Central limit theorems for squared increment sums of fractional Brownian fields based on a Delaunay triangulation in $2D$

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Abstract

An isotropic fractional Brownian field (with Hurst parameter $H < 1/2$) is observed in a family of points in the unit square $\mathbf{C} = (-1/2, 1/2]^2$. These points are assumed to come from a realization of a homogeneous Poisson point process with intensity N. We consider normalized increments (resp. pairs of increments) along the edges of the Delaunay triangulation generated by the Poisson point process (resp. pairs of edges within triangles). Central limit theorems are established for the respective centered squared increment sums as $N \to \infty$.

Keywords: Isotropic fractional Brownian fields, Squared increment sums, Poisson point process, Delaunay triangulation.

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1 Introduction

Fractional Brownian motion is a self-similar stochastic process that sharply contrasts with traditional Brownian motion, semimartingales and Markov processes. As a centered Gaussian process, it is distinguished by the stationarity of its increments and exhibits a medium- or long-memory property. Fractional Brownian motion has gained popularity in applications where classical models fail to capture these complex characteristics; for example, long memory, also referred to as persistence, is crucial in analyzing financial data and internet traffic (see e.g. [12]).

There is no unique way to extend it to self-similar random fields parametrized by d-dimensional spaces ($d > 1$), because there exist at least two ways to define increments for random fields (see e.g. Section 3.3 in [6]). In the following, we consider the case $d = 2$. A first natural extension of the stationarity of

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the increments of a random field $W := (W(x))_{x\in\mathbf{R}^2}$ is to say that W has *linear stationary increments* if the law of

$$(W(x+x_0) - W(x_0))_{x\in\mathbf{R}^2}$$

does not depend on the choice of $x_0 \in \mathbf{R}^2$. An example of a self-similar random field with linear stationary increments is the *isotropic fractional Brownian field* defined as the centered Gaussian random field such that $W(0) = 0$ a.s. and

$$\mathrm{cov}(W(x), W(y)) = \frac{\sigma^2}{2}\left(||x||^{2H} + ||y||^{2H} - ||y-x||^{2H}\right), \tag{1.1}$$

for some $H \in (0,1)$ and $\sigma^2 > 0$, with $\|x\|$ the Euclidean norm of $x \in \mathbf{R}^2$. The parameter σ is called the *scale parameter* while H is known as the *Hurst parameter* and relates to the Hölder continuity exponent of W. *Rectangular increments* for a 2-dimensional field W are defined as

$$W(x+x_0) - W(x_1e_1 + x_0) - W(x_2e_2 + x_0) + W(x_0), \tag{1.2}$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, $x_0 \in \mathbf{R}^2$, $e_1 = (1,0)$ and $e_2 = (0,1)$. A second extension of the stationarity of the increments of W is to say that W has *rectangular stationary increments* if the law of the process of the rectangular increments (1.2) does not depend on the choice of $x_0 \in \mathbf{R}^2$. An example of a self-similar random field with rectangular stationary increments is the *isotropic fractional Brownian sheet* defined as the centered Gaussian random field such that $W(x) = 0$ for any x in $\{(x_1, x_2) \in \mathbf{R}^2, x_1 = 0 \text{ or } x_2 = 0\}$ and

$$\mathrm{cov}(W(x), W(y)) = \frac{\sigma^2}{2}\left(|x_1|^{2H} + |y_1|^{2H} - |y_1 - x_1|^{2H}\right)\left(|x_2|^{2H} + |y_2|^{2H} - |y_2 - x_2|^{2H}\right)$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, for some $H \in (0,1)$ and $\sigma^2 > 0$.

Quadratic variations of stochastic processes (also known as squared increment sums) play an important role in both stochastic analysis (e.g., [17], pp. 66-77) and applications such as estimation of model parameters. For this reason the topic has been extensively studied in the literature, in particular for fractional Brownian motions (see e.g. [13, 14]) or general Gaussian sequences (see e.g. the survey [19]). Extensions of quadratic variations for random fields naturally depends on the retained definition of increments. Quadratic variations based on rectangular increments observed on a regular grid have been first considered. Strong laws of large numbers were studied e.g. in [8], while, more recently, [16] obtained functional limit theorems for generalized variations of the fractional Brownian sheet. Quadratic variations based on linearly filtered increments observed on a regular grid (generalizing rectangular increments) have been introduced in [20] in the case of fractional Brownian fields in order to build estimators of their fractal dimension. For irregularly spaced data, the literature is relatively sparse. According to [11], observations were assumed to be taken along a smooth curve in space (on a line transect), and second-order quadratic variations were introduced to estimate the smoothness of certain Gaussian random fields. But, to the best of our knowledge, there are no papers that study quadratic variations of random fields when they are observed on points distributed randomly in space.

In this paper we consider an isotropic fractional Brownian field W with covariance given in (1.1) and, independently of this, a Delaunay graph, denoted as $\mathrm{Del}(P_N)$, based on a homogeneous Poisson point process P_N with intensity N in $\mathbf{R}^2$ (see e.g. p. 478 in [18] and Section 2 of our paper for a definition and properties of the Poisson Delaunay graph). The originality of our article is twofold. On one hand, unlike classical literature, we assume that we observe W not on a regular grid but on the points of P_N that belongs to the unit square $\mathbf{C} = (-1/2, 1/2]^2$. On the other hand, the (normalized) increments (used for the quadratic variations) are based on edges (or pairs of edges) of the Delaunay graph. The main results of our paper provide central limit theorems (CLTs) for the centered squared increment sums as $N \to \infty$.

The reason we consider the Delaunay triangulation arises from a statistical problem, specifically inference for the parameters of a max-stable field, which is based on fractional Brownian fields. In [4], we construct composite maximum likelihood estimators based on pairs and triples to estimate the parameters of such a field. The pairs and triples are selected from the Delaunay triangulation. Choosing such a triangulation is natural because it is the most regular in the sense that its minimal angle is greater than the minimal angle of any other triangulation. The study of the asymptotic behavior of the estimators requires establishing central limit theorems for a single fractional Brownian field, which is the focus of this article. Assuming that the points from which we construct our increments are based on a point process rather than on a deterministic grid is, in a certain sense, natural. Indeed, in terms of inference, this means that our statistics are observed at random nodes. Moreover we chose the Poisson point process because it is more natural and has a formula (the Slivnyak-Mecke formula) that allows for the explicit calculation of expectations.

Squared normalized increment sums To state our main theorem, we first introduce some notation. Let us remind that W is a fractional Brownian field with covariance given in (1.1) and P_N is a Poisson point process, independent of W, with intensity N in $\mathbf{R}^2$. When $x_1, x_2 \in P_N$ are Delaunay neighbors, we write $x_1 \sim x_2$ in $\mathrm{Del}(P_N)$. We denote by E_N the set of couples (x_1, x_2) such that the following conditions hold:

$$x_1 \sim x_2 \text{ in } \mathrm{Del}(P_N), \quad x_1 \in \mathbf{C}, \quad \text{and} \quad x_1 \preceq x_2,$$

where $\preceq$ denotes the lexicographic order. Let DT_N be the set of triples (x_1, x_2, x_3) satisfying the following properties

$$\Delta(x_1, x_2, x_3) \in \mathrm{Del}(P_N), \quad x_1 \in \mathbf{C}, \quad \text{and} \quad x_1 \preceq x_2 \preceq x_3,$$

where $\Delta(x_1, x_2, x_3)$ is the convex hull of $\{x_1, x_2, x_3\}$. For any distinct points $x_1, x_2 \in \mathbf{R}^2$, we denote by

$$U^{(W)}_{x_1,x_2} = \sigma^{-1} d_{1,2}^{-H} \left(W(x_2) - W(x_1)\right)$$

the *normalized increment* between x_1 and x_2 with respect to (w.r.t.) W, with $d_{1,2} = \|x_2 - x_1\|$. The normalization $\sigma^{-1} d_{1,2}^{-H}$ has been chosen in such a way that $U^{(W)}_{x_1,x_2}$ follows a standard Gaussian distribution. We consider two types of squared increment sums: on the one hand, the one based on the edges of the Delaunay triangulation, defined as

$$V^{(W)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2) \in E_N} \left((U^{(W)}_{x_1,x_2})^2 - 1 \right),$$

where $|E_N|$ denotes the cardinal number of E_N, and on the other hand the one based on pairs of edges, defined as

$$V^{(W)}_{3,N} = \frac{1}{\sqrt{|DT_N|}} \times \sum_{(x_1,x_2,x_3) \in DT_N} \left(\begin{pmatrix} U^{(W)}_{x_1,x_2} & U^{(W)}_{x_1,x_3} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U^{(W)}_{x_1,x_2} \\ U^{(W)}_{x_1,x_3} \end{pmatrix} - 2 \right),$$

where

$$R_{x_1,x_2,x_3} = \operatorname{corr}(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_1,x_3}) = \frac{d_{1,2}^{2H} + d_{1,3}^{2H} - d_{2,3}^{2H}}{2\,(d_{1,2}d_{1,3})^{H}}, \tag{1.3}$$

with $d_{1,3} = \|x_3 - x_1\| > 0$ and $d_{2,3} = \|x_3 - x_2\| > 0$. Limiting increments in the definition of $V^{(W)}_{3,N}$ to pairs of edges (namely $[x_1, x_2]$ and $[x_1, x_3]$) rather than considering all edges of a Delaunay triangle (including the edge $[x_2, x_3]$) is sufficient because the third increment can be deduced from the other two. Notice that the term $V^{(W)}_{3,N}$ can be expressed as a sum of squared increments in the sense that

$$V^{(W)}_{3,N} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \left([(\tilde{U}^{(W)}_{x_1,x_2,x_3})^2 - 1] + [(\tilde{U}^{(W)}_{x_1,x_3})^2 - 1] \right),$$

where

$$\tilde{U}^{(W)}_{x_1,x_2,x_3} = (1 - R^2_{x_1,x_2,x_3})^{-1/2} \left(U^{(W)}_{x_1,x_2} - R_{x_1,x_2,x_3} U^{(W)}_{x_1,x_3} \right) \quad \text{and} \quad \tilde{U}^{(W)}_{x_1,x_3} = U^{(W)}_{x_1,x_3}.$$

According to [2], the quantity $\tilde{U}^{(W)}_{x_1,x_2,x_3}$ is a normalized increment based on the three points x_1, x_2, x_3 and satisfying

$$\operatorname{corr}(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_1,x_3}) = 0.$$

Main result Our main theorem states that the squared normalized increment sums satisfy central limit theorems.

Theorem 1 *Let W be a fractional Brownian random field with covariance given by (1.1), with $H \in (0, 1/2)$ and $\sigma^2 > 0$. There exist finite constants $\sigma^2_{V_2} > 0$ and $\sigma^2_{V_3} > 0$ such that, as $N \to \infty$,*

$$V^{(W)}_{2,N} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{V_2}\right) \quad \text{and} \quad V^{(W)}_{3,N} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{V_3}\right).$$

The rates of convergence of both sums are the same as in Theorem 3.2 of [2] or in Theorem 1 of [20] where statistics based on square increments on regular grids have been considered. The asymptotic variances appearing in the above theorem have explicit integral representations but are quite intricate.

Our result is stated within a fixed window and for a Poisson process with intensity tending to infinity. By rescaling, this is equivalent to considering an arbitrarily large window with a fixed intensity for the Poisson point process. Nonetheless, we chose to present our result within a fixed window, as it is in this context that we will establish our results in two companion papers [5] and [4]. The main ingredient to prove Theorem 1 is an extension of the Breuer-Major theorem due to Nourdin and Peccati (Theorem 7.2.4 in [15]). One of the challenges is to check the conditions of this theorem. Indeed, the latter are expressed in terms of chaos decomposition and requires delicate estimates for the variances and mixing properties for suitable marked point processes.

The weak convergences of $V^{(W)}_{2,N}$ and $V^{(W)}_{3,N}$ to Gaussian limit distributions in Theorem 1 can only be established for $H \in (0, 1/2)$. The main reason is that the increments in the sums are of order 0 in the sense of [2, 3]. In the univariate case, and in the context of regular grids, it was proved that under a suitable normalization the limit is a Rosenblatt's process [9] when $H > 3/4$. In dimension 2, it is well known that there is a regime change depending on whether H is less than or greater than $1/2$ (see e.g. Remark 3.2 in [2]). In this spirit, with an appropriately chosen renormalization, we think that the limit should also be a Rosenblatt distribution when $H > 1/2$ for the squared increment sums which are considered in our paper. Furthermore, considering increments of order 1 (necessarily based on four vertices) rather than of order 0 would lead to Gaussian limit distributions for any $H \in (0, 1)$. Such a result has already been established in the context of regular grids (see e.g. [1, 3]).

Our paper is organized as follows. In Section 2 we review some established concepts related to the Delaunay triangulation. Section 3 contains a detailed proof of Theorem 1 for the squared increment sum along the edges while Section 4 only presents a sketch of proof for the pairs of edges, since the arguments are similar. We conclude our paper with technical results which will be used to prove Theorem 1. Throughout the paper, for the sake of simplicity, we denote $\alpha := 2H$.

2 Preliminaries

In this section, we recall some known results on Poisson-Delaunay triangulations.

Let P_N be a Poisson point process with intensity N in $\mathbf{R}^2$. The *Delaunay triangulation* $\mathrm{Del}(P_N)$ is the unique triangulation with vertices in P_N such that the circumball of each triangle contains no point of P in its interior, see e.g. p. 478 in [18]. Such a triangulation is the most regular one in the sense that it is the one which maximises the minimum of the angles of the triangles.

To define the mean behavior of the Delaunay triangulation (associated with a Poisson point process P_1 of intensity 1), the notion of typical cell is defined as follows. With each cell $C \in \mathrm{Del}(P_1)$, we associate the circumcenter $z(C)$ of C. Now, let $\mathbf{B}$ be a Borel subset in $\mathbf{R}^2$ with area $a(\mathbf{B}) \in (0, \infty)$. The *cell intensity* β_2 of $\mathrm{Del}(P_1)$ is defined as the mean number of cells per unit area, i.e.

$$\beta_2 = \frac{1}{a(\mathbf{B})}\mathbb{E}\left[|\{C \in \mathrm{Del}(P_1) : z(C) \in \mathbf{B}\}|\right],$$

where we recall that $|\cdot|$ denotes the cardinality. It is well-known that $\beta_2 = 2$, see e.g. Theorem 10.2.9. in [18]. Then, we define the *typical cell* as a random triangle $\mathcal{C}$ with distribution given as follows: for any positive measurable and translation invariant function $g : \mathcal{K}_2 \to \mathbf{R}$,

$$\mathbb{E}\left[g(\mathcal{C})\right] = \frac{1}{\beta_2 a(\mathbf{B})}\mathbb{E}\left[\sum_{C\in \mathrm{Del}(P_1):z(C)\in\mathbf{B}} g(C)\right],$$

where $\mathcal{K}_2$ denotes the set of convex compact subsets in $\mathbf{R}^2$, endowed with the Fell topology (see Section 12.2 in [18] for the definition). The distribution of $\mathcal{C}$ has the following integral representation (see e.g. Theorem 10.4.4. in [18]):

$$\mathbb{E}\left[g(\mathcal{C})\right] = \frac{1}{6}\int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,u_3)) g(\Delta(ru_1,ru_2,ru_3))\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\sigma(\mathrm{d}u_3)\mathrm{d}r, \qquad (2.1)$$

where $\mathbf{S}^1$ is the unit sphere of $\mathbf{R}^2$ and σ is the spherical Lebesgue measure on $\mathbf{S}^1$ with normalization $\sigma\left(\mathbf{S}^1\right) = 2\pi$. In other words, $\mathcal{C}$ is equal in distribution to $R\Delta(U_1,U_2,U_3)$, where R and (U_1,U_2,U_3) are independent with probability density functions given respectively by $2\pi^2 r^3 e^{-\pi r^2}$ and $a(\Delta(u_1,u_2,u_3))/(12\pi^2)$.

