Limit theorems for squared increment sums of the maximum of two isotropic fractional Brownian fields over a fixed-domain

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February 14, 2025

Abstract

The pointwise maximum of two independent and identically distributed isotropic fractional Brownian fields (with Hurst parameter $H < 1/2$) is observed in a family of points in the unit square $\mathbf{C} = (-1/2, 1/2]^2$. We assume that these points come from the realization of a homogeneous Poisson point process with intensity N. We consider normalized increments (resp. pairs of increments) along the edges of the Delaunay triangulation generated by the Poisson point process (resp. pairs of edges within triangles). We investigate the asymptotic behaviors of the squared increment sums as $N \to \infty$. We show that the normalizations differ from the case of a unique isotropic fractional Brownian field as obtained in [3] and that the sums converge to the local time of the difference of the two isotropic fractional Brownian fields up to constant factors.

Keywords: Isotropic fractional Brownian fields, Pointwise maximum, Squared increment sums, Poisson point process, Delaunay triangulation.

AMS (2020): 60F05, 60G22, 60D05, 60G55.

1 Introduction

In this paper we choose the same theoretical framework as we considered in [3] where an isotropic fractional Brownian field (with Hurst parameter $H < 1/2$) was observed in a family of points in the unit square $\mathbf{C} = (-1/2, 1/2]^2$. These points are assumed to come from a realization of a homogeneous Poisson point process with intensity N. In [3] we provided central limit theorems for two types of centered squared increment sums as $N \to \infty$. We considered either normalized increments along the edges of the Delaunay triangulation generated by the Poisson point process or the normalized pairs of increments

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along the pairs of edges within triangles. We established that the centered squared increment sums (properly normalized) converge in law to centered Gaussian distributions whose asymptotic variances may be characterized by explicit (although intricate) integrals. The rates of convergences of both sums are the same as in Theorem 3.2 of [1] or in Theorem 1 of [10] where statistics based on square increments on regular grids have been considered.

Instead, we now consider the pointwise maximum of two independent isotropic fractional Brownian fields and study the asymptotic behaviors of centered squared increment sums. The reason arises from a statistical problem, specifically inference for the parameters of a max-stable field, which is based on fractional Brownian fields. In [2], we construct composite maximum likelihood estimators based on pairs and triples to estimate the parameters of such a field. The pairs and triples are selected from the Delaunay triangulation. Choosing such a triangulation is natural because it is the most regular in the sense that its minimal angle is greater than the minimal angle of any other triangulation. The study of the asymptotic behavior of the estimators requires in particular establishing limit theorems for some squared increment sums of the pointwise maximum of two independent isotropic fractional Brownian fields, which is the focus of this article. Assuming that the points used to construct our increments are based on a point process rather than a deterministic grid is, in a certain sense, natural. Indeed, in terms of inference, this means that our statistics are observed at random nodes. Moreover we chose the Poisson point process because it is more natural and has a formula (the Slivnyak-Mecke formula) that allows for the explicit calculation of expectations.

The contribution of our article compared to [3] is twofold. On the one hand, we show that the normalizations differ from the case of a single isotropic fractional Brownian field. On the other hand, we establish that the convergence is no longer in distribution but in probability toward the local time of the difference of the two isotropic fractional Brownian fields, up to a constant factor. Such a result extends Proposition 2 in [8] where an asymptotic theory for sums of powers of absolute increments of Brown-Resnick max-stable processes whose spectral processes are continuous exponential martingales was developed. Our proof methodology is inspired by the approach developed in [7] for a fractional Brownian motion and some results in [6].

Our paper is organized as follows. In Section 2, we recall some known results concerning the local time of the difference of two isotropic fractional Brownian fields and the Poisson-Delaunay triangulation. In Section 3, we present our main results by first defining the normalized squared increment sums and then characterizing their asymptotic behavior as the intensity of the Poisson point process converges to infinity. The proofs are deferred to Section 4. In the appendix, we provide some intermediary technical results that are used in Section 4.

2 Preliminaries

In this section, we recall some known results on local times and Poisson Delaunay triangulation.

2.1 Local time of the difference of two isotropic fractional Brownian fields

An isotropic fractional Brownian field (see e.g. Section 3.3 in [4]) is a centered Gaussian random field such that $W(0) = 0$ a.s. and

$$\operatorname{cov}(W(x), W(y)) = \frac{\sigma^2}{2}\left(||x||^{2H} + ||y||^{2H} - ||y-x||^{2H}\right), \tag{2.1}$$

for some $H \in (0,1)$ and $\sigma^2 > 0$, with $\|x\|$ the Euclidean norm of $x \in \mathbf{R}^2$. The parameter σ is called the *scale parameter* while H is known as the *Hurst parameter* and relates to the Hölder continuity exponent of W. It is a self-similar random field with linear stationary increments in the sense that the law of

$$(W(x + x_0) - W(x_0))_{x \in \mathbf{R}^2}$$

does not depend on the choice of $x_0 \in \mathbf{R}^2$.

Let $W^{(1)}$ and $W^{(2)}$ be two independent and identically distributed fractional Brownian random fields in $\mathbf{R}^2$, with covariance given by (2.1), and let

$$W^{(2\backslash 1)}(x) = W^{(2)}(x) - W^{(1)}(x), \quad x \in \mathbf{R}^2.$$

To measure the portion of the square $\mathbf{C} = (-1/2, 1/2]^2$ for which the two random fields coincide, the notion of local time is introduced as follows. Let $\nu^{(2\backslash 1)}$ be the *occupation measure* of $W^{(2\backslash 1)}$ over $\mathbf{C}$, i.e.

$$\nu^{(2\backslash 1)}(A) = \int_{\mathbf{C}} \mathbb{I}\left[W^{(2\backslash 1)}(x) \in A\right] \mathrm{d}x,$$

for any Borel measurable set $A \subset \mathbf{R}$. Observe that, for any $s, t \in [0,1]^2$,

$$\Delta(s,t) := \mathbb{E}\left[(W^{(2\backslash 1)}(s) - W^{(2\backslash 1)}(t))^2\right] = 2\sigma^2 \|s-t\|^{2H}.$$

Because $\int_{\mathbf{C}} (\Delta(s,t))^{-1/2} \mathrm{d}s$ is finite for all $t \in \mathbf{C}$, it follows from Section 22 in [5] that the occupation measure $\nu^{(2\backslash 1)}$ admits a Lebesgue density. The *local time* at level ℓ is then defined as

$$L_{W^{(2\backslash 1)}}(\ell) := \frac{d\nu^{(2\backslash 1)}}{d\ell}(\ell).$$

An immediate consequence of the existence of the local time is the occupation time formula, which states that

$$\int_{\mathbf{C}} g(W^{(2\backslash 1)}(x)) \mathrm{d}x = \int_{\mathbf{R}} g(\ell) L_{W^{(2\backslash 1)}}(\ell) \mathrm{d}\ell$$

for any Borel function g on $\mathbf{R}$. Adapting the proof of Lemma 1.1 in [6], we can easily show that, for any $\ell \in \mathbf{R}$,

$$L_{W^{(2\backslash 1)}}(\ell) = \lim_{\varepsilon \to 0} \int_{\mathbf{C}} \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{1}{2\varepsilon}\left(W^{(2\backslash 1)}(x) - \ell\right)^2\right) \mathrm{d}x$$

or

$$L_{W^{(2\backslash 1)}}(\ell) = \frac{1}{2\pi} \lim_{M \to \infty} \int_{-[M,M]} \int_{\mathbf{R}} e^{\mathrm{i}\xi(W^{(2\backslash 1)}(x) - \ell)} \mathrm{d}x \mathrm{d}\xi, \tag{2.2}$$

where the limits hold in L^2.

