*[M]$ is defined as the aggregate produced by $A_n^*[M-1]$ and the new sites added by particles launched at level M. By construction, $(A_n^*[M])_{M\geq 0}$ is increasing with respect to inclusion, so we can define $A_n^*[\infty]$ as

$$A_n^*[\infty] :=\uparrow \bigcup_{M\geq 0} A_n^*[M] \quad \text{a.s.}$$

A consequence of the Abelian Property (see [9], p. 97) is that for all $M\geq 0$,

$$A_n^\dagger[M] \overset{\text{law}}{=} A_n^*[M]. \tag{A.1}$$

Since both families of aggregates $(A_n^*[M])_{M\geq 0}$ and $(A_n^\dagger[M])_{M\geq 0}$ are increasing, we deduce from (A.1) that $A_n^\dagger[\infty] \overset{\text{law}}{=} A_n^*[\infty]$.

The global upper bound provided by Proposition 2.2 is a refined version of Theorem 4.1 of [5]. Although our result is finer and deals with a random number of emitted particles, the strategy of the proof is essentially the same.

We will be proving Proposition 2.2 by induction over n. Since $A_n^\dagger[\infty] \overset{\text{law}}{=} A_n^*[\infty]$, we will show the result for $A_n^*[\infty]$ instead, since this aggregate is built by sending particles in the *usual order*. We define the event $\mathbf{Over}_\alpha^*$ in the same way as $\mathbf{Over}_\alpha^\dagger$ in (2.1) but with respect to the aggregate $A_n^*[\infty]$. We show that if for some fixed n, the aggregate $A_n^*[\infty]$ is contained within $\mathscr{C}_\varepsilon^\alpha$ for some $\varepsilon>0$, and if we launch N_z' additional particles from each source z of $\mathcal{H}$, where $(N_z')_{z\in\mathcal{H}}$ is an independent family of Poisson variables of parameter 1, then the resulting aggregate is contained within a slightly larger cone $\mathscr{C}_{\varepsilon'}^\alpha$ ($\varepsilon'>\varepsilon$) with high probability. We keep the same notations as in the proof of Theorem 4.1 of [5] and define the sequences $(M_n)_{n\geq 0}$ and $(\varepsilon_n)_{n\geq 0}$ in the same way. Just like for Theorem 4.1 of [5], it is sufficient to prove the following proposition:

Proposition A.1. *For all $\alpha\in(1-1/d,1)$, for all $\varepsilon>0$, for all $n\geq 1$, there exists a constant $C=C(\varepsilon,n,\alpha,d)>0$ such that for all $M\geq 1$ and all $L>1$,*

$$\mathbb{P}\left(\mathbf{Over}_\alpha^*(M_n,\varepsilon_n)\right) \leq \frac{C}{M^L}.$$

Proof of Proposition A.1: We show our result by induction over n. Take $L>1$. Our induction statement is the following:
$\forall n\geq 0,\ \mathcal{P}(n): \forall\alpha\in(1-1/d,1),\ \forall\varepsilon\in(0,1),\ \forall M\geq 1,\ \exists C=C(\varepsilon,n,\alpha,d)>0,$

$$\mathbb{P}\left(\mathbf{Over}_\alpha^*(M_n,\varepsilon_n)\right) \leq \frac{C}{M^L}.$$

When $n=0$, this is clear since $A_n^*[\infty] \overset{\text{a.s}}{=} \emptyset$, hence $A_n^*[\infty]\cap\mathbb{Z}_M^c \overset{\text{a.s}}{\subset} \mathscr{C}_\varepsilon^\alpha$.
Let $n\geq 1$ and suppose $\mathcal{P}(n)$ holds. Fix $\alpha\in(1-1/d,1)$. We write:

$$\mathbb{P}\left(\mathbf{Over}_\alpha^*(M_{n+1},\varepsilon_{n+1})\right) \leq \mathbb{P}\left(\mathbf{Over}_\alpha^*(M_{n+1},\varepsilon_{n+1})\cap\mathbf{Over}_\alpha^*(M_n,\varepsilon_n)^c\right) + \mathbb{P}\left(\mathbf{Over}_\alpha^*(M_n,\varepsilon_n)\right).$$