In a similar way, we can define the notion of typical edge. The *edge intensity* β_1 of $\mathrm{Del}(P_1)$ is defined as the mean number of edges per unit area and is equal to $\beta_1 = 3$ (see e.g. Theorem 10.2.9. in [18]). The distribution of the length of the *typical edge* is the same as the distribution of $D = R||U_1 - U_2||$.

Its probability density function f_D satisfies the following equality

$$\begin{aligned}\mathbb{P}[D \leq \ell] &= \int_0^\ell f_D(d)\mathrm{d}d \\ &= \frac{\pi}{3}\int_0^\infty \int_{(\mathbf{S}^1)^2} r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,e_1))\mathbb{I}[r\|u_1-u_2\| \leq \ell]\,\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\mathrm{d}r, \end{aligned} \tag{2.2}$$

where $e_1 = (1,0)$ and $\ell > 0$. Following Eq. (2.1), a *typical couple* of (distinct) Delaunay edges with a common vertex can be defined as a 3-tuple of random variables (D_1, D_2, Θ), where $D_1, D_2 \geq 0$ and $\Theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, with distribution given by

$$\begin{aligned}\mathbb{P}[(D_1,D_2,\Theta) \in B] = \frac{1}{6}\int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,u_3)) \\ \times \mathbb{I}[(r||u_3-u_2||, r||u_2-u_1||, \arcsin(\cos(\theta_{u_1,u_2}/2))) \in B]\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\sigma(\mathrm{d}u_3)\mathrm{d}r,\end{aligned}$$

where θ_{u_1,u_2} is the measure of the angle (u_1,u_2) and where B is any Borel subset in $\mathbf{R}_+^2 \times [-\frac{\pi}{2}, \frac{\pi}{2})$. The random variables D_1, D_2 (resp. Θ) can be interpreted as the lengths of the two typical edges (resp. as the angle between the edges). In particular, the length of a typical edge is equal in distribution to $D = R||U_2 - U_1||$ with distribution given in Eq. (2.2).

Throughout the paper, we identify $\mathrm{Del}(P_1)$ to its skeleton. When $x_1, x_2 \in P_N$ are Delaunay neighbors, we write $x_1 \sim x_2$ in $\mathrm{Del}(P_N)$. For a Borel subset $\mathbf{B}$ in $\mathbf{R}^2$, we denote by $E_{N,\mathbf{B}}$ the set of couples (x_1, x_2) such that the following conditions hold:

$$x_1 \sim x_2 \text{ in } \mathrm{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2,$$

where $\preceq$ denotes the lexicographic order. In particular, when $\mathbf{B} = \mathbf{C} = (-1/2, 1/2]^2$, we have $E_{N,\mathbf{C}} = E_N$. For a Borel subset $\mathbf{B}$ in $\mathbf{R}^2$, let $DT_{N,\mathbf{B}}$ be the set of triples (x_1,x_2,x_3) satisfying the following properties

$$\Delta(x_1,x_2,x_3) \in \mathrm{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2 \preceq x_3,$$

where $\Delta(x_1,x_2,x_3)$ is the convex hull of (x_1,x_2,x_3). When $\mathbf{B} = \mathbf{C}$, we have $DT_{N,\mathbf{C}} = DT_N$.

3 Proof of Theorem 1 for $V_{2,N}^{(W)}$

Our aim is to prove that, when $\alpha \in (0,1)$, as $N \to \infty$, the statistic

$$V_{2,N}^{(W)} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} ((U_{x_1,x_2}^{(W)})^2 - 1)$$

converges in distribution to a Gaussian random variable with finite variance $\sigma_{V_2}^2$. Because $N^{1/2}P_N \overset{\mathcal{D}}{=} P_1$ and, by self-similarity property, $(W(x))_{x\in\mathbf{R}^2} \overset{\mathcal{D}}{=} \left(W(N^{-1/2}x)/N^{-\alpha/4}\right)_{x\in\mathbf{R}^2}$, the random variable $V_{2,N}^{(W)}$ has the same distribution as

$$V_{2,N}^{(W)\prime} = \frac{1}{\sqrt{|E_N'|}} \sum_{(x_1,x_2)\in E_N'} ((U_{x_1,x_2}^{(W)})^2 - 1),$$

where $E_N' = E_{1,\mathbf{C}_N}$ is the set of pairs (x_1,x_2) of the Delaunay triangulation associated with P_1, such that $x_1 \preceq x_2$ and $x_1 \in \mathbf{C}_N = (-N^{1/2}/2, N^{1/2}/2]^2$.

We now denote by $(\Omega_P, \mathcal{F}_P, \mathbb{P}_P)$ the probability space associated with the data-site generating process. The point process P_1 is defined on $(\Omega_P, \mathcal{F}_P, \mathbb{P}_P)$. We denote by ω_P an element of Ω_P. Let $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ be the probability space associated with the fractional Brownian random field W. We denote by ω_W an element of Ω_W. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is the product space $(\Omega_P \times \Omega_W, \mathcal{F}_P \otimes \mathcal{F}_W, \mathbb{P}_P \times \mathbb{P}_W)$ and we denote by ω an element of Ω. All the random variables in this proof can be defined relatively to the product space $(\Omega, \mathcal{F}, \mathbb{P})$. Hence, all the probabilistic statements hold w.r.t. this product space, unless it is stated otherwise.

We will first prove that, for almost all ω_P, the random variable $V_{2,N}^{(W)\prime}$ converges in distribution to a Gaussian random variable with variance $\sigma_{V_2}^2$ which does not depend on ω_P, i.e. $\mathbb{E}\left[\exp(\mathrm{i}uV_{2,N}^{(W)\prime}) \middle| P_1\right]$ converges for almost all ω_P to $\exp\left(-\sigma_{V_2}^2 u^2/2\right)$. Using the dominated convergence on $(\Omega_P, \mathcal{F}_P, \mathbb{P}_P)$, this will show that $\mathbb{E}\left[\exp(\mathrm{i}uV_{2,N}^{(W)\prime})\right]$ converges to $\exp\left(-\sigma_{V_2}^2 u^2/2\right)$.

Since, P_1 and $(W(x))_{x\in\mathbf{R}^2}$ are independent, the conditional distribution of the random variable $\left(U_{x_1,x_2}^{(W)}\right)_{(x_1,x_2)\in E_N'}$ given P_1 is still Gaussian. The main idea to deal with $V_{2,N}^{(W)\prime}$ is then to apply a modification of the Breuer-Major theorem (see e.g. Theorem 7.2.4 in [15]).

In what follows, $E_{1,\mathbf{R}^2}$ is the set of pairs (x_1, x_2) of the Delaunay triangulation associated with P_1, such that $x_1 \preceq x_2$. Since there is a countable number of such pairs, there exists a one-to-one function $\varphi : \mathbf{Z} \to E_{1,\mathbf{R}^2}$ such that, for any $(x_1, x_2) \in E_{1,\mathbf{R}^2}$, there exists a unique k such that $U_{\varphi(k)}^{(W)} = U_{x_1,x_2}^{(W)}$. The random variable $U_{\varphi(k)}^{(W)}$ is now denoted by $U^{(k)}$. Moreover we let $e_N' = \{k \in \mathbf{Z} : \varphi(k) \in E_N'\}$ and remark that $|e_N'| = |E_N'|$.

For ω_P an element of Ω_P and according to Proposition 7.2.3 in [15], there exists a real separable Hilbert space $\mathfrak{H}$, as well as an isonormal Gaussian process over $\mathfrak{H}$, written $\{X(h) : h \in \mathfrak{H}\}$, with the property that there exists a set $E = \{\varepsilon_k : k \in \mathbf{Z}\} \subset \mathfrak{H}$ such that (i) E generates $\mathfrak{H}$; (ii) $\langle \varepsilon_k, \varepsilon_l \rangle_{\mathfrak{H}} = \operatorname{corr}\left(U^{(k)}, U^{(l)}\right)$ for every $k, l \in \mathbf{Z}$; and (iii) $U^{(k)} = X(\varepsilon_k)$ for every $k \in \mathbf{Z}$.

Let

$$f_{N,2} = \frac{1}{\sqrt{|e_N'|}} \sum_{k\in e_N'} \varepsilon_k^{\otimes 2}$$

such that $V_{2,N}^{(W)\prime} = I_2\left(f_{N,2}\right)$, where I_2 is the second multiple integral of $f_{N,2}$. Our CLT is a consequence of a theorem due to Nourdin and Peccati (Theorem 6.3.1 of [15]). For sake of completeness, we rewrite their result below using the same notations (see p. 205-206 in [15]).

Theorem 2 *(Nourdin & Peccati) Let $(F_n)_{n\geq 1}$ be a sequence in $L^2(\Omega)$, the space of square-integrable random variables, such that $\mathbb{E}[F_n] = 0$ for all n. Consider the chaos decomposition*

$$F_n = \sum_{q=0}^{\infty} I_q(f_{n,q}) \quad \text{with } f_{n,q} \in \mathfrak{H}^{\odot q}, q \geq 1, n \geq 1.$$

Assume in addition that

(a) for every fixed $q \geq 1$, $q!||f_{n,q}||_{\mathfrak{H}^{\odot q}}^2 \to \sigma_q^2$ as $n \to \infty$ (for some $\sigma_q^2 \geq 0$);

(b) $\sigma^2 := \sum_{q=1}^{\infty} \sigma_q^2 < \infty$;

(c) for all $q \geq 2$ and $r = 1, \ldots, q-1$, $||f_{n,q} \otimes_r f_{n,q}||_{\mathfrak{H}^{\otimes 2q-2r}} \to 0$ as $n \to \infty$;

(d) $\lim_{N\to\infty} \sup_{n\geq 1} \sum_{q=N+1}^{\infty} q!||f_{n,q}||_{\mathfrak{H}^{\otimes q}}^2 = 0$.

Then $F_n \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$ as $n \to \infty$.

To apply Theorem 2, we proceed in the same way as in Section 7.2 of [15], but only for $q = 2$ and not for every $q \geq 1$ since our chaos decomposition has only one term: $I_2(f_{N,2})$. Note that, given $\omega_P \in \Omega_P$, the quantity $|e'_N|$ is a sequence that tends to ∞, since $\mathbb{P}[|e'_N| \to \infty | P_1] = 1$ for almost all ω_P.

- Condition *(a)* holds according to Proposition 4 in Section 3.1 and the fact that, for almost all ω_P,

$$2! \left\| f_{N,2} \right\|^2_{\mathfrak{H}^{\otimes 2}} = \mathbb{E}\left[V_{2,N}^{(W)\prime} | P_1\right] \underset{N\to\infty}{\rightarrow} \sigma^2_{V_2};$$

- Condition *(b)* holds according to Lemma 3 in Section 3.1;
- Condition *(c)* holds as proved in Section 3.2;
- Condition *(d)* holds since our chaos decomposition has only one term: $I_2(f_{N,2})$.

According to Theorem 2, this implies $V_{2,N}^{(W)\prime} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{V_2}\right)$ and therefore $V_{2,N}^{(W)} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2_{V_2}\right)$.

3.1 Checking Condition *(a)*

We begin by giving a decomposition of $\mathbb{E}[(V_{2,N}^{(W)\prime})^2 | P_1]$ that will be used to provide an integral representation of the asymptotic variance of $V_{2,N}^{(W)}$. We first notice that

$$\mathbb{E}\left[(V_{2,N}^{(W)\prime})^2 | P_1\right] = \frac{2}{|E'_N|} \sum_{(x_1,x_2)\in E'_N} \sum_{(x_3,x_4)\in E'_N} \left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2.$$

The term above can be decomposed as follows:

$$\mathbb{E}\left[(V_{2,N}^{(W)\prime})^2 | P_1\right] = \frac{2}{|E'_N|} \sum_{\substack{x_1\in\mathbf{C}_N \\ x_1\sim x_2, x_1\preceq x_2}} \sum_{\substack{x_3\sim x_4 \\ x_3\preceq x_4}} \left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 - \frac{2}{|E'_N|} \sum_{\substack{x_1\in\mathbf{C}_N \\ x_1\sim x_2, x_1\preceq x_2}} \sum_{\substack{x_3\notin\mathbf{C}_N \\ x_3\sim x_4, x_3\preceq x_4}} \left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2.$$

This gives

$$\mathbb{E}\left[(V_{2,N}^{(W)\prime})^2 | P_1\right] = 2 + \frac{2}{|E'_N|} \sum_{\substack{x_1\in\mathbf{C}_N \\ x_1\sim x_2, x_1\preceq x_2}} \sum_{\substack{x_3\sim x_4 \\ x_3\preceq x_4}} \left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \mathbb{I}\left[(x_1,x_2) \neq (x_3,x_4)\right] - \frac{2}{|E'_N|} \sum_{\substack{x_1\in\mathbf{C}_N \\ x_1\sim x_2, x_1\preceq x_2}} \sum_{\substack{x_3\notin\mathbf{C}_N \\ x_3\sim x_4, x_3\preceq x_4}} \left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2.$$

The first double sums appearing in the right-hand side can be written as:

$$\begin{aligned}
&\frac{2}{|E'_N|}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4}\right)\right)^2\mathbb{I}\left[(x_1,x_2)\neq(x_3,x_4)\right]\\
&\qquad=\frac{2}{|E'_N|}\sum_{\substack{x_1\in\mathbf{C}_N, x_1\sim x_2, x_1\sim x_4,\\ x_1\preceq x_2, x_1\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_1,x_4}\right)\right)^2\mathbb{I}\left[x_2\neq x_4\right]\\
&\qquad+\frac{2}{|E'_N|}\sum_{\substack{x_1\in\mathbf{C}_N, x_1\sim x_2, x_2\sim x_4,\\ x_1\preceq x_2, x_2\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_2,x_4}\right)\right)^2\mathbb{I}\left[x_1\neq x_4\right]\\
&\qquad+\frac{2}{|E'_N|}\sum_{\substack{x_1\in\mathbf{C}_N, x_1\sim x_2, x_3\sim x_1,\\ x_1\preceq x_2, x_3\preceq x_1}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_1}\right)\right)^2\mathbb{I}\left[x_2\neq x_3\right]\\
&\qquad+\frac{2}{|E'_N|}\sum_{\substack{x_1\in\mathbf{C}_N, x_1\sim x_2, x_3\sim x_2,\\ x_1\preceq x_2, x_3\preceq x_2}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_2}\right)\right)^2\mathbb{I}\left[x_1\neq x_3\right]\\
&\qquad+\frac{2}{|E'_N|}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4}\right)\right)^2\mathbb{I}\left[\{x_1,x_2\}\cap\{x_3,x_4\}=\emptyset\right].
\end{aligned}$$

Notice that $\lim_{N\to\infty}|E'_N|/N=3$ for almost all ω_P. Indeed, according to [10], the Poisson-Voronoi tessellation, and thus the Poisson-Delaunay tessellation, satisfies a strong mixing property. This, together with the stationarity of P_1, the ergodic theorem given in Corollary 12.2.V in [7] and the fact that the 1-face intensity of the Poisson-Delaunay tessellation is 3 (see e.g. Theorem 10.2.9 in [18]) leads to the conclusion that $\lim_{N\to\infty}|E'_N|/N=3$ for almost all ω_P. As a consequence, the asymptotic behaviors are the same if we replace $2/|E'_N|$ by $2/(3N)$.