2.2 Poisson-Delaunay graph

Let P_N be a Poisson point process with intensity N in $\mathbf{R}^2$. The *Delaunay graph* $\mathrm{Del}(P_N)$ is the unique triangulation with vertices in P_N such that the circumball of each triangle contains no point of P in its interior, see e.g. p. 478 in [9]. Such a triangulation is the most regular one in the sense that it is the one which maximizes the minimum angle among all triangles.

To define the mean behavior of the Delaunay graph (associated with a Poisson point process P_1 of intensity 1), the notion of typical cell is defined as follows. With each cell $C \in \mathrm{Del}(P_1)$, we associate the circumcenter $z(C)$ of C. Now, let $\mathbf{B}$ be a Borel subset in $\mathbf{R}^2$ with area $a(\mathbf{B}) \in (0, \infty)$. The *cell intensity*

β_2 of $\mathrm{Del}(P_1)$ is defined as the mean number of cells per unit area, i.e.

$$\beta_2 = \frac{1}{a(\mathbf{B})}\mathbb{E}\left[|\{C \in \mathrm{Del}(P_1) : z(C) \in \mathbf{B}\}|\right],$$

where $|\cdot|$ denotes the cardinality. It is well-known that $\beta_2 = 2$, see e.g. Theorem 10.2.9. in [9]. Then, we define the *typical cell* as a random triangle $\mathcal{C}$ with distribution given as follows: for any positive measurable and translation invariant function $g : \mathcal{K}_2 \to \mathbf{R}$,

$$\mathbb{E}\left[g(\mathcal{C})\right] = \frac{1}{\beta_2 a(\mathbf{B})}\mathbb{E}\left[\sum_{C\in\mathrm{Del}(P_1):z(C)\in\mathbf{B}} g(C)\right],$$

where $\mathcal{K}_2$ denotes the set of convex compact subsets in $\mathbf{R}^2$, endowed with the Fell topology (see Section 12.2 in [9] for the definition). The distribution of $\mathcal{C}$ has the following integral representation (see e.g. Theorem 10.4.4. in [9]):

$$\mathbb{E}\left[g(\mathcal{C})\right] = \frac{1}{6}\int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,u_3)) g(\Delta(ru_1,ru_2,ru_3))\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\sigma(\mathrm{d}u_3)\mathrm{d}r, \tag{2.3}$$

where $\mathbf{S}^1$ is the unit sphere of $\mathbf{R}^2$ and σ is the spherical Lebesgue measure on $\mathbf{S}^1$ with normalization $\sigma\left(\mathbf{S}^1\right) = 2\pi$. It means that $\mathcal{C}$ is equal in distribution to $R\Delta(U_1,U_2,U_3)$, where R and (U_1,U_2,U_3) are independent with probability density functions given respectively by $2\pi^2 r^3 e^{-\pi r^2}$ and $a(\Delta(u_1,u_2,u_3))/(12\pi^2)$.

Similarly, we can define the concept of typical edge. The *edge intensity* β_1 of $\mathrm{Del}(P_1)$ is defined as the mean number of edges per unit area and is equal to $\beta_1 = 3$ (see e.g. Theorem 10.2.9. in [9]). The distribution of the length of the *typical edge* is the same as the distribution of $D = R||U_1 - U_2||$. Its probability density function f_D satisfies the following equality

$$\begin{aligned}\mathbb{P}\left[D \leq \ell\right] &= \int_0^\ell f_D(d)\mathrm{d}d \\ &= \frac{\pi}{3}\int_0^\infty \int_{(\mathbf{S}^1)^2} r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,e_1))\mathbb{I}\left[r\left\|u_1-u_2\right\| \leq \ell\right]\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\mathrm{d}r,\end{aligned} \tag{2.4}$$

where $e_1 = (1,0)$ and $\ell > 0$. Following Eq. (2.3), a *typical couple* of (distinct) Delaunay edges with a common vertex can be defined as a 3-tuple of random variables (D_1,D_2,Θ), where $D_1, D_2 \geq 0$ and $\Theta \in [-\frac{\pi}{2},\frac{\pi}{2})$, with distribution given by

$$\begin{aligned}\mathbb{P}[(D_1,D_2,\Theta)\in B] = \frac{1}{6}\int_0^\infty \int_{(\mathbf{S}^1)^3} & r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,u_3)) \\ & \times \mathbb{I}[(r||u_3-u_2||, r||u_2-u_1||, \arcsin\left(\cos(\theta_{u_1,u_2}/2)\right)) \in B]\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\sigma(\mathrm{d}u_3)\mathrm{d}r,\end{aligned}$$

where θ_{u_1,u_2} is the measure of the angle (u_1,u_2) and where B is any Borel subset in $\mathbf{R}_+^2 \times [-\frac{\pi}{2},\frac{\pi}{2})$. The random variables D_1, D_2 (resp. Θ) can be interpreted as the lengths of the two typical edges (resp. as the angle between the edges). In particular, the length of a typical edge is equal in distribution to $D = R||U_2 - U_1||$ with distribution given in Eq. (2.4).

Throughout the paper, we identify $\mathrm{Del}(P_1)$ to its skeleton. When $x_1, x_2 \in P_N$ are Delaunay neighbors, we write $x_1 \sim x_2$ in $\mathrm{Del}(P_N)$. For a Borel subset $\mathbf{B}$ in $\mathbf{R}^2$, we denote by $E_{N,\mathbf{B}}$ the set of couples (x_1,x_2)

such that the following conditions hold:

$$x_1 \sim x_2 \text{ in } \mathrm{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2,$$

where $\preceq$ denotes the lexicographic order. When $\mathbf{B} = \mathbf{C}$, with $\mathbf{C} := (-1/2, 1/2]^2$, we only write $E_N = E_{N,\mathbf{C}}$. For a Borel subset $\mathbf{B}$ in $\mathbf{R}^2$, let $DT_{N,\mathbf{B}}$ be the set of triples (x_1, x_2, x_3) satisfying the following properties

$$\Delta(x_1, x_2, x_3) \in \mathrm{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2 \preceq x_3,$$

where $\Delta(x_1, x_2, x_3)$ is the convex hull of (x_1, x_2, x_3). When $\mathbf{B} = \mathbf{C}$, we let $DT_N = DT_{N,\mathbf{C}}$.

3 Main results

Throughout the paper, for the sake of clarity, we let $\alpha := 2H$ and assume that $\alpha \in (0, 1)$ to remain within the same framework as in [3].

Notation Let us recall that $W^{(1)}$ and $W^{(2)}$ are two independent and identically distributed fractional Brownian random fields, with covariance function given by (2.1), and let $W_\vee$ be the pointwise maximum, i.e.

$$W_\vee(x) = W^{(1)}(x) \vee W^{(2)}(x), \quad x \in \mathbf{R}^2.$$

Similarly to [3], for two distinct points $x_1, x_2 \in \mathbf{R}^2$, we denote by

$$U^{(W_\vee)}_{x_1,x_2} = \sigma^{-1} d_{1,2}^{-\alpha/2} \left(W_\vee(x_2) - W_\vee(x_1)\right)$$

the *normalized* increment between x_1 and x_2 w.r.t. $W_\vee$, where $d_{1,2} := ||x_2 - x_1||$. Note that $\sigma d_{1,2}^{\alpha/2}$ is the standard error of $\left(W^{(i)}(x_2) - W^{(i)}(x_1)\right)$ for $i = 1, 2$, but not of $(W_\vee(x_2) - W_\vee(x_1))$. This choice is however motivated by the fact that as $d_{1,2}$ tends to 0, the pointwise maximum $W_\vee$ in x_1 and x_2 is either equal to $W^{(1)}$ in x_1 and x_2, or is equal to $W^{(2)}$ in x_1 and x_2. We also introduce as in [3] two types of normalized sums of squared increments

$$V^{(W_\vee)}_{2,N} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \left((U^{(W_\vee)}_{x_1,x_2})^2 - 1 \right)$$

and

$$V^{(W_\vee)}_{3,N} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \left(\begin{pmatrix} U^{(W_\vee)}_{x_1,x_2} & U^{(W_\vee)}_{x_1,x_3} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U^{(W_\vee)}_{x_1,x_2} \\ U^{(W_\vee)}_{x_1,x_3} \end{pmatrix} - 2 \right),$$

where

$$R_{x_1,x_2,x_3} = \mathrm{corr}(U^{(W)}_{x_1,x_2}, U^{(W)}_{x_1,x_3}) = \frac{d^\alpha_{1,2} + d^\alpha_{1,3} - d^\alpha_{2,3}}{2\left(d_{1,2}d_{1,3}\right)^H} \tag{3.1}$$

and $d_{i,j} = ||x_j - x_i||$, $i \neq j$.