The last term is handled by our induction hypothesis. We switch our focus to the first term. On the event $\mathbf{Over}_\alpha^*(M_{n+1},\varepsilon_{n+1})\cap\mathbf{Over}_\alpha^*(M_n,\varepsilon_n)^c$, we have $A_n^*[\infty]\cap\mathbb{Z}_{M_n}^c\subset\mathscr{C}_{\varepsilon_n}^\alpha$, but when launching N_z' new particles from each source $z\in\mathcal{H}$, the new aggregate obtained spills over $\mathscr{C}_{\varepsilon_{n+1}}^\alpha$ on $\mathbb{Z}_{M_{n+1}}^c$. This implies the existence of three random sites $(Z,Z^*,Z_{n+1})\in\mathbb{Z}^d$, and aggregates

A_{Z^*} and $A^-_{Z^*}$, defined just like in [5], but defined with respect to $\mathscr{C}^\alpha_\varepsilon$ rather than $\mathscr{C}_\varepsilon$. Note now that we have : $Z_{n+1} = Z \pm (\varepsilon_{n+1}\|Z\|^\alpha)\cdot e_1$, where $e_1 = (1,0,\dots,0)$.

We must control that no unreasonable amount of particles is emitted from $\mathcal{H}$. To do so, we introduce the following event:

$$\mathscr{E}_M(\gamma) := \bigcap_{l\geq 0} \{\forall z \in \mathcal{H},\ \|z\| = l,\ N'_z \leq 1 + \max\{l, M\}^\gamma\},$$

We can show using (2.5) that $\mathbb{P}(\mathscr{E}^c_M(\gamma))$ decreases faster than any power of M^{-1}. Fix $\gamma \in (0,1)$ such that $\gamma < (\alpha-1)d+1$ (such a value exists since $\alpha \in (1-1/d, 1)$). We explain this choice later. We write

$$\begin{aligned}
&\mathbb{P}\left(\mathbf{Over}^*_\alpha(M_{n+1},\varepsilon_{n+1}) \cap \mathbf{Over}^*_\alpha(M_n,\varepsilon_n)^c\right)\\
&\quad\leq \mathbb{P}\left(\mathbf{Over}^*_\alpha(M_{n+1},\varepsilon_{n+1}) \cap \mathbf{Over}^*_\alpha(M_n,\varepsilon_n)^c \cap \mathscr{E}_M(\gamma)\right) + \mathbb{P}\left(\mathscr{E}^c_M(\gamma)\right)\\
&\quad\leq \sum_{l\geq M_{n+1}} \sum_{\|z\|=l} \mathbb{P}\left(Z = z,\ \mathbf{Over}^*_\alpha(M_{n+1},\varepsilon_{n+1}) \cap \mathbf{Over}^*_\alpha(M_n,\varepsilon_n)^c \cap \mathscr{E}_M(\gamma)\right) + \mathcal{O}(1).
\end{aligned}$$

Fix $l \geq M_{n+1}$ and $z \in \mathcal{H}$ such that $\|z\| = l$, and let $z_{n+1} = z \pm (\varepsilon_{n+1}\|z\|^\alpha)\cdot e_1$. We consider the case where a ball of particles has settled around z_{n+1}, and the case where a thin tentacle reaches out to z_{n+1}. We use an adaptation of Lemma 2 of [15] to deal with the event of tentacles.