We now need to introduce some probability functions to characterize $\sigma^2_{V_2}$. For any distinct points $x_1,x_2,x_3,x_4\in\mathbf{R}^2$ such that $x_1\preceq x_2$ and $x_3\preceq x_4$, and any $N>0$, we write

$$p_{2,N}(x_1,x_2,x_3,x_4)=\mathbb{P}\left[x_1\sim x_2,\ x_3\sim x_4 \text{ in } \mathrm{Del}(P_N\cup\{x_1,x_2,x_3,x_4\}) \text{ and } x_1\preceq x_2,\ x_3\preceq x_4\right].$$

In this definition, it is implicit that the two edges (x_1,x_2) and (x_3,x_4) of $\mathrm{Del}(P_N\cup\{x_1,x_2,x_3,x_4\})$ have no points in common. To take into account the cases where they may have a common point, we define

$$\mathcal{P}_2=\{(3,1),(3,2),(4,1),(4,2)\}.$$

The set $\mathcal{P}_2$ deals with the couples (j,i) for which $x_j=x_i$ with $x_j\in\{x_3,x_4\}$ and $x_i\in\{x_1,x_2\}$. Then we define, for $(j,i)\in\mathcal{P}_2$,

$$q^{(j\leftrightarrow i)}_{2,N}(\vec{x}_{\{1:4\}\setminus\{j\}})=\mathbb{P}\left[x_1\sim x_2,\ x_3\sim x_4 \text{ in } \mathrm{Del}(P_N\cup\{x_1,x_2,x_3,x_4\}),\ x_1\preceq x_2, x_3\preceq x_4 \text{ and } x_j=x_i\right], \tag{3.1}$$

where $\vec{x}_{\{1:4\}\backslash\{j\}} = \{x_1, x_2, x_3, x_4\}\backslash\{x_j\}$. More specifically, we have

$$q_{2,N}^{(3\leftrightarrow 1)}(x_1,x_2,x_4) = \mathbb{P}\left[x_1 \sim x_2, x_1 \sim x_4 \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_4\}) \text{ and } x_1 \preceq x_2, x_1 \preceq x_4\right] \tag{3.2a}$$

$$q_{2,N}^{(3\leftrightarrow 2)}(x_1,x_2,x_4) = \mathbb{P}\left[x_1 \sim x_2, x_2 \sim x_4 \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_4\}) \text{ and } x_1 \preceq x_2 \preceq x_4\right] \tag{3.2b}$$

$$q_{2,N}^{(4\leftrightarrow 1)}(x_1,x_2,x_3) = \mathbb{P}\left[x_1 \sim x_2, x_1 \sim x_3 \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_3\}) \text{ and } x_3 \preceq x_1 \preceq x_2\right] \tag{3.2c}$$

$$q_{2,N}^{(4\leftrightarrow 2)}(x_1,x_2,x_3) = \mathbb{P}\left[x_1 \sim x_2, x_2 \sim x_3 \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_3\}) \text{ and } x_1 \preceq x_2, x_3 \preceq x_2\right]. \tag{3.2d}$$

The following quantities are proportional to the correlations of the typical squared increments associated with pairs of Delaunay neighbors accordingly to the different cases. When there are no common points, let

$$\sigma_{0,V_2}^2 = \int_{(\mathbf{R}^2)^3} \mathrm{corr}(U_{x_1,x_2}^{(W)}, U_{x_3,x_4}^{(W)})^2 p_{2,1}(x_1,x_2,x_3,x_4)\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{1:4\}\backslash\{1\}}$$

and when there is one common point, let, for $(j,i) \in \mathcal{P}_2$,

$$\sigma_{1,(j\leftrightarrow i),V_2}^2 = \int_{(\mathbf{R}^2)^2} \mathrm{corr}(U_{x_1,x_2}^{(W)}, U_{x_{(i)},\vec{x}_{(\{3:4\}\backslash\{j\})}}^{(W)})^2 q_{2,1}^{(j\leftrightarrow i)}(\vec{x}_{\{1:4\}\backslash\{j\}})\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{1:4\}\backslash\{1,j\}}$$

where $(x_{(i)}, \vec{x}_{(\{3:4\}\backslash\{j\})})$ is the couple associated with the pair of points $\{x_i, \vec{x}_{\{3:4\}\backslash\{j\}}\}$ w.r.t. the lexicographic order (the first coordinate is the smallest point). More specifically, we have

$$\begin{aligned}
\sigma_{0,V_2}^2 &= \int_{(\mathbf{R}^2)^3} \left(\mathrm{corr}\left(U_{0,x_2}, U_{x_3,x_4}\right)\right)^2 p_{2,1}(0,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 \\
\sigma_{1,(3\leftrightarrow 1),V_2}^2 &= \int_{(\mathbf{R}^2)^2} \mathrm{corr}(U_{0,x_2}, U_{0,x_4})^2 q_{2,1}^{(3\leftrightarrow 1)}(0,x_2,x_4)\mathrm{d}x_2\mathrm{d}x_4 \\
\sigma_{1,(3\leftrightarrow 2),V_2}^2 &= \int_{(\mathbf{R}^2)^2} \mathrm{corr}(U_{0,x_2}, U_{x_2,x_4})^2 q_{2,1}^{(3\leftrightarrow 2)}(0,x_2,x_4)\mathrm{d}x_2\mathrm{d}x_4 \\
\sigma_{1,(4\leftrightarrow 1),V_2}^2 &= \int_{(\mathbf{R}^2)^2} \mathrm{corr}(U_{0,x_2}, U_{x_3,0})^2 q_{2,1}^{(4\leftrightarrow 1)}(0,x_2,x_3)\mathrm{d}x_2\mathrm{d}x_3 \\
\sigma_{1,(4\leftrightarrow 2),V_2}^2 &= \int_{(\mathbf{R}^2)^2} \mathrm{corr}(U_{0,x_2}, U_{x_3,x_2})^2 q_{2,1}^{(4\leftrightarrow 2)}(0,x_2,x_3)\mathrm{d}x_2\mathrm{d}x_3.
\end{aligned}$$

We will prove in Proposition 4 that the asymptotic variance of $V_{2,N}^{(W)}$ is given by

$$\sigma_{V_2}^2 = \frac{2}{3}\left(\sigma_{0,V_2}^2 + \sum_{(j,i)\in\mathcal{P}_2} \sigma_{1,(j\leftrightarrow i),V_2}^2\right) + 2.$$

Note that $2\sigma_{0,V_2}^2/3$ corresponds to the case where the two edges (x_1,x_2) and (x_3,x_4) have no common points, $2\sum_{(j,i)\in\mathcal{P}_2}\sigma_{1,(j\leftrightarrow i),V_2}^2/3$ to the case where they have one common point. The value 2 corresponds to the case where $(x_1,x_2) = (x_3,x_4)$, i.e. when they have two common points (since, for any distinct points $x_1, x_2 \in \mathbf{R}^2$, we have $\mathrm{var}[(U_{x_1,x_2}^{(W)})^2] = 2$). Let us first check that $\sigma_{V_2}^2$ is finite.

Lemma 3 *With the above notation, $\sigma_{V_2}^2$ is finite.*

Proof of Lemma 3. We prove only here that σ_{0,V_2}^2 is finite since we can deal with $\sigma_{1,(j\leftrightarrow i),V_2}^2$, $(j,i) \in \mathcal{P}_2$, in a similar way. Let $\varepsilon \in (0, 1/2)$ and d_0 be as in Lemma 6 *(ii)*. Let g be the function defined by

$$g(x_2, x_3, x_4) = 2(\mathrm{corr}(U^{(W)}_{0,x_2}, U^{(W)}_{x_3,x_4}))^2 p_{2,1}(0, x_2, x_3, x_4)\mathbb{I}\left[||x_4 - x_3|| \le ||x_2||\right],$$

so that $\sigma^2_{0,V_2} = \int_{(\mathbf{R}^2)^3} g(x_2, x_3, x_4)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$. To prove that σ^2_{0,V_2} is finite, we discuss three (non disjoint) cases.

Case 1. Assume that $||x_2|| \le ||x_3||^\varepsilon$ and $||x_3|| \ge d_0$. Then, according to Lemmas 6 *(ii)* and 7, we have

$$g(x_2, x_3, x_4) \le c||x_2||^{6-2\alpha}||x_3||^{2\alpha-4}e^{-\frac{\pi}{4}||x_2||^2}\mathbb{I}\left[||x_4 - x_3|| \le ||x_2||\right]$$

Integrating over x_2, x_3, x_4, we get

$$\int_{(\mathbf{R}^2)^3} ||x_2||^{6-2\alpha}||x_3||^{2\alpha-4}e^{-\frac{\pi}{4}||x_2||^2}\mathbb{I}\left[||x_4 - x_3|| \le ||x_2||\right]\mathbb{I}\left[||x_3|| \ge d_0\right]\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$$

$$= \pi\int_{\mathbf{R}^2} ||x_2||^{8-2\alpha}e^{-\frac{\pi}{4}||x_2||^2}\mathrm{d}x_2 \times \int_{\mathbf{R}^2} ||x_3||^{2\alpha-4}\mathbb{I}\left[||x_3|| \ge d_0\right]\mathrm{d}x_3.$$

Because $\alpha < 1$, the right-hand side is finite, which proves that

$$\int_{(\mathbf{R}^2)^3} g(x_2, x_3, x_4)\mathbb{I}\left[||x_2|| \le ||x_3||^\varepsilon\right]\mathbb{I}\left[||x_3|| \ge d_0\right]\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 < \infty.$$

Case 2. Assume that $||x_3|| < d_0$. Then

$$g(x_2, x_3, x_4) \le c||x_2||^2 e^{-\frac{\pi}{4}||x_2||^2}\mathbb{I}\left[||x_4 - x_3|| \le ||x_2||\right]\mathbb{I}\left[||x_3|| < d_0\right].$$

Because the right-hand side is in L^1, this proves that

$$\int_{(\mathbf{R}^2)^3} g(x_2, x_3, x_4)\mathbb{I}\left[||x_3|| < d_0\right]\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 < \infty.$$

Case 3. Assume that $||x_2|| > ||x_3||^\varepsilon$. Then

$$g(x_2, x_3, x_4) \le c||x_2||^2 e^{-\frac{\pi}{4}||x_2||^2}\mathbb{I}\left[||x_4 - x_3|| \le ||x_2||\right]\mathbb{I}\left[||x_3|| < ||x_2||^{1/\varepsilon}\right].$$

Since the right-hand side is also in L^1, this proves that

$$\int_{(\mathbf{R}^2)^3} g(x_2, x_3, x_4)\mathbb{I}\left[||x_2|| > ||x_3||^\varepsilon\right]\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 < \infty.$$

Splitting into the three cases discussed above, we conclude that σ^2_{0,V_2} is finite. □

We are now well equipped to prove the following proposition.

Proposition 4 *For $\alpha \in (0, 1)$, the second conditional moment $\mathbb{E}[(V^{(W)\prime}_{2,N})^2|P_1]$ converges for almost all ω_P, as $N \to \infty$, to*

$$\sigma^2_{V_2} = \frac{2}{3}\left(\sigma^2_{0,V_2} + \sum_{(j,i)\in\mathcal{P}_2}\sigma^2_{1,(j\leftrightarrow i),V_2}\right) + 2.$$

We subdivide the proof of Proposition 4 into three steps. In the first one, we introduce marked point processes $\left\{\left(x, m_{1,(j\leftrightarrow i),x}\right), x\in P_1\right\}$, $(j,i)\in\mathcal{P}_2$, and $\{(x, m_{0,x}), x\in P_1\}$ whose marks are given as follows

$$m_{1,(j\leftrightarrow i),x} = 2\sum_{\substack{x_1\sim x_2, x_3\sim x_4,\\ x_1\preceq x_2, x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x_{(i)},\vec{x}_{(1:2)\setminus\{i\}}}, U^{(W)}_{x_{(i)},\vec{x}_{(\{3:4\}\setminus\{i\})}}\right)\right)^2\mathbb{I}\left[\vec{x}_{\{1:2\}\setminus\{i\}}\neq\vec{x}_{\{3:4\}\setminus\{i\}}\right]\mathbb{I}\left[x_i = x\right],$$

and

$$m_{0,x} = 2\sum_{\substack{x\sim x_2\\ x\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2\mathbb{I}\left[\{x,x_2\}\cap\{x_3,x_4\}=\emptyset\right].$$

More specifically, we have

$$\begin{aligned}
m_{1,(3\leftrightarrow 1),x} &= 2\sum_{\substack{x\sim x_2, x\sim x_4,\\ x\preceq x_2, x\preceq x_4}}\left(\mathrm{corr}\left(U_{x,x_2}, U_{x,x_4}\right)\right)^2\mathbb{I}\left[x_2\neq x_4\right],\\
m_{1,(4\leftrightarrow 1),x} &= 2\sum_{\substack{x\sim x_2, x_3\sim x,\\ x\preceq x_2, x_3\preceq x}}\left(\mathrm{corr}\left(U_{x,x_2}, U_{x_3,x}\right)\right)^2\mathbb{I}\left[x_2\neq x_3\right]\\
m_{1,(3\leftrightarrow 2),x} &= 2\sum_{\substack{x_1\sim x, x\sim x_4,\\ x_1\preceq x, x\preceq x_4}}\left(\mathrm{corr}\left(U_{x_1,x}, U_{x,x_4}\right)\right)^2\mathbb{I}\left[x_1\neq x_4\right]\\
m_{1,(4\leftrightarrow 2),x} &= 2\sum_{\substack{x_1\sim x, x_3\sim x,\\ x_1\preceq x, x_3\preceq x}}\left(\mathrm{corr}\left(U_{x_1,x}, U_{x_3,x}\right)\right)^2\mathbb{I}\left[x_1\neq x_3\right].
\end{aligned}$$

We prove that the above stationary marked point processes satisfy the mixing property in the sense of [7]. In a second step, we apply Corollary 12.2.V in [7] and show that

$$\frac{2}{N}\sum_{x\in P_1\cap\mathbf{C}_N} m_{1,(j\leftrightarrow i),x}\underset{N\to\infty}{\overset{a.s.}{\to}} 2\sigma^2_{1,(j\leftrightarrow i),V_2}\quad\text{and}\quad\frac{2}{N}\sum_{x\in P_1\cap\mathbf{C}_N} m_{0,x}\underset{N\to\infty}{\overset{a.s.}{\to}} 2\sigma^2_{0,V_2}.$$

In a third step, we prove that

$$\frac{1}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\notin\mathbf{C}_N\\ x_3\sim x_4, x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2\underset{N\to\infty}{\overset{a.s.}{\to}} 0.$$

3.1.1 Mixing of the marked point processes

We prove only the mixing property of the marked point process $\{(x, m_{0,x}), x\in P_1\}$ since the mixing property of $\left\{\left(x, m_{1,(j\leftrightarrow i),x}\right), x\in P_1\right\}$, $(j,i)\in\mathcal{P}_2$, may be derived with the same types of arguments.