The main result of our paper deals with the asymptotic behaviors of the squared increment sums. To state it, we first provide a decomposition of these sums.

Decomposition of the squared increment sums Recall that we denote the difference between the fractional Brownian fields by $W^{(2\backslash 1)}(x) = W^{(2)}(x) - W^{(1)}(x)$ for any $x \in \mathbf{R}^2$.

Similarly to Section 5.1 in [8], we observe that, for any real measurable function $f : \mathbf{R} \to \mathbf{R}$ and for any $(x_1, x_2) \in E_N$,

$$f(U^{(W_\vee)}_{x_1,x_2}) = f(U^{(1)}_{x_1,x_2})\mathbb{I}[W^{(2\backslash 1)}(x_1) < 0] + f(U^{(2)}_{x_1,x_2})\mathbb{I}[W^{(2\backslash 1)}(x_1) > 0] + \Psi_f\left(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, W^{(2\backslash 1)}(x_1)/(\sigma d_{1,2}^{\alpha/2})\right), \quad (3.2)$$

where

$$U^{(1)}_{x_1,x_2} = \frac{1}{\sigma d_{1,2}^{\alpha/2}}\left(W^{(1)}(x_2) - W^{(1)}(x_1)\right), \quad U^{(2)}_{x_1,x_2} = \frac{1}{\sigma d_{1,2}^{\alpha/2}}\left(W^{(2)}(x_2) - W^{(2)}(x_1)\right)$$

and

$$\Psi_f\,(x, y, w) = (f(y + w) - f(x))\mathbb{I}\,[x - y \le w \le 0] + (f(x - w) - f(y))\mathbb{I}\,[0 \le w \le x - y]\,.$$

Now, let H_2 be the Hermite polynomial with degree 2, i.e. $H_2(u) = u^2 - 1$ for all $u \in \mathbf{R}$. Taking $f = H_2$ in the above decomposition, we get

$$V^{(W_\vee)}_{2,N} = V^{(1)}_{2,N} + V^{(2)}_{2,N} + V^{(2/1)}_{2,N}, \quad (3.3)$$

where

$$V^{(1)}_{2,N} = \frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N, W^{(2\backslash 1)}(x_1)<0}\left((U^{(1)}_{x_1,x_2})^2 - 1\right)$$
$$V^{(2)}_{2,N} = \frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N, W^{(2\backslash 1)}(x_1)>0}\left((U^{(2)}_{x_1,x_2})^2 - 1\right)$$
$$V^{(2/1)}_{2,N} = \frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N}\Psi_{H_2}(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, W^{(2\backslash 1)}\,(x_1)\,/(\sigma d_{1,2}^{\alpha/2})).$$

To obtain a similar decomposition for the triples, let us denote, for $-1 < R < 1$, by Ω the following function

$$\begin{aligned}\Omega(u_1, v_1, u_2, v_2, w_1, w_2; R) = {} & \frac{1}{1 - R^2}\left[\Psi_{H_2}\,(u_1, v_1, w_1) + \Psi_{H_2}\,(u_2, v_2, w_2)\right] \\ & - 2\frac{R}{1 - R^2}\Psi_I\,(u_1, v_1, w_1)\,\Psi_I\,(u_2, v_2, w_2) \\ & -2\frac{R}{1 - R^2}\left[u_1\Psi_I\,(u_2, v_2, w_2) + u_2\Psi_I\,(u_1, v_1, w_1)\right]\mathbb{I}[w_1 < 0] \\ & -2\frac{R}{1 - R^2}\left[v_1\Psi_I\,(u_2, v_2, w_2) + v_2\Psi_I\,(u_1, v_1, w_1)\right]\mathbb{I}[w_1 > 0]\end{aligned} \quad (3.4)$$

with $I\,(u) = u$ for all $u \in \mathbf{R}$. Then we have

$$V^{(W_\vee)}_{3,N} = V^{(1)}_{3,N} + V^{(2)}_{3,N} + V^{(2/1)}_{3,N}, \quad (3.5)$$

where

$$V^{(1)}_{3,N} = \frac{1}{\sqrt{|DT_N|}}\sum_{\substack{(x_1,x_2,x_3)\in DT_N,\\ W^{(2\backslash 1)}(x_1)<0}}\left(\begin{pmatrix} U^{(1)}_{x_1,x_2} & U^{(1)}_{x_1,x_3}\end{pmatrix}\begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1\end{pmatrix}^{-1}\begin{pmatrix} U^{(1)}_{x_1,x_2} \\ U^{(1)}_{x_1,x_3}\end{pmatrix} - 2\right)$$

$$V_{3,N}^{(2)} = \frac{1}{\sqrt{|DT_N|}} \sum_{\substack{(x_1,x_2,x_3)\in DT_N,\\ W^{(2\backslash 1)}(x_1)>0}} \left(\begin{pmatrix} U_{x_1,x_2}^{(2)} & U_{x_1,x_3}^{(2)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1,x_2}^{(2)} \\ U_{x_1,x_3}^{(2)} \end{pmatrix} - 2 \right)$$

$$V_{3,N}^{(2/1)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \Omega\left(U_{x_1,x_2}^{(1)}, U_{x_1,x_3}^{(1)}, U_{x_1,x_2}^{(2)}, U_{x_1,x_3}^{(2)}, \frac{W^{(2\backslash 1)}(x_1)}{\sigma d_{1,2}^{\alpha/2}}, \frac{W^{(2\backslash 1)}(x_1)}{\sigma d_{1,3}^{\alpha/2}}; R_{x_1,x_2,x_3} \right).$$

Asymptotic behaviors of $V_{2,N}^{(W_\vee)}$ and $V_{3,N}^{(W_\vee)}$ To state the main result of this section, we need to introduce some additional notation. In the following, we denote by F_2 the function defined, for any $z \in \mathbf{R}$, by

$$F_2(z) = \int_{\mathbf{R}^2\times\mathbf{R}_+} \Psi_{H_2}(x,y,z/d^{\alpha/2}) \frac{1}{2\pi} e^{-(x^2+y^2)/2} f_D\left(d\right) \mathrm{d}x\mathrm{d}y\mathrm{d}d,$$

where f_D is the density function of the length of the typical edge defined in Eq. (2.4). We also define

$$\begin{aligned} F_3(z) = \int_{\mathbf{R}^4\times(\mathbf{R}_+)^3} & \Omega(x_1,y_1,x_2,y_2,z/d_1^{\alpha/2}, z/d_3^{\alpha/2}; R(d_1,d_2,d_3)) \\ & \times \varphi_2\left(x_1,y_1; R(d_1,d_2,d_3)\right) \varphi_2\left(x_2,y_2; R(d_1,d_2,d_3)\right) \\ & \times f_{D_1,D_2,D_3}\left(d_1,d_2,d_3\right) \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}d_1\mathrm{d}d_2\mathrm{d}d_3, \end{aligned}$$

where

$$\begin{aligned} \varphi_2\left(x,y;R\right) &= \frac{1}{2\pi}\frac{1}{(1-R^2)} \exp\left(-\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right), \\ R(d_1,d_2,d_3) &= \frac{d_1^\alpha + d_3^\alpha - d_2^\alpha}{2\left(d_1 d_3\right)^{\alpha/2}}, \end{aligned}$$

and where f_{D_1,D_2,D_3} is the density function of the edge lengths of the typical Delaunay triangle $\mathcal{C}$. Moreover, we write

$$c_{V_2} = \int_{\mathbf{R}} F_2(z)\mathrm{d}z \quad \text{and} \quad c_{V_3} = \int_{\mathbf{R}} F_3(z)\mathrm{d}z.$$

The following proposition deals with the asymptotic behaviors of $V_{2,N}^{(2/1)}$ and $V_{3,N}^{(2/1)}$.