Lemma A.2. *There exist positive universal constants $b,\ K_0,\ c$ such that for all real numbers $r > 0$ and all $z \in \mathcal{H}$ with $0 \notin \mathbb{B}(z_{n+1}, r)$,*

$$\mathbb{P}\Big(Z = z,\ \mathbf{Over}^*_\alpha(M_{n+1},\varepsilon_{n+1}) \cap \mathbf{Over}^*_\alpha(M_n,\varepsilon_n)^c,\ \#\left(A_{Z^*} \cap \mathbb{B}(z_{n+1}, r)\right) \leq br^d\Big) \leq K_0 e^{-cr^2}.$$

We apply this Lemma with $r = r_{n+1} = \frac{\varepsilon l^\alpha}{2^{n+1}}$, in the same spirit as in [5]. This gives:

$$\begin{aligned}
&\mathbb{P}\left(Z = z,\ \mathbf{Over}^*_\alpha(M_{n+1},\varepsilon_{n+1}) \cap \mathbf{Over}^*_\alpha(M_n,\varepsilon_n)^c \cap \mathscr{E}_M(\gamma)\right)\\
\leq&\mathbb{P}\Big(Z = z,\ \mathbf{Over}^*_\alpha(M_{n+1},\varepsilon_{n+1}) \cap \mathbf{Over}^*_\alpha(M_n,\varepsilon_n)^c,\\
&\qquad \mathscr{E}_M(\gamma),\ \#\left(A_{Z^*} \cap \mathbb{B}(z_{n+1}, r_{n+1})\right) > br^d_{n+1}\Big) + K_0 e^{-c_1 l^{2\alpha}},
\end{aligned}\tag{A.2}$$

where $c_1 = c_1(n,\varepsilon) = \frac{c\varepsilon^2}{4^{n+1}}$. The term (A.2) requires a little more work than in the deterministic global upper bound. Note that r^d_{n+1} is of order $l^{\alpha d}$. We are working on the event where (roughly) more than $l^{\alpha d}$ particles have settled around z_{n+1}, knowing that each source $\tilde{z} \in \mathcal{H}$ has emitted at most $1 + \|\tilde{z}\|^\gamma$ particles. We show that this implies that $\|Z - Z^*\| \geq Kl^\eta$, where $\eta > 1$ and K denotes some positive constant. Let us now explain why $\eta > 1$. Suppose, contrary to our claim, that $\eta \leq 1$. The number of sources inside $\mathbb{B}(z, Kl^\eta) \cap \mathcal{H}$ is of the order $l^{\eta(d-1)}$ since $\mathcal{H}$ is a hyperplane of dimension $d-1$. Working on the event $\mathscr{E}_M$, the largest amount of particles emitted by a single source within this $(d-1)$-dimensional ball is $1 + (l + Kl^\eta)^\gamma$, which is of the order l^γ when $\eta \leq 1$. In the worst case scenario, if all of the sources within $\mathbb{B}(z, Kl^\eta) \cap \mathcal{H}$ emit of the order of l^γ particles, the total number of particles emitted will be of the order of $l^{\eta(d-1)} \times l^\gamma = l^{\eta(d-1)+\gamma}$. Now, for this to be of the order (or greater) than $l^{\alpha d}$, it is thus necessary that

$$\eta(d-1) + \gamma \geq \alpha d \Longleftrightarrow \eta \geq \frac{\alpha d - \gamma}{d-1}.$$

However, due to our choice of γ, this necessarily implies that $\eta > 1$, which contradicts our assumption that $\eta \leq 1$. Therefore, in order for *more* than $l^{\alpha d}$ particles to settle inside $\mathbb{B}(z_{n+1}, r_{n+1})$, it is necessary that one of these particles has been emitted from a source outside of $\mathbb{B}(z, Kl^\eta) \cap \mathcal{H}$, with $\eta > 1$. It is thus necessary that $\|Z - Z^*\| \geq Kl^\eta$. Since Z^* is defined as the source from which the *first* overflowing particle is emitted, this implies that the aggregate before sending from Z^* is strictly contained within $\mathscr{C}_{n+1}^{\alpha}$, allowing us to use a donut argument (similar to the one used in the proof of Lemma 4.4) to control the trajectory of the overflowing particle. This implies that one of the particles sent from z', with $\|z'-z\| \geq Kl^{\eta}$, has crossed multiple donuts. We detail below how we compute a lower bound on the total number of donuts a particle needs to cross from z' to z. Fix $h \geq Kl^{\eta}$ and z' such that $\|z'-z\| = h$. We build donuts from level $\|z'\|$ to $\|z\| = l$. The first donut, which is the *largest* donut, has dimensions at most $2\varepsilon_{n+1}(l+h)^{\alpha} \leq 2\varepsilon_{n+1}(2h)^{\alpha} \leq 4\varepsilon_{n+1}h^{\alpha}$. All other donuts will have smaller dimensions, which implies that the number of donuts $k = k(h, l, \varepsilon_{n+1}, \alpha)$ between z' and z is such that