First note that

$$m_{0,x} = m_{0,1,x} + m_{0,2,x},$$

where

$$m_{0,1,x} = 2\sum_{\substack{x\sim x_2\\ x\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2\mathbb{I}\left[\{x,x_2\}\cap\{x_3,x_4\}=\emptyset\right]\mathbb{I}\left[\|x_4-x_3\|\leq\|x_2-x\|\right]$$

and

$$m_{0,2,x} = 2\sum_{\substack{x\sim x_2\\ x\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U^{(W)}_{x,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2\mathbb{I}\left[\{x,x_2\}\cap\{x_3,x_4\}=\emptyset\right]\mathbb{I}\left[\|x_2-x\|\leq\|x_4-x_3\|\right].$$

We only consider the marks $m_{0,1,x}$ because the same reasoning holds for the marks $m_{0,2,x}$. To simplify the notation, we replace $m_{0,1,x}$ by m_x. Let

$$\xi(\mathbf{A}) = \sum_{x \in P_1 \cap \mathbf{A}} m_x$$

be the random measure associated with $\{(x, m_x), x \in P_1\}$. By Theorem 10.3.VI in in [7], to prove that $\{(x, m_x), x \in P_1\}$ is mixing, it suffices to show that, for any measurable, positive and bounded functions h_1 and h_2

$$L[h_1 + T_y h_2] \underset{\|y\| \to \infty}{\to} L[h_1] L[h_2],$$

where $L[h] = \mathbb{E}\left[\exp\left(-\sum_{x \in P_1} h(x) m_x\right)\right]$ is the Laplace functional of ξ, and T_y is the shift operator defined by $T_y h(x) = h(x+y)$.

For any $x \in P_1$, let

$$m_x^{(y)} = 2 \sum_{\substack{x \sim x_2 \\ x \preceq x_2}} \sum_{\substack{x_3 \sim x_4 \\ x_3 \preceq x_4}} \left(\operatorname{corr}\left(U_{x,x_2}^{(W)}, U_{x_3,x_4}^{(W)}\right)\right)^2 \mathbb{I}[\|x_4 - x_3\| \leq \|x_2 - x\|] \\ \times \mathbb{I}[\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset] \, \mathbb{I}[x_2, x_3, x_4 \in B(x, \|y\|/4)].$$

Notice that the expression of $m_x^{(y)}$ is very close to the one of m_x, except that we only consider sites x_2, x_3, x_4 in a neighborhood of x.

Let $h_1(x) = \mathbb{I}[x \in \mathbf{A}]$ and $h_2(x) = \mathbb{I}[x \in \mathbf{B}]$ for two bounded sets $\mathbf{A}, \mathbf{B} \subset \mathbf{R}^2$. We have

$$\begin{aligned} L[h_1 + T_y h_2] &= \mathbb{E}\left[\exp\left(-\sum_{x \in P_1 \cap \mathbf{A}} m_x - \sum_{x \in P_1 \cap (\mathbf{B} - y)} m_x\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\sum_{x \in P_1 \cap \mathbf{A}} m_x^{(y)} - \sum_{x \in P_1 \cap (\mathbf{B} - y)} m_x^{(y)} - R_{\mathbf{A}}^{(y)} - R_{\mathbf{B}}^{(y)}\right)\right] \end{aligned}$$

with

$$R_{\mathbf{A}}^{(y)} = \sum_{x \in P_1 \cap \mathbf{A}} (m_x - m_x^{(y)}) \geq 0 \quad \text{and} \quad R_{\mathbf{B}}^{(y)} = \sum_{x \in P_1 \cap (\mathbf{B} - y)} (m_x - m_x^{(y)}) \geq 0.$$

Therefore

$$L[h_1 + T_y h_2] = \mathbb{E}\left[\exp\left(-\sum_{x \in P_1 \cap \mathbf{A}} m_x^{(y)} - \sum_{x \in P_1 \cap (\mathbf{B} - y)} m_x^{(y)}\right)\left(\exp(-R_{\mathbf{A}}^{(y)} - R_{\mathbf{B}}^{(y)}) - 1\right)\right] \\ + \mathbb{E}\left[\exp\left(-\sum_{x \in P_1 \cap \mathbf{A}} m_x^{(y)} - \sum_{x \in P_1 \cap (\mathbf{B} - y)} m_x^{(y)}\right)\right]. \quad (3.3)$$

Let us start by showing that the top term of Eq. (3.3) converges to 0. Since $m_x^{(y)}$, $R_{\mathbf{A}}^{(y)}$ and $R_{\mathbf{B}}^{(y)}$ are positive, we have

$$\left|\exp\left(-\sum_{x \in P_1 \cap \mathbf{A}} m_x^{(y)} - \sum_{x \in P_1 \cap (\mathbf{B} - y)} m_x^{(y)}\right)\left(\exp(-R_{\mathbf{A}}^{(y)} - R_{\mathbf{B}}^{(y)}) - 1\right)\right| \leq \left(1 - \exp(-R_{\mathbf{A}}^{(y)} - R_{\mathbf{B}}^{(y)}\right) \\ \leq R_{\mathbf{A}}^{(y)} + R_{\mathbf{B}}^{(y)}.$$

We now prove that $\mathbb{E}\left[R_{\mathbf{A}}^{(y)}\right]$ and $\mathbb{E}\left[R_{\mathbf{B}}^{(y)}\right]$ converge to 0 as $\|y\| \to \infty$. To do this, we will prove that $\mathbb{E}\left[R_{A}^{(y)}\right]$ is bounded by $c\|y\|^{-2(1-\alpha)}$. First, we notice that $\mathbb{E}\left[R_{A}^{(y)}\right] = \mathbb{E}\left[\sum_{x\in P_1\cap\mathbf{A}}(m_x - m_x^{(y)})\right]$ is equal to

$$2\mathbb{E}\Big[\sum_{x\in P_1\cap\mathbf{A}}\sum_{\substack{x\sim x_2\\ x\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U_{x,x_2},U_{x_3,x_4}\right)\right)^2\mathbb{I}\left[\|x_4-x_3\|\le\|x_2-x\|\right] \times\mathbb{I}\left[\exists j: 2\le j\le 4, x_j\notin B\left(x,\|y\|/4\right)\right]\Big],$$

which implies

$$\mathbb{E}\left[R_{A}^{(y)}\right]\le 2\sum_{j=2}^{4}\mathbb{E}\Big[\sum_{x\in P_1\cap\mathbf{A}}\sum_{\substack{x\sim x_2\\ x\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U_{x,x_2},U_{x_3,x_4}\right)\right)^2\mathbb{I}\left[\|x_4-x_3\|\le\|x_2-x\|\right] \times\mathbb{I}\left[x_j\notin B\left(x,\|y\|/4\right)\right]\Big].$$

By Lemma 7, the term for $j=2$ tends to 0 exponentially fast in $\|y\|$, and the terms for $j=3$ and 4 are of the same order. So we now consider

$$\mathbb{E}\Big[\sum_{x\in P_1\cap\mathbf{A}}\sum_{\substack{x\sim x_2\\ x\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}}\left(\mathrm{corr}\left(U_{x,x_2}^{(W)},U_{x_3,x_4}^{(W)}\right)\right)^2\mathbb{I}\left[\|x_4-x_3\|\le\|x_2-x\|\right]\mathbb{I}\left[x_3\notin B\left(x,\|y\|/4\right)\right]\Big].$$

Thanks to the Slivnyak-Mecke formula (see e.g. Theorem 3.2.5 in [18]), the last term can be expressed as

$$\int_{\mathbf{A}}\int_{(\mathbf{R}^2)^3}\left(\mathrm{corr}\left(U_{x,x_2}^{(W)},U_{x_3,x_4}^{(W)}\right)\right)^2\mathbb{I}\left[\|x_4-x_3\|\le\|x_2-x\|\right]\mathbb{I}\left[\|x_3-x\|>\|y\|/4\right]p_{2,1}(x,x_2,x_3,x_4)\mathrm{d}\vec{x}.$$

By Lemmas 6 and 7, this term is bounded by

$$c\int_{\mathbf{A}}\int_{(\mathbf{R}^2)^3}\left(\|x_2-x\|\,\|x_4-x_3\|\right)^{2-\alpha}\|x_3-x\|^{2\alpha-4}\,||x_2-x||^2 e^{-\frac{\pi}{4}||x_2-x||^2} \times\mathbb{I}\left[\|x_4-x_3\|\le\|x_2-x\|\right]\mathbb{I}\left[\|x_4-x\|>\|y\|/4\right]\mathrm{d}\vec{x} \le c\int_{\|y\|/4}^{\infty}r^{2\alpha-4}r\mathrm{d}r = c\|y\|^{-2(1-\alpha)},$$

where c denotes a generic constant. This proves that $\mathbb{E}\left[R_{\mathbf{A}}^{(y)}\right]$ converges to 0 as $\|y\|\to\infty$. The same reasoning holds for $\mathbb{E}\left[R_{\mathbf{B}}^{(y)}\right]$.

Let us now consider the bottom term of Eq. (3.3). Note that $\sum_{x\in P_1\cap\mathbf{A}}m_x^{(y)}$ and $\sum_{x\in P_1\cap(\mathbf{B}-y)}m_x^{(y)}$ are independent since $\sum_{x\in P_1\cap\mathbf{A}}m_x^{(y)}$ is measurable w.r.t. $\sigma\left(P_1\cap\mathbf{A}\oplus B\left(0,\|y\|/4\right)\right)$, $\sum_{x\in P_1\cap(\mathbf{B}-y)}m_x^{(y)}$ is measurable w.r.t. $\sigma\left(P_1\cap(\mathbf{B}-y)\oplus B\left(0,\|y\|/4\right)\right)$ and

$$\mathbf{A}\oplus B\left(0,\|y\|/4\right)\cap(\mathbf{B}-y)\oplus B\left(0,\|y\|/4\right)=\emptyset$$

for large $\|y\|$. Therefore we have

$$\mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x^{(y)} - \sum_{x\in P_1\cap(\mathbf{B}-y)} m_x^{(y)}\right)\right]$$

$$= \mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x^{(y)}\right)\right] \times \mathbb{E}\left[\left(-\sum_{x\in P_1\cap(\mathbf{B}-y)} m_x^{(y)}\right)\right].$$

Let us now prove that $\mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x^{(y)}\right)\right]$ converges to $\mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x\right)\right]$. We have

$$\left|\mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x^{(y)}\right)\right] - \mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x\right)\right]\right|$$

$$= \mathbb{E}\left[\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x^{(y)}\right)\left(1-\exp(-R_{\mathbf{A}}^{(y)})\right)\right].$$

Once again, we note that $m_x^{(y)} \geq 0$ and $R_{\mathbf{A}}^{(y)} \geq 0$ so that

$$\left|\exp\left(-\sum_{x\in P_1\cap\mathbf{A}} m_x^{(y)}\right)\left(1-\exp(-R_{\mathbf{A}}^{(y)})\right)\right| \leq R_{\mathbf{A}}^{(y)}$$

and we conclude using the fact that $\lim_{\|y\|\to\infty} \mathbb{E}\left[R_{\mathbf{A}}^{(y)}\right] = 0$. In conclusion this proves that the marked point process $\{(x, m_{0,x}), x \in P_1\}$ has the mixing property.

3.1.2 Ergodic means

The main ingredient to deal with ergodic means is the following result, which is a direct consequence of Corollary 12.2.V in [7].

Proposition 5 *Let $\{(x, m_x), x \in P\}$ be a stationary marked point process with $P \subset \mathbf{R}^2$ and with marks m_x in $\mathbf{R}_+$. Let $\mathbb{P}_0$ be the Palm mark distribution, i.e.*

$$\mathbb{P}_0(B) = \mathbb{E}\left[\sum_{x\in P\cap[0,1]^2} \mathbb{I}\left[m_x \in B\right]\right]$$

for any Borel subset $B \subset \mathbf{R}_+$. Let $h : \mathbf{R}_+ \to \mathbf{R}$ be an integrable function w.r.t. $\mathbb{P}_0$. Then, a.s.,

$$\frac{1}{N}\sum_{x\in P\cap\mathbf{C}_N} h(m_x) \underset{N\to\infty}{\longrightarrow} \int_{\mathbf{R}_+} h(m)\mathrm{d}\mathbb{P}_0(m).$$

Now, consider the marked point process $\{(x, m_{0,x}) : x \in P_1\}$. Since P_1 is a stationary point process, it is clear that $\{(x, m_{0,x}), x \in P_1\}$ is also a stationary marked point process. According to Section 3.1.1, we know that it has a mixing property and thus that it is ergodic. It follows from Proposition 5 that

$$\frac{1}{N}\sum_{x\in P_1\cap\mathbf{C}_N} m_{0,x} \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left[\sum_{x\in P_1\cap\mathbf{C}_1} m_{0,x}\right].$$

Moreover, according to the Slivnyak-Mecke formula, we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{x\in P_1\cap\mathbf{C}_1} m_{0,x}\right]\\
&\quad = 2\mathbb{E}\left[\sum_{\substack{x_1\in\mathbf{C}_1, x_1\sim x_2, x_3\sim x_4,\\ x_1\preceq x_2, x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \mathbb{I}\left[\{x_1,x_2\}\cap\{x_3,x_4\}=\emptyset\right]\right]\\
&\quad = 2\int_{\mathbf{C}_1}\int_{(\mathbf{R}^2)^3} \mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_1,x_4}\right)^2 p_{2,1}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_1\\
&\quad = 2\int_{\mathbf{C}_1}\int_{(\mathbf{R}^2)^3} \mathrm{corr}\left(U^{(W)}_{0,x_2-x_1}, U^{(W)}_{0,x_4-x_1}\right)^2 p_{2,1}(0,x_2-x_1,x_3-x_1,x_4-x_1)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4\mathrm{d}x_1\\
&\quad = 2\int_{(\mathbf{R}^2)^3} \mathrm{corr}\left(U^{(W)}_{0,x_2'}, U^{(W)}_{x_3',x_4'}\right)^2 p_{2,1}(0,x_2',x_3',x_4')\mathrm{d}x_2'\mathrm{d}x_3'\mathrm{d}x_4'\\
&\quad = 2\sigma^2_{0,V_2}.
\end{aligned}$$

It follows from the definition of $m_{0,x}$ that

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \mathbb{I}\left[\{x_1,x_2\}\cap\{x_3,x_4\}=\emptyset\right] \underset{N\to\infty}{\overset{a.s.}{\to}} 2\sigma^2_{0,V_2}.$$

We proceed in a similar way for the marks $m_{1,(j\leftrightarrow i),x}$, $(j,i)\in\mathcal{P}_2$. This implies

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\sim x_4\\ x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \mathbb{I}\left[(x_1,x_2)\neq(x_3,x_4)\right] \underset{N\to\infty}{\overset{a.s.}{\to}} \sum_{(j,i)\in\mathcal{P}_2}\sigma^2_{1,(j\leftrightarrow i),V_2}+\sigma^2_{0,V_2}.$$

3.1.3 Deviations in x_3.

We prove below that

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\notin\mathbf{C}_N\\ x_3\sim x_4, x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \underset{N\to\infty}{\overset{a.s.}{\to}} 0.$$

To do it, it is sufficient to show that

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\in\mathbf{C}_N\\ x_3\sim x_4, x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \underset{N\to\infty}{\overset{a.s.}{\to}} 2\sigma^2_{V_2}. \tag{3.4}$$

For $d>0$ and $N>(2d)^2$, let $\mathbf{A}_N(d)=(-N^{1/2}/2+d, N^{1/2}/2-d]^2$ and denote by $D_d(x_1)$ the square of $\mathbf{R}^2$ with center x_1 and edges of size $2d$ parallel to the x- and y-axis. We have

$$\begin{aligned}
&\frac{2}{N}\sum_{\substack{x_1\in\mathbf{A}_N(d)\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\in D_d(x_1)\\ x_3\sim x_4, x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2\\
&\qquad\qquad\qquad \leq \frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2, x_1\preceq x_2}}\sum_{\substack{x_3\in\mathbf{C}_N\\ x_3\sim x_4, x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2
\end{aligned}$$

and

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2,x_1\preceq x_2}}\sum_{\substack{x_3\in\mathbf{C}_N\\ x_3\sim x_4,x_3\preceq x_4}}\left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4}\right)\right)^2 \leq \frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2,x_1\preceq x_2}}\sum_{x_3\sim x_4,x_3\preceq x_4}\left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4}\right)\right)^2.$$