Proposition 1 *Let $W^{(1)}$ and $W^{(2)}$ be two independent and identically distributed fractional Brownian random fields, with covariance given by (2.1), with $\sigma^2 > 0$ and $\alpha \in (0,1)$. Then, as $N \to \infty$,*

$$\begin{aligned} \frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V_{2,N}^{(2/1)} &\xrightarrow{\mathbb{P}} c_{V_2} L_{W^{(2\backslash 1)}}(0), \\ \frac{\sqrt{2}}{2} N^{-(2-\alpha)/4} V_{3,N}^{(2/1)} &\xrightarrow{\mathbb{P}} c_{V_3} L_{W^{(2\backslash 1)}}(0). \end{aligned}$$

In the above proposition, the expression $\xrightarrow{\mathbb{P}}$ denotes the convergence in probability as N goes to infinity. Notice that the factors $\sqrt{3}/3$ and $\sqrt{2}/2$ come from the facts that $|E_N|/N \xrightarrow{a.s.} 3$ and $|DT_N|/N \xrightarrow{a.s.} 2$ as $N \to \infty$, respectively (see p.9 in [3] and Theorem 10.2.9 in [9]). An adaptation of the proof of Theorem 1 in [3] shows that, for $\alpha \in (0,1)$, as $N \to \infty$,

$$V_{2,N}^{(1)} + V_{2,N}^{(2)} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_{V_2}^2\right) \tag{3.6}$$

and

$$V_{3,N}^{(1)} + V_{3,N}^{(2)} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_{V_3}^2\right), \tag{3.7}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution as N goes to infinity.

As a direct consequence, we obtain from Proposition 1 and Eq. (3.3) and (3.5) the following theorem.

Theorem 2 *Under the same assumptions as in Proposition 1, as $N \to \infty$,*

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V_{2,N}^{(W_\vee)} \xrightarrow{\mathbb{P}} c_{V_2} L_{W^{(2\setminus 1)}}(0),$$
$$\frac{\sqrt{2}}{2} N^{-(2-\alpha)/4} V_{3,N}^{(W_\vee)} \xrightarrow{\mathbb{P}} c_{V_3} L_{W^{(2\setminus 1)}}(0).$$

The rates of convergence of $V_{2,N}^{(W_\vee)}$ and $V_{3,N}^{(W_\vee)}$ differ from those of $V_{2,N}^{(W)}$ and $V_{3,N}^{(W)}$. Indeed, the sums of square increments in $V_{2,N}^{(2/1)}$ and $V_{3,N}^{(2/1)}$ are the dominant terms. As a first observation, it could appear as surprising since it means that the dominant terms only concern Delaunay neighbors, say x_1, x_2, for which the maximal trajectories are different, i.e. such that $x_1 \in C_1$ and $x_2 \in C_2$ simultaneously, or such that $x_1 \in C_2$ and $x_2 \in C_1$ simultaneously, where C_i, $i \in \{1, 2\}$, denotes the cell

$$C_i := \{x \in \mathbf{R^2} : W_\vee(x) = W_i(x)\}.$$

In particular, these edges have to intersect the boundary of the cells, i.e. $C_1 \cap C_2$. But the Hausdorff dimension of $C_1 \cap C_2$ is $2 - \alpha/2$, which is strictly larger than 1. Moreover, on the boundaries of the cells, the random field $W_\vee$ is very irregular, which provides in particular large increments. Roughly, this explains why $V_{2,N}^{(2/1)}$ and $V_{3,N}^{(W_\vee)}$ are the dominant terms and why they reveal the local time at level 0.

4 Proofs

4.1 Proof of Proposition 1

4.1.1 Proof for $V_{2,N}^{(2/1)}$

Without loss of generality, we assume that $\sigma = 1$. Let, for any measurable function $f : \mathbf{R} \to \mathbf{R}$,

$$G_N^{(2\setminus 1)}[f] = \frac{1}{3} N^{\alpha/4-1} \sum_{(x_1,x_2)\in E_N} \Psi_f(U_{x_1,x_2}^{(1)}, U_{x_1,x_2}^{(2)}, W^{(2\setminus 1)}(x_1)/d_{1,2}^{\alpha/2})$$

and

$$G_{N,*}^{(2/1)}[f] = N^{\alpha/4-1} \sum_{x \in P_N \cap \mathbf{C}} F_f\left(N^{\alpha/4} W^{(2\setminus 1)}(x)\right)$$

with

$$F_f(z) = \int_{\mathbf{R}^2 \times \mathbf{R}_+} \Psi_f(x, y, z/d^{\alpha/2}) \frac{1}{2\pi} e^{-(x^2+y^2)/2} f_D(d)\, \mathrm{d}x \mathrm{d}y \mathrm{d}d.$$

Note that

$$G_N^{(2\setminus 1)}[H_2] = \frac{1}{3} N^{\alpha/4-1} \sqrt{|E_N|} V_{2,N}^{(2/1)}$$

and therefore $G_N^{(2\setminus 1)}[H_2]$ is of the same order as $N^{\alpha/4-1/2} V_{2,N}^{(2/1)}$, since $\frac{|E_N|}{N} \xrightarrow{a.s.} 3$ as $N \to \infty$.

We want to prove that, as $N \to \infty$, $G_N^{(2/1)}[H_2] \xrightarrow{\mathbb{P}} c_{V_2} L_{W^{(2\setminus 1)}}(0)$. To do it, we subdivide our proof into three parts. First, we show that, for any f such that F_f belongs to the Schwartz space, as $N \to \infty$,

$$G_{N,*}^{(2/1)}[f] \xrightarrow{L^2} c_f L_{W^{(2\setminus 1)}}(0) \tag{4.1}$$

with $c_f = \int_{\mathbf{R}} F_f(z)\mathrm{d}z$, and deduce that, as $N \to \infty$, $G_{N,*}^{(2/1)}[f] \xrightarrow{\mathbb{P}} c_f L_{W^{(2\setminus 1)}}(0)$. Then we prove that, for $f = H_2$, as $N \to \infty$,

$$G_N^{(2/1)}[H_2] - G_{N,*}^{(2/1)}[H_2] \xrightarrow{\mathbb{P}} 0, \tag{4.2}$$

and we conclude by proving that F_{H_2} belongs to the Schwartz space.