$$k \geq \frac{h}{4\varepsilon_{n+1}h^{\alpha}} = \frac{h^{1-\alpha}}{4\varepsilon_{n+1}}.$$

Now, the number of particles sent from z' at distance h of z is (up to a multiplicative factor) at most $h^{d-2+\gamma}$. Therefore, the donut argument gives:

$$\begin{aligned}&\sum_{h\geq Kl^{\eta}} \sum_{\|z'-z\|=h} \mathbb{P}\left(\left\{\begin{array}{c}\text{a particle sent from } z' \text{ reaches level } l \\ \text{while staying within } \mathscr{C}_{\varepsilon_{n+1}}^{\alpha}\end{array}\right\} \cap \mathscr{E}_M(\gamma)\right) \\ \leq & K_d \sum_{h\geq Kl^{\eta}} h^{d-2+\gamma}(1-c)^k = K_d \sum_{h\geq K_d l^{\eta}} h^{d-2+\gamma} \exp\left(-c_0\left(\varepsilon_{n+1}\right) h^{1-\alpha}\right),\end{aligned}$$

where $c_0\left(\varepsilon_{n+1}\right) = -\frac{\log(1-c)}{4\varepsilon_{n+1}}$. Throughout the rest of the proof, K_d will denote a generic constant depending only on d, whose value may vary from line to line. Now, using the fact that $1-\alpha > 0$, standard computations yield:

$$K_d \sum_{h\geq Kl^{\eta}} h^{d-2+\gamma} \exp\left(-c_0\left(\varepsilon_{n+1}\right) h^{1-\alpha}\right) \leq K_d \exp\left(-\frac{1}{2}c_0(\varepsilon_{n+1}) l^{\eta(1-\alpha)}\right).$$

Combining this result with (A.2), we get

$$\begin{aligned}&\mathbb{P}\left(\mathbf{Over}_{\alpha}^{*}(M_{n+1}, \varepsilon_{n+1}) \cap \mathbf{Over}_{\alpha}^{*}(M_n, \varepsilon_n)^c \cap \mathscr{E}_M(\gamma)\right) \\ &\qquad \leq \sum_{l\geq M_{n+1}} \sum_{\|z\|=l} K_0 e^{-c_1 l^{2\alpha}} + \sum_{l\geq M_{n+1}} \sum_{\|z\|=l} K_d \exp\left(-\frac{1}{2}c_0(\varepsilon_{n+1}) l^{\eta(1-\alpha)}\right) \\ &\qquad \leq K_d \sum_{l\geq M_{n+1}} l^{d-2} e^{-c_1 l^{2\alpha}} + K_d \sum_{l\geq M_{n+1}} l^{d-2} \exp\left(-\frac{1}{2}c_0(\varepsilon_{n+1}) l^{\eta(1-\alpha)}\right) \\ &\qquad \leq K_d \exp\left(-\frac{c_1 M_{n+1}^{2\alpha}}{2}\right) + K_d \exp\left(-\frac{1}{4}c_0(\varepsilon_{n+1}) M_{n+1}^{\eta(1-\alpha)}\right).\end{aligned}$$

Since $M \leq M_{n+1}$, it is clear that both of these terms can be bounded by $\frac{C'}{M^L}$ for some constant $C' = C(\varepsilon, n, \alpha, d) > 0$. □

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