According to (3.4), we get

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{C}_N\\ x_1\sim x_2,x_1\preceq x_2}}\sum_{x_3\sim x_4,x_3\preceq x_4}\left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4}\right)\right)^2 \underset{N\to\infty}{\to} 2\sigma^2_{V_2}$$

for almost all $\omega_P\in\Omega_P$. Using the same arguments as above, we can also prove that

$$\frac{2}{N}\sum_{\substack{x_1\in\mathbf{A}_N(d)\\ x_1\sim x_2,x_1\preceq x_2}}\sum_{\substack{x_3\in D_d(x_1)\\ x_3\sim x_4,x_3\preceq x_4}}\left(\operatorname{corr}\left(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4}\right)\right)^2 \underset{N\to\infty}{\to} 2\sigma^2_{V_2}(d),$$

for almost all $\omega_P\in\Omega_P$, where

$$\sigma^2_{V_2}(d)=2+\frac{2}{3}\left(\sigma^2_{0,V_2}(d)+\sum_{(j,i)\in\mathcal{P}_2}\sigma^2_{1,(j\leftrightarrow i),V_2}\right),$$

with

$$\sigma^2_{0,V_2}(d)=\int_{(\mathbf{R}^2)^3}\operatorname{corr}(U^{(W)}_{x_1,x_2},U^{(W)}_{x_3,x_4})^2p_{2,1}(x_1,x_2,x_3,x_4)\mathbb{I}\left[x_3\in D_d\left(x_1\right)\right]\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{2:4\}}$$

and

$$\begin{aligned}\sigma^2_{1,(j\leftrightarrow i),V_2}(d)=\mathbb{I}\left[j=3\right]&\int_{(\mathbf{R}^2)^2}\operatorname{corr}(U^{(W)}_{x_1,x_2},U^{(W)}_{x_{(i)},\vec{x}_{(\{3:4\}\setminus\{j\})}})^2q^{(j\leftrightarrow i)}_{2,1}(\vec{x}_{\{1:4\}\setminus\{j\}})\\&\times\mathbb{I}\left[x_i\in D_d\left(x_1\right)\right]\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:4\}\setminus\{1,j\}}\\&+\mathbb{I}\left[j=4\right]\int_{(\mathbf{R}^2)^2}\operatorname{corr}(U^{(W)}_{x_1,x_2},U^{(W)}_{x_{(i)},\vec{x}_{(\{3:4\}\setminus\{j\})}})^2q^{(j\leftrightarrow i)}_{2,1}(\vec{x}_{\{1:4\}\setminus\{j\}})\\&\times\mathbb{I}\left[x_3\in D_d\left(x_1\right)\right]\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:4\}\setminus\{1,j\}}.\end{aligned}$$

More specifically, we have

$$\begin{aligned}\sigma^2_{0,V_2}(d)&=\int_{(\mathbf{R}^2)^3}\left(\operatorname{corr}\left(U_{0,x_2},U_{x_3,x_4}\right)\right)^2p_{2,1}(0,x_2,x_3,x_4)\mathbb{I}\left[x_3\in D_d\left(0\right)\right]\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4,\\ \sigma^2_{1,(3\leftrightarrow1),V_2}(d)&=\sigma^2_{1,(3\leftrightarrow1),V_2},\\ \sigma^2_{1,(3\leftrightarrow2),V_2}(d)&=\int_{(\mathbf{R}^2)^2}\operatorname{corr}(U_{0,x_2},U_{x_2,x_4})^2q^{(3\leftrightarrow2)}_{2,1}(0,x_2,x_4)\mathbb{I}\left[x_2\in D_d\left(0\right)\right]\mathrm{d}x_2\mathrm{d}x_4,\\ \sigma^2_{1,(4\leftrightarrow1),V_2}(d)&=\int_{(\mathbf{R}^2)^2}\operatorname{corr}(U_{0,x_2},U_{x_3,0})^2q^{(4\leftrightarrow1)}_{2,1}(0,x_2,x_3)\mathbb{I}\left[x_3\in D_d\left(0\right)\right]\mathrm{d}x_2\mathrm{d}x_3,\\ \sigma^2_{1,(4\leftrightarrow2),V_2}(d)&=\int_{(\mathbf{R}^2)^2}\operatorname{corr}(U_{0,x_2},U_{x_3,x_2})^2q^{(4\leftrightarrow2)}_{2,1}(0,x_2,x_3)\mathbb{I}\left[x_3\in D_d\left(0\right)\right]\mathrm{d}x_2\mathrm{d}x_3.\end{aligned}$$

By the monotone convergence theorem, we also deduce that $\sigma^2_{V_2}(d) \underset{d\to\infty}{\to} \sigma^2_{V_2}$ for almost all $\omega_P \in \Omega_P$. We therefore derive that

$$\frac{2}{N} \sum_{\substack{x_1\in\mathbf{C}_N \\ x_1\sim x_2, x_1\preceq x_2}} \sum_{\substack{x\in\mathbf{C}_N \\ x_3\sim x_4, x_3\preceq x_4}} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2 \underset{N\to\infty}{\to} 2\sigma^2_{V_2},$$

for almost all $\omega_P \in \Omega_P$.

Proof of Proposition 4. From the previous sub-sections and the fact that $\lim_{N\to\infty} |E'_N|/N = 3$ for almost all $\omega_P \in \Omega_P$, we derive that

$$\mathbb{E}\left[(V^{(W)\prime}_{2,N})^2|P_1\right] = \frac{2}{|E'_N|} \sum_{(x_1,x_2)\in E'_N} \sum_{(x_3,x_4)\in E'_N} \left(\mathrm{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right)\right)^2$$

converges for almost all $\omega_P \in \Omega_P$ to $\sigma^2_{V_2}$. □

3.2 Checking Condition *(c)*

Since $V^{(W)\prime}_{2,N} = I_2(f_{N,2})$, we only consider the case $r = 1$. Because

$$f_{N,2} \otimes_1 f_{N,2} = \frac{1}{|e'_N|} \sum_{k\in e'_N} \sum_{l\in e'_N} \varepsilon_k^{\otimes 2} \otimes \varepsilon_l^{\otimes 2}$$

we have

$$\|f_{N,2} \otimes_1 f_{N,2}\|^2_{\mathfrak{H}^{\otimes 2}} = \frac{1}{|e'_N|^2} \sum_{k,l,i,j\in e'_N} \mathrm{corr}\left(U^{(k)}, U^{(l)}\right) \mathrm{corr}\left(U^{(i)}, U^{(j)}\right) \mathrm{corr}\left(U^{(k)}, U^{(i)}\right) \mathrm{corr}\left(U^{(l)}, U^{(j)}\right).$$

Let $\mathrm{d}_{k,l}$ denote the distance between the first point in $\mathbf{R}^2$ of the pair $\varphi(k)$ and the first point in $\mathbf{R}^2$ of the pair $\varphi(l)$, and let $\|e_m\|$ be the distance between the points of the pair $\varphi(m)$ for $m = i, j, k, l$.

Let $\varepsilon \in (0, 1/2)$ and d_0 be as in Lemma 6 *(ii)*. We assume that the terms appearing in the sums are such that

$$\mathrm{d}_{k,l}, \mathrm{d}_{i,j}, \mathrm{d}_{k,i}, \mathrm{d}_{l,j} \geq \max\{d_0, \|e_i\|^{1/\varepsilon}\},$$

with $\|e_i\| \geq \|e_j\|, \|e_k\|, \|e_l\|$. The above assumption is not restrictive since it deals with the most difficult case. According to Lemma 6, is is sufficient to prove that

$$\frac{1}{|e'_N|^2} \sum_{k,l,i,j\in e'_N} \mathrm{d}^{\alpha-2}_{k,l}\mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}^{\alpha-2}_{i,j}\mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}^{\alpha-2}_{k,i}\mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right] \mathrm{d}^{\alpha-2}_{l,j}\mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right] \|e_i\|^{8(2-\alpha)}$$

converges a.s. to 0 as N goes to infinity.

We consider two cases depending on whether $\|e_i\|$ is larger or smaller than N^{ε_0} for some $\varepsilon_0 > 0$.

Case 1. Assume that $\|e_i\| \geq N^{\varepsilon_0}$, with $\varepsilon_0 > 0$. Then

$$\frac{1}{|e'_N|^2} \sum_{k,l,i,j \in e'_N} \mathrm{d}_{k,l}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{k,i}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right]$$
$$\times \|e_i\|^{8(2-\alpha)} \mathbb{I}\left[\|e_i\| \geq N^{\varepsilon_0}\right]$$
$$\leq c|e'_N| \sum_{i \in e'_N} \|e_i\|^{8(2-\alpha)} \mathbb{I}\left[\|e_i\| \geq N^{\varepsilon_0}\right].$$

According to Lemma 7 (given in Section 5) and the Slivnyak-Mecke formula, we can easily show that

$$\mathbb{E}\left[N \sum_{i \in e'_N} \|e_i\|^{8(2-\alpha)} \mathbb{I}\left[\|e_i\| \geq N^{\varepsilon_0}\right]\right] = O\left(e^{-\frac{\pi}{4} N^{\varepsilon_0}}\right),$$

which is the term of a convergent series. According to the Borel-Cantelli lemma, this shows that

$$N \sum_{i \in e'_N} \|e_i\|^{8(2-\alpha)} \mathbb{I}\left[\|e_i\| \geq N^{\varepsilon_0}\right]$$

converges to 0 a.s.. The same holds for $c|e'_N| \sum_{i \in e'_N} \|e_i\|^{8(2-\alpha)} \mathbb{I}\left[\|e_i\| \geq N^{\varepsilon_0}\right]$ since $\frac{|e'_N|}{N} \underset{N\to\infty}{\longrightarrow} 3$ a.s..

Case 2. We now assume that $\|e_i\| \leq N^{\varepsilon_0}$. Then, taking $\tilde{\varepsilon}_0 = 8\varepsilon_0(2-\alpha)$, we have $\|e_i\|^{8(2-\alpha)} \leq N^{\tilde{\varepsilon}_0}$. It is sufficient to prove that, a.s.,

$$\frac{N^{\tilde{\varepsilon}_0}}{|e'_N|^2} \sum_{k,l,i,j \in e'_N} \mathrm{d}_{k,l}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{k,i}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right] \underset{N\to\infty}{\longrightarrow} 0.$$

Since

$$\mathrm{d}_{k,l}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}_{k,i}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right] \leq \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] + \mathrm{d}_{k,i}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right],$$

we have

$$\sum_{k,l,i,j \in e'_N} \mathrm{d}_{k,l}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{k,i}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right]$$
$$\leq \sum_{k,l,i,j \in e'_N} \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right] \left(\mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] + \mathrm{d}_{k,i}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right]\right)$$
$$= \sum_{k,l,i,j \in e'_N} \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right] \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right]$$
$$+ \sum_{k,l,i,j \in e'_N} \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right] \mathrm{d}_{k,i}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,i} \geq d_0\right]$$
$$= 2 \sum_{k,l,i,j \in e'_N} \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right].$$

Let us prove that, for $0 < \alpha < 1$ and for any $\eta > 0$,

$$N^{-\eta} \sup_{l \in e'_N} \sum_k \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \underset{N\to\infty}{\longrightarrow} 0, \qquad a.s..$$

In this way, we will be able to bound $\sup_{l \in e'_N} \sum_k \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right]$ by N^η for N large enough. To do this it is sufficient to prove that, for any $\eta > 0$,

$$N^{-\eta} \sup_{y \in P_1 \cap \mathbf{C}_N} \sum_{x \in P_1} \|y - x\|^{2(\alpha-2)} \mathbb{I}\left[\|y - x\| \geq d_0\right] \underset{N\to\infty}{\longrightarrow} 0, \qquad a.s..$$

To establish this convergence, we will use the Borel-Cantelli lemma. Let $\varepsilon > 0$ be fixed. We have

$$\begin{aligned}
&\mathbb{P}\left[N^{-\eta} \sup_{y \in P_1 \cap \mathbf{C}_N} \sum_{x \in P_1} \|y - x\|^{2(\alpha-2)} \mathbb{I}\left[\|y - x\| \geq d_0\right] > \varepsilon\right] \\
&\qquad = \mathbb{P}\left[\exists y \in P_1 \cap \mathbf{C}_N : N^{-\eta} \sum_{x \in P_1} \|y - x\|^{2(\alpha-2)} \mathbb{I}\left[\|y - x\| \geq d_0\right] > \varepsilon\right] \\
&\qquad \leq \mathbb{E}\left[\sum_{y \in P_1 \cap \mathbf{C}_N} \mathbb{I}\left[N^{-\eta} \sum_{x \in P_1} \|y - x\|^{2(\alpha-2)} \mathbb{I}\left[\|y - x\| \geq d_0\right] > \varepsilon\right]\right] \\
&\qquad = \int_{\mathbf{C}_N} \mathbb{P}\left[N^{-\eta} \sum_{x \in P_1} \|y - x\|^{2(\alpha-2)} \mathbb{I}\left[\|y - x\| \geq d_0\right] > \varepsilon\right] \mathrm{d}y \\
&\qquad = N\mathbb{P}\left[N^{-\eta} \sum_{x \in P_1} \|x\|^{2(\alpha-2)} \mathbb{I}\left[\|x\| \geq d_0\right] > \varepsilon\right],
\end{aligned}$$

where, in the last line, we used the fact that P_1 is stationary. According to Chernoff's inequality, we have

$$\mathbb{P}\left[N^{-\eta} \sum_{x \in P_1 \cap B(0, \sqrt{2}N^{1/2})} \|x\|^{2(\alpha-2)} > \varepsilon\right] \leq \exp\left(-\varepsilon N^{\eta}\right) \mathbb{E}\left[\exp\left(\sum_{x \in P_1} \|x\|^{2(\alpha-2)} \mathbb{I}\left[\|x\| \geq d_0\right]\right)\right].$$

By Theorem 3.2.4 in [18], we have

$$\mathbb{E}\left[\exp\left(\sum_{x \in P_1} \|x\|^{2(\alpha-2)} \mathbb{I}\left[\|x\| \geq d_0\right]\right)\right] = \exp\left(\int_{\mathbf{R}^2} \left(\|x\|^{2(\alpha-2)} \mathbb{I}\left[\|x\| \geq d_0\right] - 1\right) \mathrm{d}x\right),$$

which is finite. We can easily deduce that

$$\mathbb{P}\left[N^{-\eta} \sup_{y \in P_1 \cap \mathbf{C}_N} \sum_{x \in P_1} \|y - x\|^{2(\alpha-2)} \mathbb{I}\left[\|y - x\| \geq d_0\right] > \varepsilon\right]$$

is the term of a convergent series and consequently that

$$N^{-\eta} \sup_{l \in e'_N} \sum_{k} \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \underset{N\to\infty}{\longrightarrow} 0, \qquad a.s..$$

Therefore, we have, for large N,

$$\begin{aligned}
&\sum_{k,l,i,j \in e'_N} \mathrm{d}_{k,l}^{2(\alpha-2)} \mathbb{I}\left[\mathrm{d}_{k,l} \geq d_0\right] \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right] \\
&\qquad \leq N^{\eta} \sum_{l,i,j \in e'_N} \mathrm{d}_{i,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{i,j} \geq d_0\right] \mathrm{d}_{l,j}^{\alpha-2} \mathbb{I}\left[\mathrm{d}_{l,j} \geq d_0\right], \qquad a.s..
\end{aligned}$$