Part 1. Proof of the L^2 convergence in Eq. (4.1)

Let f be a function such that F_f belongs to the Schwartz space. Our aim is to prove that

$$\lim_{N\to\infty} \mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f] - c_f L_{W^{(2\setminus 1)}}(0)\right)^2\right] = 0$$

or equivalently that

$$\lim_{N\to\infty} \left(\mathbb{E}\left[(G_{N,*}^{(2/1)}[f])^2\right] - 2c_f\mathbb{E}\left[G_{N,*}^{(2/1)}[f]L_{W^{(2\setminus 1)}}(0)\right] + \mathbb{E}\left[(c_f L_{W^{(2\setminus 1)}}(0))^2\right]\right) = 0.$$

We proceed in the same vein as [6]. First, according to Eq. (2.2), we notice that

$$\begin{aligned}
\mathbb{E}\left[(c_f L_{W^{(2\setminus 1)}}(0))^2\right] &= \frac{c_f^2}{(2\pi)^2} \lim_{M\to\infty} \mathbb{E}\left[\left(\int_{[-M,M]}\int_{\mathbf{C}} e^{\mathrm{i}\xi W^{(2\setminus 1)}(x)}\mathrm{d}x\mathrm{d}\xi\right)^2\right] \\
&= \frac{c_f^2}{(2\pi)^2} \lim_{M\to\infty} \int_{[-M,M]^2}\int_{\mathbf{C}^2} \mathbb{E}\left[e^{\mathrm{i}\left(\xi W^{(2\setminus 1)}(x)+\xi' W^{(2\setminus 1)}(x')\right)}\right]\mathrm{d}\vec{x}\mathrm{d}\vec{\xi} \\
&= \frac{c_f^2}{(2\pi)^2} \int_{\mathbf{R}^2}\int_{\mathbf{C}^2} e^{-\frac{1}{2}\vec{\xi}^\intercal \Sigma_{x,x'}\vec{\xi}}\mathrm{d}\vec{x}\mathrm{d}\vec{\xi},
\end{aligned} \tag{4.3}$$

where $\vec{\xi} = (\xi, \xi')$, $\mathrm{d}\vec{\xi} = \mathrm{d}\xi\mathrm{d}\xi'$ and $\mathrm{d}\vec{x} = \mathrm{d}x\mathrm{d}x'$.

Secondly, we deal with $\mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f]\right)^2\right]$. To do it, let us denote by $\hat{F}_f$ the Fourier transform of F_f, i.e.

$$\hat{F}_f(\xi) = \int_{\mathbf{R}} F_f(y)e^{-\mathrm{i}\xi y}\mathrm{d}y, \quad \xi \in \mathbf{R}.$$

Because F_f belongs to the Schwartz space, the function $\hat{F}_f$ also belongs to the Schwartz space and, for any $w \in \mathbf{R}$,

$$\begin{aligned}
F_f\left(N^{\alpha/4}w\right) &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{F}_f(\xi)e^{\mathrm{i}\xi N^{\alpha/4}w}\mathrm{d}\xi \\
&= \frac{1}{2\pi N^{\alpha/4}}\int_{\mathbf{R}}\int_{\mathbf{R}} F_f(y)e^{\mathrm{i}\xi(w-y/N^{\alpha/4})}\mathrm{d}y\mathrm{d}\xi.
\end{aligned} \tag{4.4}$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f]\right)^2\right] &= \frac{1}{(2\pi)^2 N^2} \\
&\times \mathbb{E}\left[\sum_{(x,x')\in(P_N\cap\mathbf{C})^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} e^{\mathrm{i}\left(\xi W^{(2\setminus 1)}(x)+\xi'\left(W^{(2\setminus 1)}\left(x'\right)\right)\right.}e^{-\mathrm{i}\left(\vec{\xi}^\intercal\vec{y}\right)/N^{\alpha/4}}F_f(y)F_f(y')\mathrm{d}\vec{y}\mathrm{d}\vec{\xi}\right].
\end{aligned}$$

We can easily prove that the right-hand side has the same order as if we only consider couples of *distinct* points (x, x'). Such a property allows us to apply the Slivnyak-Mecke formula (see e.g. Theorem 3.2.5 in [9]), which gives

$$\mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f]\right)^2\right] \underset{N\to\infty}{\sim} \frac{1}{(2\pi)^2} \times \mathbb{E}\left[\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} e^{\mathrm{i}\left(\xi W_\alpha^{(2\setminus 1)}(x)+\xi'(W^{(2\setminus 1)}(x')\right)} e^{-\mathrm{i}\left(\vec{\xi}^\intercal \vec{y}\right)/N^{\alpha/4}} F_f(y)F_f(y')\mathrm{d}\vec{y}\mathrm{d}\vec{\xi}\mathrm{d}\vec{x}\right].$$

Now, it follows from Fubini's theorem which can be applied because F_f belongs to the Schwartz space, that

$$\mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f]\right)^2\right] \underset{N\to\infty}{\sim} \frac{1}{(2\pi)^2} \times \int_{\mathbf{C}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \mathbb{E}\left[e^{\mathrm{i}\left(\xi W^{(2\setminus 1)}(x)+\xi'(W^{(2\setminus 1)}(x')\right)} e^{-\mathrm{i}\left(\vec{\xi}^\intercal \vec{y}\right)/N^{\alpha/4}}\right] F_f(y)F_f(y')\mathrm{d}\vec{y}\mathrm{d}\vec{\xi}\mathrm{d}\vec{x},$$

which gives

$$\mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f]\right)^2\right] \underset{N\to\infty}{\sim} \frac{1}{(2\pi)^2}\int_{\mathbf{C}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} e^{-\frac{1}{2}\vec{\xi}^\intercal\Sigma_{x,x'}\vec{\xi}} F_f(y)F_f(y')\mathrm{d}\vec{y}\mathrm{d}\vec{\xi}\mathrm{d}\vec{x}.$$

According to the Lebesgue's dominated convergence theorem, we deduce that

$$\mathbb{E}\left[\left(G_{N,*}^{(2/1)}[f]\right)^2\right] \underset{N\to\infty}{\to} \frac{c_f^2}{(2\pi)^2}\int_{\mathbf{R}^2}\int_{\mathbf{C}^2} e^{-\frac{1}{2}\vec{\xi}^\intercal\Sigma_{x,x'}\vec{\xi}}\mathrm{d}\vec{x}\mathrm{d}\vec{\xi}. \tag{4.5}$$

It remains to deal with $\mathbb{E}\left[G_{N,*}^{(2/1)}[f]c_f L_{W^{(2\setminus 1)}}(0)\right]$. According to Eq. (4.4), we have

$$\mathbb{E}\left[G_{N,*}^{(2/1)}[f]c_f L_{W^{(2\setminus 1)}}(0)\right] = \frac{c_f}{(2\pi)^2 N} \times \lim_{M\to\infty}\mathbb{E}\left[\sum_{x\in P_N\cap\mathbf{C}}\int_{\mathbf{R}}\int_{\mathbf{R}} e^{\mathrm{i}\xi W^{(2\setminus 1)}(x)} e^{-\mathrm{i}\xi y/N^{\alpha/4}} F_f(y)\mathrm{d}y\mathrm{d}\xi \int_{[-M,M]}\int_{\mathbf{C}} e^{\mathrm{i}\xi'^{(2\setminus 1)}(x')}\mathrm{d}x'\mathrm{d}\xi'\right].$$

Applying the Slivnyak-Mecke formula, Fubini's theorem and the Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}
&\mathbb{E}\left[G_{N,*}^{(2/1)}[f]c_f L_{W^{(2\setminus 1)}}(0)\right] \\
&\quad= \frac{c_f}{(2\pi)^2}\lim_{M\to\infty}\int_{\mathbf{C}^2}\int_{\mathbf{R}\times[-M,M]}\int_{\mathbf{R}} \mathbb{E}\left[e^{\mathrm{i}\left(\xi W^{(2\setminus 1)}(x)+\xi'^{(2\setminus 1)}(x')\right)}\right] e^{-\mathrm{i}\xi y/N^{\alpha/4}} F_f(y)\mathrm{d}y\mathrm{d}\vec{\xi}\mathrm{d}\vec{x} \\
&\quad= \frac{c_f}{(2\pi)^2}\int_{\mathbf{C}^2}\int_{\mathbf{R}^2}\int_{\mathbf{R}} e^{-\frac{1}{2}\vec{\xi}^\intercal\Sigma_{x,x'}\vec{\xi}} e^{-\mathrm{i}\xi y/N^{\alpha/4}} F_f(y)\mathrm{d}y\mathrm{d}\vec{\xi}\mathrm{d}\vec{x} \\
&\quad\underset{N\to\infty}{\to} \frac{c_f^2}{(2\pi)^2}\int_{\mathbf{R}^2}\int_{\mathbf{C}^2} e^{-\frac{1}{2}\vec{\xi}^\intercal\Sigma_{x,x'}\vec{\xi}}\mathrm{d}\vec{x}\mathrm{d}\vec{\xi}.
\end{aligned}$$

This together with Eq. (4.3) and Eq. (4.5) implies Eq. (4.1).