Then, note that

$$\sum_{l,i,j\in e'_N} \mathrm{d}_{i,j}^{\alpha-2}\mathbb{I}\left[\mathrm{d}_{i,j}\geq d_0\right]\mathrm{d}_{l,j}^{\alpha-2}\mathbb{I}\left[\mathrm{d}_{l,j}\geq d_0\right] = \sum_{j\in e'_N}\sum_{l,i\in e'_N} \mathrm{d}_{i,j}^{\alpha-2}\mathbb{I}\left[\mathrm{d}_{i,j}\geq d_0\right]\mathrm{d}_{l,j}^{\alpha-2}\mathbb{I}\left[\mathrm{d}_{l,j}\geq d_0\right]$$

$$\leq \sum_{j\in e'_N}\left(\sum_{i\in B\left(j,\sqrt{2}N^{1/2}\right)} \mathrm{d}_{i,j}^{\alpha-2}\mathbb{I}\left[\mathrm{d}_{i,j}\geq d_0\right]\right)^2.$$

Let $\eta_0 > 0$ and ε_0 be such that $\tilde{\varepsilon}_0 + (\alpha-1) + \eta_0/2 < 0$. It remains to prove that

$$\frac{N^{\tilde{\varepsilon}_0+\eta_0/2}}{|e'_N|^2}\sum_{j\in e'_N}\left(\sum_{i\in B(j,\sqrt{2}N^{1/2})} \mathrm{d}_{i,j}^{\alpha-2}\mathbb{I}\left[\mathrm{d}_{i,j}\geq d_0\right]\right)^2$$

converges to 0 as $N\to\infty$ a.s.. Since $\frac{|e'_N|}{N} \underset{N\to\infty}{\longrightarrow} 3$ a.s., we have to prove that, a.s.,

$$\frac{1}{N^{2-\tilde{\varepsilon}_0-\eta_0/2}}\sum_{y\in P_1\cap\mathbf{C}_N}\left(\sum_{x\in P_1\cap B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}\,\mathbb{I}\left[\|y-x\|\geq d_0\right]\right)^2 \underset{N\to\infty}{\longrightarrow} 0.$$

Because $\frac{|P_1\cap\mathbf{C}_N|}{N} \underset{N\to\infty}{\longrightarrow} 1$ a.s. and $0<\alpha<1$, it is sufficient to prove that, for any $\eta>0$, a.s.,

$$N^{-(\alpha/2+\eta)}\sup_{y\in P_1\cap\mathbf{C}_N}\sum_{x\in P_1\cap B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}\,\mathbb{I}\left[\|y-x\|\geq d_0\right] \underset{N\to\infty}{\longrightarrow} 0.$$

To do this, we will use again the Borel-Cantelli lemma. Let $\varepsilon>0$ be fixed. We have

$$\mathbb{P}_{P_1}\left[N^{-(\alpha/2+\eta)}\sup_{y\in P_1\cap\mathbf{C}_N}\sum_{x\in B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}\,\mathbb{I}\left[\|y-x\|\geq d_0\right]>\varepsilon\right]$$

$$\leq \mathbb{E}_{P_1}\left[\sum_{y\in P_1\cap\mathbf{C}_N}\mathbb{I}\left[N^{-(\alpha/2+\eta)}\sum_{x\in P_1\cap B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}\,\mathbb{I}\left[\|y-x\|\geq d_0\right]>\varepsilon\right]\right]$$

$$=\int_{\mathbf{C}_N}\mathbb{P}_{P_1}\left[N^{-(\alpha/2+\eta)}\sum_{x\in P_1\cap B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}\,\mathbb{I}\left[\|y-x\|\geq d_0\right]>\varepsilon\right]\mathrm{d}y$$

$$=N\mathbb{P}_{P_1}\left[N^{-(\alpha/2+\eta)}\sum_{x\in P_1\cap B(0,\sqrt{2}N^{1/2})}\|x\|^{\alpha-2}\,\mathbb{I}\left[\|x\|\geq d_0\right]>\varepsilon\right],$$

where, in the last line, we used the fact that P_1 is stationary. According to Chernoff's inequality, we have

$$\mathbb{P}\left[N^{-(\alpha/2+\eta)}\sum_{x\in P_1\cap B(0,\sqrt{2}N^{1/2})}\|x\|^{\alpha-2}\,\mathbb{I}\left[\|x\|\geq d_0\right]>\varepsilon\right]$$

$$\begin{aligned}&\leq \exp\left(-\varepsilon N^{\alpha/2+\eta}\right)\mathbb{E}\left[\exp\left(\sum_{x\in P_1\cap B(0,\sqrt{2}N^{1/2})}\|x\|^{\alpha-2}\,\mathbb{I}\left[\|x\|\geq d_0\right]\right)\right]\\ &= \exp\left(-\varepsilon N^{\alpha/2+\eta}\right)\mathbb{E}\left[\exp\left(\sum_{i\leq\mathcal{N}}R_i^{\alpha-2}\mathbb{I}\left[R_i\geq d_0\right]\right)\right]\\ &= \exp\left(-\varepsilon N^{\alpha/2+\eta}\right)\mathbb{E}\left[\left(\mathbb{E}\left[e^{R_1^{\alpha-2}\mathbb{I}[R_i\geq d_0]}\right]\right)^{\mathcal{N}}\right],\end{aligned}$$

where $\mathcal{N}=\left|P_1\cap B(0,\sqrt{2}N^{1/2})\right|$ has a Poisson distribution with parameter $2\pi N$ and $R_i=\|U_i\|$ for $i\geq 1$ with $(U_i)_{i\geq 1}$ a sequence of i.i.d. random variables with uniform distribution on $B(0,\sqrt{2}N^{1/2})$. Note that the random variables R_i's have the same distribution as in Eq. (5.1). By Lemma 8, we know that, for large N,

$$\mathbb{E}\left[e^{R_1^{\alpha-2}\mathbb{I}[R_1\geq d_0]}\right]\leq 1+c_1(\sqrt{N})^{\alpha-2}+c_2N^{-1}.$$

It follows (by using the moment generating function of Poisson distributions) that, for large N,

$$\mathbb{E}\left[\left(\mathbb{E}\left[e^{R_1^{\alpha-2}\mathbb{I}[\|R_1\|\geq d_0]}\right]\right)^{\mathcal{N}}\right]\leq \exp\left(2\pi c_1(\sqrt{N})^{\alpha}+2\pi c_2\right).$$

Therefore

$$\mathbb{P}\left[N^{-(\alpha/2+\eta)}\sum_{x\in P_1\cap B(0,\sqrt{2}N^{1/2})}\|x\|^{\alpha-2}>\varepsilon\right]\leq \exp\left(-\varepsilon N^{\alpha/2+\eta}+2\pi c_1(\sqrt{N})^{\alpha}+2\pi c_2\right).$$

Since $\eta>0$, the bound converges to 0 exponentially fast in N. This proves that

$$\mathbb{P}\left[N^{-(\alpha/2+\eta)}\sup_{y\in P_1\cap\mathbf{C}_N}\sum_{x\in B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}>\varepsilon\right]$$

is the term of a convergent series and consequently that

$$N^{-(\alpha/2+\eta)}\sup_{y\in P_1\cap\mathbf{C}_N}\sum_{x\in B(y,\sqrt{2}N^{1/2})}\|y-x\|^{\alpha-2}\underset{N\to\infty}{\longrightarrow}0\text{ a.s.}.$$

Considering the two above cases, we deduce that a.s. $\|f_{N,2}\otimes_1 f_{N,2}\|^2_{\mathfrak{H}^{\otimes 2}}\to 0$ as $N\to\infty$, which shows that Condition *(c)* is satisfied.

4 Proof of Theorem 1 for $V_{3,N}^{(W)}$

We proceed in the same way as $V_{2,N}^{(W)}$. Let P_1 be a Poisson point process of intensity 1 in $\mathbf{R}^2$. Because $N^{1/2}P_N\overset{\mathcal{D}}{=}P_1$ and, by self-similarity property, $(W(x))_{x\in\mathbf{R}^2}\overset{\mathcal{D}}{=}\left(W(N^{-1/2}x)/N^{-\alpha/4}\right)_{x\in\mathbf{R}^2}$, the random variable $V_{3,N}^{(W)}$ has the same distribution as

$$V_{3,N}^{(W)\prime} = \frac{1}{\sqrt{|DT'_N|}} \times \sum_{(x_1,x_2,x_3)\in DT'_N} \left(\begin{pmatrix} U_{x_1,x_2}^{(W)} & U_{x_1,x_3}^{(W)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1,x_2}^{(W)} \\ U_{x_1,x_3}^{(W)} \end{pmatrix} - 2 \right),$$

where $DT'_N = DT_{1,\mathbf{C}_N}$ is the set of triples (x_1, x_2, x_3) of the Delaunay triangulation associated with P_1, such that $\Delta(x_1, x_2, x_3) \in$Del$(P_1)$, $x_1 \preceq x_2 \preceq x_3$, and $x_1 \in \mathbf{C}_N = (-N^{1/2}/2, N^{1/2}/2]^2$.

With

$$\tilde{U}_{x_1,x_2,x_3}^{(W)} = (1 - R_{x_1,x_2,x_3}^2)^{-1/2} \left(U_{x_1,x_2}^{(W)} - R_{x_1,x_2,x_3} U_{x_1,x_3}^{(W)} \right) \quad \text{and} \quad \tilde{U}_{x_1,x_3}^{(W)} = U_{x_1,x_3}^{(W)},$$

we have

$$V_{3,N}^{(W)\prime} = \frac{1}{\sqrt{|DT'_N|}} \sum_{(x_1,x_2,x_3)\in DT'_N} \left([(\tilde{U}_{x_1,x_2,x_3}^{(W)})^2 - 1] + [(\tilde{U}_{x_1,x_3}^{(W)})^2 - 1] \right).$$

The correlations between the terms appearing in the sum can be made explicit in the following sense. First, for any $(x_1, x_2, x_3) \in DT'_N$, we have

$$\text{corr}\left(\tilde{U}_{x_1,x_2,x_3}^{(W)}, \tilde{U}_{x_1,x_3}^{(W)} \right) = 0.$$

Second, for any $(x_1, x_2, x_3) \in DT'_N$ and $(x_4, x_5, x_6) \in DT'_N$, we have

- $$\text{corr}\left(\tilde{U}_{x_1,x_2,x_3}^{(W)}, \tilde{U}_{x_4,x_5,x_6}^{(W)} \right) = (1 - R_{x_1,x_2,x_3}^2)^{-1/2} (1 - R_{x_4,x_5,x_6}^2)^{-1/2} \times \left[\begin{array}{c} \text{corr}\left(U_{x_1,x_2}^{(W)}, U_{x_4,x_5}^{(W)} \right) - R_{x_4,x_5,x_6} \text{corr}\left(U_{x_1,x_2}^{(W)}, U_{x_4,x_6}^{(W)} \right) \\ -R_{x_1,x_2,x_3} \text{corr}\left(U_{x_1,x_3}^{(W)}, U_{x_4,x_5}^{(W)} + R_{x_1,x_2,x_3} R_{x_4,x_5,x_6} \text{corr}\left(U_{x_1,x_3}^{(W)}, U_{x_4,x_6}^{(W)} \right) \right) \end{array} \right]$$

- $$\text{corr}\left(\tilde{U}_{x_1,x_2,x_3}^{(W)}, \tilde{U}_{x_4,x_6}^{(W)} \right) = (1 - R_{x_1,x_2,x_3}^2)^{-1/2} \left[\text{corr}\left(U_{x_1,x_2}^{(W)}, U_{x_4,x_6}^{(W)} \right) - R_{x_1,x_2,x_3} \text{corr}\left(U_{x_1,x_3}^{(W)}, U_{x_4,x_6}^{(W)} \right) \right];$$

- $$\text{corr}\left(\tilde{U}_{x_1,x_3}^{(W)}, \tilde{U}_{x_4,x_5,x_6}^{(W)} \right) = (1 - R_{x_4,x_5,x_6}^2)^{-1/2} \left[\text{corr}\left(U_{x_1,x_3}^{(W)}, U_{x_4,x_5}^{(W)} \right) - R_{x_4,x_5,x_6} \text{corr}\left(U_{x_1,x_3}^{(W)}, U_{x_4,x_6}^{(W)} \right) \right];$$

- $$\text{corr}\left(\tilde{U}_{x_1,x_3}^{(W)}, \tilde{U}_{x_4,x_6}^{(W)} \right) = \text{corr}\left(U_{x_1,x_3}^{(W)}, U_{x_4,x_6}^{(W)} \right),$$

where the correlations appearing in the right-hand side of the equalities are given in Section 3.

Let $DT_{1,\mathbf{R}^2}$ be the set of triples (x_1, x_2, x_3) in P_1, such that $\Delta(x_1, x_2, x_3) \in \text{Del}(P_1)$ and $x_1 \preceq x_2 \preceq x_3$. Let $EDT_{1,\mathbf{R}^2} = \{((x_1, x_2, x_3), (x_1, x_3)) : (x_1, x_2, x_3) \in DT_{1,\mathbf{R}^2}\}$. Since there is a countable number of triangles, there exists a one-to-one function $\varphi : \mathbf{Z}\times\{1, 2\} \to EDT_{1,\mathbf{R}^2}$ such that, for any $(x_1, x_2, x_3) \in DT_{1,\mathbf{R}^2}$, there exists a unique $(k, i) \in \mathbf{Z}\times\{1, 2\}$ such that $\tilde{U}_{\varphi(k,1)}^{(W)} = \tilde{U}_{x_1,x_2,x_3}^{(W)}$ and $\tilde{U}_{\varphi(k,2)} = \tilde{U}_{x_2,x_3}$. For simplicity, the quantity $\tilde{U}^{(W)}_{\varphi(k,i)}$ is now denoted by $\tilde{U}^{(k,i)}$. Moreover we let $edt'_N = \{(k,i) \in \mathbf{Z} \times \{1,2\} : \varphi(k,i) \in EDT'_N\}$ where $EDT'_N = EDT_{1,\mathbf{C}_N}$ and remark that $|edt'_N| = |EDT'_N|$.

Given ω_P an element of Ω_P, according to Proposition 7.2.3 in [15], there exists a real separable Hilbert space $\mathfrak{H}$, as well as an isonormal Gaussian process over $\mathfrak{H}$, written $\{X(h) : h \in \mathfrak{H}\}$, with the property that there exists a set $E = \{\varepsilon_{k,i} : k \in \mathbf{Z}, i \in \{1,2\}\} \subset \mathfrak{H}$ such that *(i)* E generates $\mathfrak{H}$; *(ii)* $\langle \varepsilon_{k,i}, \varepsilon_{l,j}\rangle_{\mathfrak{H}} = \text{corr}\left(\tilde{U}^{(k,i)}, \tilde{U}^{(l,j)}\right)$ for every $k,l \in \mathbf{Z}, i,j \in \{1,2\}$; and *(iii)* $\tilde{U}^{(k,i)} = X(\varepsilon_{k,i})$ for every $k \in \mathbf{Z}, i \in \{1,2\}$. Let

$$f_{N,2} = \frac{1}{\sqrt{|edt'_N|}} \sum_{(k,i)\in edt'_N} \varepsilon_{k,i}^{\otimes 2}$$

so that $V^{(W)\prime}_{3,N} = I_2\left(f_{N,2}\right)$, where I_2 is the 2^{nd} multiple integral of $f_{N,2}$. The construction is therefore similar to the one that we used for $V^{(W)\prime}_{2,N}$.