Part 2. Proof of the convergence in probability in Eq. (4.2)

Adapting the proof of Eq. (4.1), we can easily show that, as $N \to \infty$,

$$G^{(2/1)}_{N,*}[H_2] - \frac{1}{3} N^{\alpha/4-1} \sum_{(x_1,x_2)\in E_N} F_{H_2}\left(N^{\alpha/4} W^{(2\setminus 1)}(x_1)\right) \xrightarrow{\mathbb{P}} 0.$$

The factor $\frac{1}{3}$ comes from the fact that, for a typical point x, the mean number of edges whose x is the leftmost vertex is 3. Therefore, it is sufficient to prove that

$$\mathbb{E}\left[\left(N^{\alpha/4-1} \sum_{(x_1,x_2)\in E_N} \left(\Psi_{H_2}(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, W^{(2\setminus 1)}(x_1)/d^{\alpha/2}_{1,2}) - F_{H_2}\left(N^{\alpha/4} W^{(2\setminus 1)}(x_1)\right)\right)\right)^2\right]$$

converges to 0 as N goes to infinity, or equivalently that

$$\lim_{N\to\infty} N^{\alpha/2-2} \mathbb{E}\left[\sum_{(x_1,x_2),\,(x_3,x_4)\in E_N} r_{N;x_1,x_2} r_{N;x_3,x_4}\right] = 0, \tag{4.6}$$

where

$$r_{N;x_1,x_2} = \Psi_{H_2}(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, W^{(2\setminus 1)}(x_1)/d^{\alpha/2}_{1,2}) - F_{H_2}\left(N^{\alpha/4} W^{(2\setminus 1)}(x_1)\right).$$

We only prove Eq. (4.6) when we assume that the pairs (x_1, x_2) and (x_3, x_4) appearing in the sum have no point in common since the opposite case can be dealt in a similar way. Let

$$R_N = N^{\alpha/2-2} \mathbb{E}\left[\sum_{(x_1,x_2),(x_3,x_4)\in E_N} \mathbb{I}\left[\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset\right] r_{N;x_1,x_2} r_{N;x_3,x_4}\right].$$

According to the Slivnyak-Mecke formula, we have

$$R_N = N^{\alpha/2+2} \int_{(\mathbf{R}^2)^2} \mathbb{I}\left[x_1, x_3 \in \mathbf{C}\right] \left[\int_{(\mathbf{R}^2)^2} \mathbb{E}\left[r_{N;x_1,x_2} r_{N;x_3,x_4}\right] p_{2,N}(x_1, x_2, x_3, x_4) \mathrm{d}x_2 \mathrm{d}x_4\right] \mathrm{d}x_1 \mathrm{d}x_3,$$

where $p_{2,N}(x_1, x_2, x_3, x_4)$ is given (as in [3]) by

$$p_{2,N}(x_1, x_2, x_3, x_4) = \mathbb{P}\left[x_1 \sim x_2,\ x_3 \sim x_4 \text{ in } \mathrm{Del}(P_N \cup \{x_1, x_2, x_3, x_4\}) \text{ and } x_1 \preceq x_2,\ x_3 \preceq x_4\right]. \tag{4.7}$$

Let

$$\begin{aligned}
&\psi_N\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right) \\
&\qquad = \int_{(\mathbf{R}^2)^2} \mathbb{E}\left[r_{N;x_1,x_2} r_{N;x_3,x_4} \middle| W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right] p_{2,N}(x_1, x_2, x_3, x_4) \mathrm{d}x_2 \mathrm{d}x_4 \\
&\qquad = \mathbb{E}\left[\int_{(\mathbf{R}^2)^2} r_{N;x_1,x_2} r_{N;x_3,x_4} p_{2,N}(x_1, x_2, x_3, x_4) \mathrm{d}x_2 \mathrm{d}x_4 \middle| W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right],
\end{aligned}$$

so that

$$R_N = N^{\alpha/2+2} \int_{\mathbf{C}^2} \mathbb{E}\left[\psi_N\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right] \mathrm{d}x_1 \mathrm{d}x_3.$$

Let $\delta \in (0,1)$ be fixed, define

$$\begin{aligned} R^{(1)}_{N,\delta} &= N^{\alpha/2+2} \int_{\mathbf{C}^2} \mathbb{I}\left[\|x_1 - x_3\| \le \delta\right] \mathbb{E}\left[\psi_N\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right] \mathrm{d}x_1 \mathrm{d}x_3 \\ R^{(2)}_{N,\delta} &= N^{\alpha/2+2} \int_{\mathbf{C}^2} \mathbb{I}\left[\|x_1 - x_3\| > \delta\right] \mathbb{E}\left[\psi_N\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right] \mathrm{d}x_1 \mathrm{d}x_3, \end{aligned}$$

so that $R_N = R^{(1)}_{N,\delta} + R^{(2)}_{N,\delta}$. It is sufficient to show that, for some positive constant c,

$$\limsup_{N\to\infty} |R^{(1)}_{N,\delta}| \le c\delta^{2-\alpha} \tag{4.8}$$

and

$$\lim_{N\to\infty} R^{(2)}_{N,\delta} = 0, \tag{4.9}$$

which will constitute the two main steps of our proof.

Before proving this, we provide bounds for the mean and correlation coefficients. To do it, we notice that the random vector

$$\left(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, U^{(1)}_{x_3,x_4}, U^{(2)}_{x_3,x_4}, W^{(2\setminus 1)}(x_1)/\sqrt{2}\,\|x_1\|^{\alpha/2}, W^{(2\setminus 1)}(x_3)/\sqrt{2}\,\|x_3\|^{\alpha/2}\right)$$

has a centered Gaussian distribution with covariance (correlation) matrix given by

$$\Sigma_{x_1,x_2,x_3,x_4} = \begin{bmatrix} 1 & 0 & \eta_{x_1,x_2,x_3,x_4} & 0 & \rho_{x_1,x_2} & \nu_{x_1,x_2,x_3} \\ & 1 & 0 & \eta_{x_1,x_2,x_3,x_4} & -\rho_{x_1,x_2} & -\nu_{x_1,x_2,x_3} \\ & & 1 & 0 & \nu_{x_3,x_4,x_1} & \rho_{x_3,x_4} \\ & & & 1 & -\nu_{x_3,x_4,x_1} & -\rho_{x_3,x_4} \\ & & & & 1 & \kappa_{x_1,x_3} \\ & & & & & 1 \end{bmatrix},$$

where

$$\begin{aligned} \eta_{x_1,x_2,x_3,x_4} &= \mathrm{corr}(U^{(1)}_{x_1,x_2}, U^{(1)}_{x_3,x_4}) = \mathrm{corr}(U^{(2)}_{x_1,x_2}, U^{(2)}_{x_3,x_4}) \\ &= \frac{1}{2}\frac{(||x_4 - x_1||^\alpha - ||x_3 - x_1||^\alpha) - (||x_4 - x_2||^\alpha - ||x_3 - x_2||^\alpha)}{||x_2 - x_1||^{\alpha/2}||x_4 - x_3||^{\alpha/2}} \\ \rho_{x_1,x_2} &= -\mathrm{corr}(U^{(1)}_{x_1,x_2}, W^{(1)}(x_1)) = \frac{1}{2}\frac{-||x_2||^\alpha + ||x_1||^\alpha + ||x_2 - x_1||^\alpha}{||x_2 - x_1||^{\alpha/2}||x_1||^{\alpha/2}} \\ \nu_{x_1,x_2,x_3} &= -\mathrm{corr}(U^{(1)}_{x_1,x_2}, W^{(1)}(x_3)) = -\frac{1}{2}\frac{||x_2||^\alpha - ||x_3 - x_2||^\alpha - ||x_1||^\alpha + ||x_3 - x_1||^\alpha}{||x_2 - x_1||^{\alpha/2}||x_3||^{\alpha/2}} \\ \kappa_{x_1,x_3} &= \mathrm{corr}(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)) = \frac{1}{2}\frac{||x_1||^\alpha - ||x_3 - x_1||^\alpha + ||x_3||^\alpha}{||x_1||^{\alpha/2}||x_3||^{\alpha/2}}. \end{aligned}$$

Let us write

$$\Sigma_{x_1,x_2,x_3,x_4} = \begin{bmatrix} \Sigma^{(1,1)}_{x_1,x_2,x_3,x_4} & \Sigma^{(1,2)}_{x_1,x_2,x_3,x_4} \\ \Sigma^{(2,1)}_{x_1,x_2,x_3,x_4} & \Sigma^{(2,2)}_{x_1,x_3} \end{bmatrix},$$

where $\Sigma^{(1,1)}_{x_1,x_2,x_3,x_4}$ is the covariance (correlation) matrix of $(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, U^{(1)}_{x_3,x_4}, U^{(2)}_{x_3,x_4})$.