As in the previous section, we need to introduce additional probability functions to characterize the asymptotic variance of $V^{(W)}_{3,N}$. For any distinct points $x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbf{R}^2$ and $N > 0$, we write

$$p_{3,N}(x_1,x_2,x_3,x_4,x_5,x_6) = \mathbb{P}\Big[\Delta(x_1,x_2,x_3), \Delta(x_4,x_5,x_6) \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_3,x_4,x_5,x_6\}), \\ \text{with } \ x_1 \preceq x_2 \preceq x_3 \text{ and } x_4 \preceq x_5 \preceq x_6\Big].$$

It is assumed that the two triangles $\Delta(x_1,x_2,x_3)$ and $\Delta(x_4,x_5,x_6)$ of the Delaunay triangulation $\mathrm{Del}(P_N \cup \{x_1,x_2,x_3,x_4,x_5,x_6\})$ have no points in common, i.e. x_1,x_2,x_3 and x_4,x_5,x_6 are distinct points. We also have to take into account the cases where they have one or two common points. Let

$$\mathcal{P}_{3,1} = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\},$$

resp.

$$\mathcal{P}_{3,2} = \{((4,1),(5,2)),((4,1),(5,3)),((4,1),(6,2)),((4,1),(6,3)),((4,2),(5,3)), \\ ((4,2),(6,3)),((5,1),(6,2)),((5,1),(6,3)),((5,2),(6,3))\}.$$

The above quantities deal with the couples (resp. the set of pairs of couples) for which $\Delta(x_1,x_2,x_3)$ and $\Delta(x_4,x_5,x_6)$ have one (resp. two) common points: i.e. for $(j,i) \in \mathcal{P}_{3,1}$, $x_j = x_i$ with $x_j \in \{x_4,x_5,x_6\}$ and $x_i \in \{x_1,x_2,x_3\}$ (resp. for $((j,i),(l,k)) \in \mathcal{P}_{3,2}$ $(i < k, j < l)$, $x_j = x_i$ with $x_j \in \{x_4,x_5,x_6\}$ and $x_i \in \{x_1,x_2,x_3\}$ and $x_l = x_k$ with $x_l \in \{x_4,x_5,x_6\}$ and $x_k \in \{x_1,x_2,x_3\}$). Then we define, for $(j,i) \in \mathcal{P}_{3,1}$,

$$q^{(j\leftrightarrow i)}_{3,N}(\vec{x}_{\{1:6\}\setminus\{j\}}) = \mathbb{P}\Big[\Delta(x_1,x_2,x_3), \Delta(x_4,x_5,x_6) \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_3,x_4,x_5,x_6\}), \\ \text{with } x_1 \preceq x_2 \preceq x_3, x_4 \preceq x_5 \preceq x_6 \text{ and } x_j = x_i\Big]$$

and, for $((j,i),(l,k)) \in \mathcal{P}_{3,2}$,

$$q^{(j\leftrightarrow i, l\leftrightarrow k)}_{3,N}(\vec{x}_{\{1:6\}\setminus\{j,l\}}) = \mathbb{P}\Big[\Delta(x_1,x_2,x_3), \Delta(x_4,x_5,x_6) \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_3,x_4,x_5,x_6\}), \\ \text{with } x_1 \preceq x_2 \preceq x_3, x_4 \preceq x_5 \preceq x_6 \text{ and } x_j = x_i, x_l = x_k\Big].$$

The following quantities are proportional to the correlations of the typical squared increments associated with pairs of Delaunay neighbors accordingly to the different cases. When there are no common points, we define

$$\sigma^2_{0,V_3,1} = \int_{(\mathbf{R}^2)^5} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_4,x_5,x_6}\right)\right)^2 p_{3,1}(x_1,x_2,x_3,x_4,x_5,x_6)\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{2:6\}},$$

$$\sigma^2_{0,V_3,2} = \int_{(\mathbf{R}^2)^5} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_4,x_6}\right)\right)^2 p_{3,1}(x_1,x_2,x_3,x_4,x_5,x_6)\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{2:6\}},$$

$$\sigma^2_{0,V_3,3} = \int_{(\mathbf{R}^2)^5} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_4,x_5,x_6}\right)\right)^2 p_{3,1}(x_1,x_2,x_3,x_4,x_5,x_6)\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{2:6\}},$$

$$\sigma^2_{0,V_3,4} = \int_{(\mathbf{R}^2)^5} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_4,x_6}\right)\right)^2 p_{3,1}(x_1,x_2,x_3,x_4,x_5,x_6)\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{2:6\}}.$$

When there is one common point, for $(j,i) \in \mathcal{P}_{3,1}$, we define

$$\sigma^2_{1,(j\leftrightarrow i),V_3,1} = \int_{(\mathbf{R}^2)^4} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_{(i)},\vec{x}_{(\{4:6\}\setminus\{j\})}}\right)\right)^2 q^{(j\leftrightarrow i)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j\}})\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j\}},$$

$$\begin{aligned}\sigma^2_{1,(j\leftrightarrow i),V_3,2} &= \mathbb{I}[j=5]\int_{(\mathbf{R}^2)^4} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_4,x_6}\right)\right)^2 q^{(j\leftrightarrow i)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j\}} \\ &+ \mathbb{I}\left[j\neq 5\right]\int_{(\mathbf{R}^2)^4} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_{(i)},\vec{x}_{(\{4:6\}\setminus\{j,5\})}}\right)\right)^2 q^{(j\leftrightarrow i)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j\}},\end{aligned}$$

$$\sigma^2_{1,(j\leftrightarrow i),V_3,3} = \int_{(\mathbf{R}^2)^4} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_{(i)},\vec{x}_{(\{4:6\}\setminus\{j\})}}\right)\right) q^{(j\leftrightarrow i)}_{3,N}(\vec{x}_{\{1:6\}\setminus\{j\}})\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j\}},$$

$$\begin{aligned}\sigma^2_{1,(j\leftrightarrow i),V_3,4} &= \mathbb{I}[j=5]\int_{(\mathbf{R}^2)^4} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_4,x_6}\right)\right)^2 q^{(j\leftrightarrow i)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j\}})\mathbb{I}[x_1=0]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j\}} \\ &+ \mathbb{I}\left[j\neq 5\right]\int_{(\mathbf{R}^2)^4} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_{(i)},\vec{x}_{(\{4:6\}\setminus\{j,5\})}}\right)\right)^2 q^{(j\leftrightarrow i)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j\}}.\end{aligned}$$

When there are two common points, for $((j,i),(l,k)) \in \mathcal{P}_{3,2}$, we define

$$\begin{aligned}\sigma^2_{2,(j\leftrightarrow i,l\leftrightarrow k),V_3,1} = \int_{(\mathbf{R}^2)^3} &\left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_{(i)},x_{(k)},\vec{x}_{(\{4:6\}\setminus\{j,l\})}}\right)\right)^2 \\ &\times q^{(j\leftrightarrow i,l\leftrightarrow k)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}}\end{aligned}$$

and

$$\begin{aligned}&\sigma^2_{2,(j\leftrightarrow i,l\leftrightarrow k),V_3,2} \\ &= \mathbb{I}\left[j=4,l=5\right]\int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_i,x_6}\right)\right)^2 q^{(j\leftrightarrow i,l\leftrightarrow k)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}} \\ &+ \mathbb{I}\left[j=4,l=6\right]\int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_i,x_k}\right)\right)^2 q^{(j\leftrightarrow i,l\leftrightarrow k)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}} \\ &+ \mathbb{I}\left[j=5,l=6\right]\int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_2,x_3}, \tilde{U}^{(W)}_{x_4,x_k}\right)\right)^2 q^{(j\leftrightarrow i,l\leftrightarrow k)}_{3,1}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}}\end{aligned}$$

and

$$
\begin{aligned}
&\sigma^2_{2,(j\leftrightarrow i,l\leftrightarrow k),V_3,3} \\
&\quad = \int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_{(i)},x_{(k)},\vec{x}_{(\{4:6\}\setminus\{j,l\})}}\right)\right)^2 q_{3,1}^{(j\leftrightarrow i,l\leftrightarrow k)}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sigma^2_{2,(j\leftrightarrow i,l\leftrightarrow k),V_3,4} \\
&\quad = \mathbb{I}\left[j=4,l=5\right]\int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_i,x_6}\right)\right)^2 q_{3,1}^{(j\leftrightarrow i,l\leftrightarrow k)}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}} \\
&\quad + \mathbb{I}\left[j=4,l=6\right]\int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_i,x_k}\right)\right)^2 q_{3,1}^{(j\leftrightarrow i,l\leftrightarrow k)}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}} \\
&\quad + \mathbb{I}\left[j=5,l=6\right]\int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{x_1,x_3}, \tilde{U}^{(W)}_{x_4,x_k}\right)\right)^2 q_{3,1}^{(j\leftrightarrow i,l\leftrightarrow k)}(\vec{x}_{\{1:6\}\setminus\{j,l\}})\mathbb{I}\left[x_1=0\right]\mathrm{d}\vec{x}_{\{1:6\}\setminus\{1,j,l\}}.
\end{aligned}
$$

For example, for $((j,i),(l,k)) = ((4,1),(5,2))$, we have

$$
\begin{aligned}
\sigma^2_{2,(4\leftrightarrow 1,5\leftrightarrow 2),V_3,1} &= \int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{0,x_2,x_3}, \tilde{U}^{(W)}_{0,x_2,x_6}\right)\right)^2 q_{3,1}^{(4\leftrightarrow 1,5\leftrightarrow 2)}(0,x_2,x_3,x_6)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_6 \\
\sigma^2_{2,(4\leftrightarrow 1,5\leftrightarrow 2),V_3,2} &= \int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{0,x_2,x_3}, \tilde{U}^{(W)}_{0,x_6}\right)\right)^2 q_{3,1}^{(4\leftrightarrow 1,5\leftrightarrow 2)}(0,x_2,x_3,x_6)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_6 \\
\sigma^2_{2,(4\leftrightarrow 1,5\leftrightarrow 2),V_3,3} &= \int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{0,x_3}, \tilde{U}^{(W)}_{0,x_2,x_6}\right)\right)^2 q_{3,1}^{(4\leftrightarrow 1,5\leftrightarrow 2)}(0,x_2,x_3,x_6)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_6 \\
\sigma^2_{2,(4\leftrightarrow 1,5\leftrightarrow 2),V_3,4} &= \int_{(\mathbf{R}^2)^3} \left(\operatorname{corr}\left(\tilde{U}^{(W)}_{0,x_3}, \tilde{U}^{(W)}_{0,x_6}\right)\right)^2 q_{3,1}^{(4\leftrightarrow 1,5\leftrightarrow 2)}(0,x_2,x_3,x_6)\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_6.
\end{aligned}
$$

The asymptotic variance of $V^{(W)}_{3,N}$ is then defined as

$$
\sigma^2_{V_3} = \left(\sum_{m=1}^{4}\sigma^2_{0,V_3,m} + \sum_{(j,i)\in\mathcal{P}_{3,1}}\sum_{m=1}^{4}\sigma^2_{1,(j\leftrightarrow i),V_3,m} + \sum_{((j,i),(l,k))\in\mathcal{P}_{3,2}}\sum_{m=1}^{4}\sigma^2_{2,(j\leftrightarrow i,l\leftrightarrow k),V_3,m}\right) + 4.
$$

In the same spirit as Lemma 3, we can show that $\sigma^2_{V_3}$ is finite. The CLT for $V^{(W)\prime}_{3,N}$ is then proved in the same way as for $V^{(W)\prime}_{2,N}$, by addressing the correlations given above.

5 Technical lemmas

In this section, we establish technical results which are useful to derive Theorem 1.

5.1 Asymptotic correlations between pairs of normalized increments

Let $x_1, x_2, x_3, x_4 \in \mathbf{R}^2$. The following lemma deals with the asymptotic behavior of

$$
\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right) = \frac{1}{\sigma^2\left(d_{1,2}d_{3,4}\right)^{\alpha/2}}\operatorname{cov}\left(W\left(x_2\right)-W\left(x_1\right), W\left(x_4\right)-W\left(x_3\right)\right)
$$

as the distance between x_1 and x_3 goes to infinity.

Lemma 6 *(i) Let $d_{1,2}$ and $d_{3,4}$ be fixed and let x_1, x_2, x_3, x_4 be such that $\|x_2 - x_1\| = d_{1,2}$ and $\|x_4 - x_3\| = d_{3,4}$. Then*

$$corr\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right) \underset{d\to\infty}{\sim} \frac{1}{2}(d_{1,2}d_{3,4})^{1-\alpha/2} d_{1,3}^{\alpha-2} \left(\cos\beta\cos\theta - (1-\alpha)\sin\beta\sin\theta\right)$$

where $d := d_{1,3} := \|x_3 - x_1\|$, $\theta = angle(\vec{u}, \overrightarrow{x_1x_2}) \in [0, 2\pi)$ and $\beta = angle(\vec{u}, \overrightarrow{x_3x_4}) \in [0, 2\pi)$ with $\vec{u}$ a vector orthogonal to $\overrightarrow{x_3x_1}$ such that $(\vec{u}, \overrightarrow{x_3x_1})$ is a direct coordinate system.

(ii) Let $\varepsilon \in (0, 1/2)$. Then, there exist two constants c and d_0 such that, for any $x_1, x_2, x_3, x_4 \in \mathbf{R}^2$ with $0 < ||x_4 - x_3|| \le ||x_2 - x_1|| \le ||x_3 - x_1||^{\varepsilon}$ and $||x_3 - x_1|| \ge d_0$, we have

$$|corr(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4})| \le c \cdot ||x_2 - x_1||^{2-\alpha} \cdot ||x_3 - x_1||^{\alpha-2}.$$

Proof of Lemma 6.