It follows that $(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, U^{(1)}_{x_3,x_4}, U^{(2)}_{x_3,x_4})$ given $\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)$ has a Gaussian distribution with mean

$$\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4} = \Sigma^{(1,2)}_{x_1,x_2,x_3,x_4}\left(\Sigma^{(2,2)}_{x_1,x_3}\right)^{-1}\begin{bmatrix} W^{(2\backslash 1)}(x_1)/\sqrt{2}\,\|x_1\|^{\alpha/2} \\ W^{(2\backslash 1)}(x_3)/\sqrt{2}\,\|x_3\|^{\alpha/2} \end{bmatrix}$$

and covariance matrix

$$\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4} = \Sigma^{(1,1)}_{x_1,x_2,x_3,x_4} - \Sigma^{(1,2)}_{x_1,x_2,x_3,x_4}\left(\Sigma^{(2,2)}_{x_1,x_3}\right)^{-1}\Sigma^{(2,1)}_{x_1,x_2,x_3,x_4}.$$

We have

$$\begin{aligned} \Sigma^{(1,2)}_{x_1,x_2,x_3,x_4}\left(\Sigma^{(2,2)}_{x_1,x_3}\right)^{-1} &= \frac{1}{1-\kappa^2_{x_1,x_3}}\begin{bmatrix} \rho_{x_1,x_2} & \nu_{x_1,x_2,x_3} \\ -\rho_{x_1,x_2} & -\nu_{x_1,x_2,x_3} \\ \nu_{x_3,x_4,x_1} & \rho_{x_3,x_4} \\ -\nu_{x_3,x_4,x_1} & -\rho_{x_3,x_4} \end{bmatrix}\begin{bmatrix} 1 & -\kappa_{x_1,x_3} \\ -\kappa_{x_1,x_3} & 1 \end{bmatrix} \\ &= \frac{1}{1-\kappa^2_{x_1,x_3}}\begin{bmatrix} \rho_{x_1,x_2} - \kappa_{x_1,x_3}\nu_{x_1,x_2,x_3} & -\kappa_{x_1,x_3}\rho_{x_1,x_2} + \nu_{x_1,x_2,x_3} \\ -\rho_{x_1,x_2} + \kappa_{x_1,x_3}\nu_{x_1,x_2,x_3} & \kappa_{x_1,x_3}\rho_{x_1,x_2} - \nu_{x_1,x_2,x_3} \\ \nu_{x_3,x_4,x_1} - \kappa_{x_1,x_3}\rho_{x_3,x_4} & -\kappa_{x_1,x_3}\nu_{x_3,x_4,x_1} + \rho_{x_3,x_4} \\ -\nu_{x_3,x_4,x_1} + \kappa_{x_1,x_3}\rho_{x_3,x_4} & \kappa_{x_1,x_3}\nu_{x_3,x_4,x_1} - \rho_{x_3,x_4} \end{bmatrix}. \end{aligned}$$

Therefore we can conclude that

$$\max_{1\le i\le 4}\left|\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4,i}\right| \le C\frac{\upsilon_{x_1,x_2,x_3,x_4}}{1-\kappa^2_{x_1,x_3}}\left(\frac{\left|W^{(2\backslash 1)}(x_1)\right|}{\|x_1\|^{\alpha/2}} + \frac{\left|W^{(2\backslash 1)}(x_3)\right|}{\|x_3\|^{\alpha/2}}\right), \tag{4.10}$$

where

$$\upsilon_{x_1,x_2,x_3,x_4} = \max\{|\rho_{x_1,x_2}|, |\rho_{x_3,x_4}|, |\nu_{x_1,x_2,x_3}|, |\nu_{x_3,x_4,x_1}|\} \le 1$$

and C is a positive constant that does not depend on x_1, x_2, x_3, x_4. Moreover it is easily seen that

$$\max_{1\le i,j\le 4}\{|\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4,i,j}|\} \le C \tag{4.11}$$

and that

$$\max_{1\le i,j\le 4}\{|\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4,i,j} - I_{4,i,j}|\} \le C\left(|\eta_{x_1,x_2,x_3,x_4}| + \frac{\upsilon_{x_1,x_2,x_3,x_4}}{1-\kappa^2_{x_1,x_3}}\right),$$

where I_4 is the identity matrix of dimension 4.

Note that, if x_1 and x_3 are fixed (with $d_{1,3} > 0$), we have $\upsilon_{x_1,x_2,x_3,x_4} \to 0$ and $\eta_{x_1,x_2,x_3,x_4} \to 0$ as $d_{1,2} \to 0$ and $d_{3,4} \to 0$. Thus, if $d_{1,3} > 0$ is fixed, as $d_{1,2} \to 0$ and $d_{3,4} \to 0$, we have

$$\Sigma^{(1,1)}_{x_1,x_2,x_3,x_4} \to I_4 \qquad \text{and} \qquad \Sigma^{(1,2)}_{x_1,x_2,x_3,x_4} \to 0_{4\times 2},$$

and therefore

$$\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4} \to 0_{4\times 1} \qquad \text{and} \qquad \Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4} \to I_4.$$

When $d_{1,3} \to 0$, it is actually possible to get more precise inequalities than Eq. (4.10). Let us e.g. consider the $(1,1)$ entry of $\Sigma^{(1,2)}_{x_1,x_2,x_3,x_4}(\Sigma^{(2,2)}_{x_1,x_3})^{-1}$, i.e.

$$\frac{\rho_{x_1,x_2} - \kappa_{x_1,x_3}\nu_{x_1,x_2,x_3}}{1-\kappa^2_{x_1,x_3}}.$$

If $d_{1,2} \leq \gamma d_{1,3}^{2+\varepsilon}$ with $\gamma, \varepsilon > 0$, then we have (according to Lemma 3)

$$\rho_{x_1,x_2} \underset{d_{1,3}\to 0}{\sim} \frac{1}{2} d_{1,2}^{\alpha/2} ||x_1||^{-\alpha/2},$$

$$\kappa_{x_1,x_3} \underset{d_{1,3}\to 0}{\sim} 1 - \frac{1}{2} d_{1,3}^{\alpha} ||x_1||^{-\alpha},$$

$$\nu_{x_1,x_2,x_3} \underset{d_{1,3}\to 0}{\sim} -\frac{1}{2}\alpha d_{1,2}^{1-\alpha/2} ||x_1||^{\alpha-1} ||x_3||^{-\alpha/2},$$

and therefore

$$\rho_{x_1,x_2} - \kappa_{x_1,x_3}\nu_{x_1,x_2,x_3} \underset{d_{1,3}\to 0}{\sim} \frac{1}{2} d_{1,2}^{\alpha/2} ||x_1||^{-\alpha/2}.$$

The assumption $d_{1,2} \leq \gamma d_{1,3}^{2+\varepsilon}$ is not restrictive since

$$\lim_{N\to\infty} \mathbb{P}_{P_N} \left[||x_2 - x_1|| \leq \gamma ||x_3 - x_1||^{2+\varepsilon} \right] = 1,$$

where, for any $x \in \mathbf{R}^2$,

$$\gamma(x) := \frac{1}{2}\mathrm{var}\left(W(x)\right) = \frac{1}{2}\sigma^2 \left\|x\right\|^{\alpha}$$

denotes the semi-variogram and where the subscript P_N means that (x_1, x_2) and (x_3, x_4) are edges of triangles of Delaunay triangulation generated by P_N.