(i). Since

$$\begin{aligned}
&\operatorname{cov}\left(W(x_2) - W(x_1), W(x_4) - W(x_3)\right) \\
&\qquad = \operatorname{cov}\left(W(x_2) - W(x_1), W(x_4) - W(x_1)\right) - \operatorname{cov}\left(W(x_2) - W(x_1), W(x_3) - W(x_1)\right) \\
&\qquad = \frac{1}{2}\sigma^2\left(d_{1,2}^{\alpha} + d_{1,4}^{\alpha} - d_{2,4}^{\alpha} - \left(d_{1,2}^{\alpha} + d_{1,3}^{\alpha} - d_{2,3}^{\alpha}\right)\right) \\
&\qquad = \frac{1}{2}\sigma^2\left(\left(d_{1,4}^{\alpha} - d_{1,3}^{\alpha}\right) - \left(d_{2,4}^{\alpha} - d_{2,3}^{\alpha}\right)\right),
\end{aligned}$$

we have

$$\operatorname{corr}\left(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_3,x_4}\right) = \frac{1}{2\left(d_{1,2}d_{3,4}\right)^{\alpha/2}}\left[\left(d_{1,4}^{\alpha} - d_{1,3}^{\alpha}\right) - \left(d_{2,4}^{\alpha} - d_{2,3}^{\alpha}\right)\right].$$

Let $\ell_1 = \|x_1 - x_2\|$ and $\ell_2 = \|x_3 - x_4\|$ be fixed. Without loss of generality, we assume that the points have coordinates $x_1\,(0, d)$, $x_2\,(\ell_1\cos\theta, d + \ell_1\sin\theta)$, $x_3\,(0,0)$, $x_4\,(\ell_2\cos\beta, \ell_2\sin\beta)$ with $\theta, \beta \in [0, 2\pi)$. This gives

$$\begin{aligned}
d_{1,3} &= d \\
d_{1,4} &= \sqrt{(\ell_2\cos\beta)^2 + (d - \ell_2\sin\beta)^2} = \sqrt{\ell_2^2 - 2d\ell_2\sin\beta + d^2} \\
d_{2,3} &= \sqrt{(\ell_1\cos\theta)^2 + (d + \ell_1\sin\theta)^2} = \sqrt{\ell_1^2 + 2d\ell_1\sin\theta + d^2} \\
d_{2,4} &= \sqrt{(\ell_2\cos\beta - \ell_1\cos\theta)^2 + (d - \ell_2\sin\beta + \ell_1\sin\theta)^2} \\
&= \sqrt{\ell_1^2 + \ell_2^2 + 2\left(-\ell_2\cos\beta\ell_1\cos\theta - \ell_2\sin\beta\ell_1\sin\theta\right) + 2d\left(-\ell_2\sin\beta + \ell_1\sin\theta\right) + d^2}.
\end{aligned}$$

Now note that

$$\begin{aligned}
d_{1,4}^{\alpha} - d_{1,3}^{\alpha} &= d^{\alpha}\left(1 - \alpha\ell_2\sin\beta\frac{1}{d} + \frac{\alpha\ell_2^2}{2d^2} + \frac{\alpha(\alpha-2)}{8}\left(2\ell_2\sin\beta\right)^2\frac{1}{d^2} + o\left(d^{-2}\right) - 1\right) \\
&= d^{\alpha-1}\left(-\alpha\ell_2\sin\beta + \frac{\alpha}{2d}\left(\ell_2^2 + (\alpha-2)\left(\ell_2\sin\beta\right)^2\right) + o\left(d^{-1}\right)\right)
\end{aligned}$$

and

$$d_{2,4}^{\alpha}-d_{2,3}^{\alpha}=d^{\alpha}\Big(1+\alpha\left(-\ell_2\sin\beta+\ell_1\sin\theta\right)\frac{1}{d}+\frac{\alpha}{d^2}\left(\frac{\ell_1^2}{2}+\frac{\ell_2^2}{2}-\ell_2\cos\beta\ell_1\cos\theta-\ell_2\sin\beta\ell_1\sin\theta\right)$$
$$+\frac{\alpha\left(\alpha-2\right)}{8}\left(2\left(-\ell_2\sin\beta+\ell_1\sin\theta\right)\right)^2\frac{1}{d^2}+o\left(d^{-2}\right)\Big)$$
$$-d^{\alpha}\left(1+\alpha\ell_1\sin\theta\frac{1}{d}+\frac{\alpha\ell_1^2}{2d^2}+\frac{\alpha\left(\alpha-2\right)}{8}\left(2\ell_1\sin\theta\right)^2\frac{1}{d^2}+o\left(d^{-2}\right)\right).$$

Thus

$$d_{2,4}^{\alpha}-d_{2,3}^{\alpha}=-d^{\alpha-1}\Big(\alpha\ell_2\sin\beta+\frac{\alpha}{2d}\Big(\ell_1^2+\ell_2^2+2\left(-\ell_2\cos\beta\ell_1\cos\theta-\ell_2\sin\beta\ell_1\sin\theta\right)$$
$$+(\alpha-2)(-\ell_2\sin\beta+\ell_1\sin\theta)^2-\ell_1^2-(\alpha-2)\ell_1^2(\sin\theta)^2+o\left(d^{-2}\right)\Big)\Big).$$

This implies

$$\left[\left(d_{1,4}^{\alpha}-d_{1,3}^{\alpha}\right)-\left(d_{2,4}^{\alpha}-d_{2,3}^{\alpha}\right)\right]/\left(d^{\alpha-2}\frac{\alpha}{2}\right)=2\ell_1\ell_2(\cos\beta\cos\theta-(1-\alpha)\sin\beta\sin\theta)+o(1).$$

It follows that

$$\left(d_{1,4}^{\alpha}-d_{1,3}^{\alpha}\right)-\left(d_{2,4}^{\alpha}-d_{2,3}^{\alpha}\right)\underset{d\to\infty}{\sim}d^{\alpha-2}\alpha\ell_1\ell_2\left(\cos\beta\cos\theta-(1-\alpha)\sin\beta\sin\theta\right),$$

and therefore that

$$\operatorname{corr}\left(U_{x_1,x_2}^{(W)},U_{x_3,x_4}^{(W)}\right)\underset{d\to\infty}{\sim}\frac{1}{2}(d_{1,2}d_{3,4})^{1-\alpha/2}d^{\alpha-2}\left(\cos\beta\cos\theta-(1-\alpha)\sin\beta\sin\theta\right),$$

which concludes the proof of *(i)*.

(ii). We use the same notation as in *(i)*. First, we write $d_{1,4}=d^{\alpha}f(1/d)$, with

$$f(h)=\sqrt{\ell_2^2+2d\ell_2\sin\beta+d^2}.$$

Now, take $h=1/d$. Provided than d is large enough, we have $f(h)>0$. In particular, the function f admits a third derivative and, according to the Taylor-Lagrange inequality,

$$f(h)=1+\alpha\ell_2h\sin\beta+\frac{1}{2}\alpha\ell_2^2h^2+\frac{\alpha(\alpha-2)}{8}(2\ell_2\sin\beta)^2h^2+R_{\varepsilon}(h),$$

with

$$|R_{\varepsilon}(h)|\leq\frac{1}{6}\sup_{y\in[0,h]}\left|f^{(3)}(y)\right|h^3.$$

Moreover

$$f^{(3)}(y)=\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\left(\frac{\alpha}{2}-2\right)\left(1+2\ell_2\sin\beta y+\ell_2^2y^2\right)^{\frac{\alpha}{2}-3}(2\ell_2\sin\beta+2\ell_2^2y)^3$$
$$+4\alpha\left(\frac{\alpha}{2}-1\right)\left(1+2\ell_2\sin\beta y+\ell_2^2y^2\right)^{\frac{\alpha}{2}-2}(\ell_2\sin\beta+\ell_2^2y)\ell_2^2.$$

Taking $h\leq2^{-1/(1-\varepsilon)}$, we can prove that $1+2\ell_2\sin\beta y+\ell_2^2y^2\geq1-2h^{1-\varepsilon}$ and that $|\ell_2\sin\beta+\ell_2^2y|\leq\ell_2(1+h^{1-2\varepsilon})$ for any $y\in[0,h]$. When h is small enough, these inequalities implies $|R_{\varepsilon}(h)|\leq c\ell_2^3h^3$. Therefore

$$d_{1,4}^{\alpha}-d_{1,3}^{\alpha}=d^{\alpha-1}\left(\alpha\ell_2\sin\beta+\frac{\alpha}{2d}\left(\ell_2^2+(\alpha-2)(\ell_2\sin\beta)^2\right)\right)+R_{\varepsilon}^{(1)}(d),$$

with $\left|R_\varepsilon^{(1)}(d)\right| \leq c\ell_2^3 d^{\alpha-3}$. Proceeding in the same spirit as above, we can prove that the rest $R_\varepsilon^{(2)}(d)$ in the Taylor expansion of $d_{2,4}^\alpha - d_{2,3}^\alpha$ with order 3 is such that $\left|R_\varepsilon^{(2)}(d)\right| \leq c\ell_1^2\ell_2 d^{\alpha-3}$. Following the same lines as in *(i)*, this shows that

$$\operatorname{corr}(U_{x_1,x_2}^{(W)}, U_{x_3,x_4}^{(W)}) = \frac{1}{2}(\ell_1\ell_2)^{1-\alpha/2} d^{\alpha-2}(\cos\beta\cos\theta + (1-\alpha)\sin\beta\sin\theta) + \tilde{R}_\varepsilon(d),$$

with

$$|\tilde{R}_\varepsilon(d)| \leq c(\ell_1\ell_2)^{-\alpha/2}\ell_1^2\ell_2 d^{\alpha-3} \leq c\ell_1^{3-\alpha} d^{\alpha-3}.$$

This together with the fact that

$$\left|(\ell_1\ell_2)^{1-\alpha/2} d^{\alpha-2}(\cos\beta\cos\theta + (1-\alpha)\sin\beta\sin\theta)\right| \leq 2\ell_1^{2-\alpha} d^{\alpha-2}$$

concludes the proof of *(ii)*. □

5.2 Bounds for the density functions of Delaunay neighbors

Lemma 7 *(i) Let $x_1, x_2, x_3, x_4 \in \mathbf{R}^2$. Let $q_{2,N}^{(3\leftrightarrow 1)}(x_1, x_2, x_4)$, $q_{2,N}^{(3\leftrightarrow 2)}(x_1, x_2, x_4)$, $q_{2,N}^{(4\leftrightarrow 1)}(x_1, x_2, x_3)$ and $q_{2,N}^{(4\leftrightarrow 2)}(x_1, x_2, x_3)$ be as in Eq. (3.2a), (3.2b), (3.2c) and (3.2d) respectively.*

- *With $R = \max\{||x_2 - x_1||, ||x_4 - x_1||\}$,*

$$q_{2,N}^{(3\leftrightarrow 1)}(x_1, x_2, x_4) \leq \pi N\left(1 + \frac{4}{\pi N}\right) R^2 e^{-\frac{\pi}{4}NR^2}.$$

- *With $R = \max\{||x_2 - x_1||, ||x_4 - x_2||\}$,*

$$q_{2,N}^{(3\leftrightarrow 2)}(x_1, x_2, x_4) \leq \pi N\left(1 + \frac{4}{\pi N}\right) R^2 e^{-\frac{\pi}{4}NR^2}.$$

- *With $R = \max\{||x_2 - x_1||, ||x_3 - x_1||\}$, then*

$$q_{2,N}^{(4\leftrightarrow 1)}(x_1, x_2, x_3) \leq \pi N\left(1 + \frac{4}{\pi N}\right) R^2 e^{-\frac{\pi}{4}NR^2}.$$

- *With $R = \max\{||x_2 - x_1||, ||x_3 - x_2||\}$, then*

$$q_{2,N}^{(4\leftrightarrow 2)}(x_1, x_2, x_3) \leq \pi N\left(1 + \frac{4}{\pi N}\right) R^2 e^{-\frac{\pi}{4}NR^2}.$$

(ii) Let $x_1, x_2, x_3, x_4 \in \mathbf{R}^2$ and let $p_{2,N}(x_1, x_2, x_3, x_4)$ be as in Eq. (3.1). Assume that $||x_4 - x_3|| \leq ||x_2 - x_1||$. Then

$$p_{2,N}(x_1, x_2, x_3, x_4) \leq \pi N\left(1 + \frac{4}{\pi N}\right) ||x_2 - x_1||^2 e^{-\frac{\pi}{4}N||x_2-x_1||^2}.$$

Proof of Lemma 7.

We prove only *(ii)* as the proof of *(i)* follows similar steps. First, we notice that

$$
\begin{aligned}
p_{2,N}(x_1,x_2,x_3,x_4) &= \mathbb{P}\left[x_1 \sim x_2,\, x_3 \sim x_4 \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2,x_3,x_4\}) \text{ and } x_1 \preceq x_2,\, x_3 \preceq x_4\right] \\
&\leq \mathbb{P}\left[x_1 \sim x_2 \text{ in } \mathrm{Del}(P_N \cup \{x_1,x_2\})\right] \\
&= \mathbb{P}\left[\exists y \in P_N : \Delta(x_1,x_2,y) \in \mathrm{Del}(P_N \cup \{x_1,x_2\})\right] \\
&\leq \mathbb{E}\left[\sum_{y\in P_N} \mathbb{I}\left[P_N \cap B(x_1,x_2,y) = \emptyset\right]\right] \\
&= N\int_{\mathbf{R}^2} e^{-Na(B(x_1,x_2,y))}\mathrm{d}y,
\end{aligned}
$$

where $B(x_1,x_2,y)$ denotes the disk passing through x_1, x_2 and y and where $a(B)$ denotes the area of any Borel subset $B \subset \mathbf{R}^2$. Since the radius of this disk is larger than $\frac{1}{2}\max\{||x_2-x_1||, ||y-x_1||\}$, we get

$$
p_{2,N}(x_1,x_2,x_3,x_4) \leq N\int_{\mathbf{R}^2} e^{-\frac{1}{4}\pi N \max\{||x_2-x_1||,||y-x_1||\}^2}\mathrm{d}y.
$$

This gives

$$
\begin{aligned}
p_{2,N}(x_1,x_2,x_3,x_4) \leq N e^{-\frac{1}{4}\pi N ||x_2-x_1||^2} &\int_{\mathbf{R}^2} \mathbb{I}\left[||y-x_1|| \leq ||x_2-x_1||\right]\mathrm{d}y \\
&+ N\int_{\mathbf{R}^2} e^{-\frac{1}{4}\pi N||y-x_1||^2}\mathbb{I}\left[||y-x_1|| > ||x_2-x_1||\right]\mathrm{d}y,
\end{aligned}
$$

and consequently,

$$
p_{2,N}(x_1,x_2,x_3,x_4) \leq \pi N\left(1+\frac{4}{\pi N}\right)||x_2-x_1||^2 e^{-\frac{\pi}{4}N||x_2-x_1||^2}.
$$

□

5.3 Bounds for some exponential moments of a uniform distribution over a disc

Let $N > 0$ and R be a positive random variable with probability distribution function given by

$$
\mathbb{P}[R \leq r] = \begin{cases} \dfrac{r^2}{N}, & \text{if } 0 \leq r \leq \sqrt{N} \\ 1 & \text{if } r > N \end{cases} \tag{5.1}
$$

Lemma 8 *Let $0 < \alpha < 1$ and $d_0 > 0$. There exist two constants c_1 and c_2 (only depending on α and d_0) such that, for large N,*

$$
\mathbb{E}\left[\exp(R^{\alpha-2}\mathbb{I}[R \geq d_0])\right] \leq 1 + c_1(\sqrt{N})^{\alpha-2} + c_2 N^{-1}.
$$

Proof of Lemma 8. We have

$$\mathbb{E}\left[\exp(R^{\alpha-2}\mathbb{I}[R\geq d_0])\right] = \int_{d_0/\sqrt{N}}^1 \exp((r\sqrt{N})^{\alpha-2})2r\mathrm{d}r$$
$$\leq 1 + 2\int_{d_0/\sqrt{N}}^1 (r\sqrt{N})^{\alpha-2} r\mathrm{d}r + 2\int_0^1 \sum_{k=2}^{\infty} \frac{(r\sqrt{N})^{k(\alpha-2)}}{k!} r\mathbb{I}[r\geq d_0/\sqrt{N}]\mathrm{d}r.$$

The first integral is lower than $\frac{2}{\alpha}(\sqrt{N})^{\alpha-2}$. For the second one, we write

$$\int_0^1 \sum_{k=2}^{\infty} \frac{(r\sqrt{N})^{k(\alpha-2)}}{k!} r\mathbb{I}[r\geq d_0/\sqrt{N}] \leq (\sqrt{N})^{2(\alpha-2)} \int_0^1 \sum_{k=0}^{\infty} r^{2(\alpha-2)} \frac{(r\sqrt{N})^{k(\alpha-2)}}{k!} r\mathbb{I}[r\geq d_0/\sqrt{N}]\mathrm{d}r$$
$$= (\sqrt{N})^{2(\alpha-2)} \int_0^1 r^{2\alpha-3} \exp\left((r\sqrt{N})^{\alpha-2}\right)\mathbb{I}[r\geq d_0/\sqrt{N}]\mathrm{d}r$$
$$\leq \exp(d_0^{\alpha-2})(\sqrt{N})^{2(\alpha-2)} \int_0^1 r^{2\alpha-3}\mathbb{I}[r\geq d_0/\sqrt{N}]\mathrm{d}r$$
$$= \frac{\exp(d_0^{\alpha-2})d_0^{2(\alpha-1)}}{2(1-\alpha)} N^{-1}.$$

This proves Lemma 8 with $c_1 = \frac{2}{\alpha}$ and $c_2 = \frac{\exp(d_0^{\alpha-2})}{(1-\alpha)} d_0^{2(\alpha-1)}$. □

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