Since, according to Lemma 3, we have $1 - \kappa_{x_1,x_3} \underset{d_{1,3}\to 0}{\sim} \frac{1}{2} d_{1,3}^{\alpha} ||x_1||^{-\alpha}$ with the assumption $x_1, x_3 \in \mathbf{C}$, we get, as $d_{1,3} \to 0$,

$$|\rho_{x_1,x_2} - \kappa_{x_1,x_3}\nu_{x_1,x_2,x_3}| = o(1 - \kappa_{x_1,x_3}).$$

So it is possible to derive that, as $d_{1,3} \to 0$ and $d_{1,2} \leq c d_{1,3}^{2+\varepsilon}$,

$$\frac{\upsilon_{x_1,x_2,x_3,x_4}}{1 - \kappa_{x_1,x_3}^2} = o(1).$$

More generally, it can be shown that, for $d_{1,2} \leq \gamma d_{1,3}^{2+\varepsilon}$, the inequality in Eq. (4.10) may be replaced by

$$\max_{1\leq i\leq 4} \left| \mu_{x_1,x_2,x_3,x_4,i}^{|x_1,x_3} \right| \leq C \left(\frac{\left| W^{(2\backslash 1)}(x_1) \right|}{\left\| x_1 \right\|^{\alpha/2}} + \frac{\left| W^{(2\backslash 1)}(x_3) \right|}{\left\| x_3 \right\|^{\alpha/2}} \right). \tag{4.12}$$

In what follows, we re-write the term $\psi_N \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right)$ as follows:

$$\psi_N \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right) = \mathbb{E}\left[\int_{(\mathbf{R}^2)^2} r_{N;x_1,x_2} r_{N;x_3,x_4} p_{2,N}(x_1,x_2,x_3,x_4) \mathrm{d}x_2 \mathrm{d}x_4 \middle| W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right].$$

We have

$$\begin{aligned}\psi_N \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right) &= \psi_N^{\Psi\Psi} \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right) - \psi_N^{\Psi F} \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right) \\ &\quad - \psi_N^{F\Psi} \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right) + \psi_N^{FF} \left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3) \right)\end{aligned}$$

with

$$\psi_N^{\Psi\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)$$
$$= \int_{(\mathbf{R}^2)^2}\int_{\mathbf{R}^4} \Psi_{H_2}(u_1, u_2, N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})$$
$$\times \Psi_{H_2}(u_3, u_4, N^{\alpha/4}W^{(2\setminus 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})$$
$$\times \phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1, u_2, u_3, u_4)\, \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1, x_2, x_3, x_4)\mathrm{d}x_2\mathrm{d}x_4,$$

$$\psi_N^{\Psi F}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)$$
$$= F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2}\int_{\mathbf{R}^4} \Psi_{H_2}(u_1, u_2, N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})$$
$$\times \phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1, u_2, u_3, u_4)\, \mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1, x_2, x_3, x_4)\mathrm{d}x_2\mathrm{d}x_4,$$

$$\psi_N^{FF}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)$$
$$= F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_1)\right)F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2} p_{2,N}(x_1, x_2, x_3, x_4)\mathrm{d}x_2\mathrm{d}x_4$$

and with $\psi_N^{F\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right) = \psi_N^{\Psi F}\left(W^{(2\setminus 1)}(x_3), W^{(2\setminus 1)}(x_1)\right)$. Note also that

$$|\Psi_{H_2}(u_1, u_2, N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})|$$
$$\leq u_1^2 \vee u_2^2\left(\mathbb{I}_{\{u_1-u_2\leq N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}\leq 0\}} + \mathbb{I}_{\{0\leq N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}\leq u_1-u_2\}}\right)$$
$$= u_1^2 \vee u_2^2 \mathbb{I}_{\{N^{\alpha/4}|W^{(2\setminus 1)}(x_1)|/(d_{1,2}N^{1/2})^{\alpha/2}\leq |u_1-u_2|\}}.$$

We are now ready to address the terms $R^{(1)}_{N,\delta}$ and $R^{(2)}_{N,\delta}$ introduced on page 12.

Step 1: proof of (4.8).
Let

$$\begin{aligned}
R^{(1),\Psi\Psi}_{N,\delta} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|\leq\delta\right]\mathbb{E}\left[\psi_N^{\Psi\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3\\
R^{(1),\Psi F}_{N,\delta} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|\leq\delta\right]\mathbb{E}\left[\psi_N^{\Psi F}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3\\
R^{(1),F\Psi}_{N,\delta} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|\leq\delta\right]\mathbb{E}\left[\psi_N^{F\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3\\
R^{(1),FF}_{N,\delta} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|\leq\delta\right]\mathbb{E}\left[\psi_N^{FF}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3.
\end{aligned}$$

We mainly consider the term $R^{(1),\Psi\Psi}_{N,\delta}$. We have

$$\begin{aligned}\psi_N^{\Psi\Psi}&\left(W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\right)\\&=\int_{(\mathbf{R}^2)^2}\int_{(\mathbf{R}^2)^2}\Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\\&\times\Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\\&\times\phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\end{aligned}$$

and therefore

$$\begin{aligned}\psi_N^{\Psi\Psi}&\left(W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\right)\\&=\int_{(\mathbf{R}^2)^2}\int_{(\mathbf{R}^2)^2}\Psi_{H_2}(\tilde{u}_1,\tilde{u}_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\\&\qquad\times\Psi_{H_2}(\tilde{u}_3,\tilde{u}_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\\&\qquad\times p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\phi_{0,I_4}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4,\end{aligned}$$

where

$$\begin{bmatrix}\tilde{u}_1\\\tilde{u}_2\\\tilde{u}_3\\\tilde{u}_4\end{bmatrix}=\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4}+(\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4})^{1/2}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}.$$

We deduce from Eq. (4.12) and Eq. (4.11) that, for $j=1,2,3,4$,

$$|\tilde{u}_j|\le C\left(\frac{\left|W^{(2\backslash 1)}(x_1)\right|}{\sqrt{2}\left\|x_1\right\|^{\alpha/2}}+\frac{\left|W^{(2\backslash 1)}(x_3)\right|}{\sqrt{2}\left\|x_3\right\|^{\alpha/2}}+\left\|\vec{u}\right\|_1\right):=B\left(x_1,x_3,\left\|\vec{u}\right\|_1\right)$$

for some positive constant C, with $\left\|\vec{u}\right\|_1=\sum_{i=1}^4|u_i|$. Then

$$\begin{aligned}&\left|\Psi_{H_2}(\tilde{u}_1,\tilde{u}_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\right|\\&\qquad\le 2B\left(x_1,x_3,\left\|\vec{u}\right\|_1\right)^2\mathbb{I}_{\{N^{\alpha/4}|W^{(2\backslash 1)}(x_1)|/(d_{1,2}N^{1/2})^{\alpha/2}\le 2B\left(x_1,x_3,\left\|\vec{u}\right\|_1\right)\}}.\end{aligned}$$

We deduce that

$$\begin{aligned}\int_{(\mathbf{R}^2)^2}&\left|\Psi_{H_2}(\tilde{u}_1,\tilde{u}_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\right.\\&\left.\times\Psi_{H_2}(\tilde{u}_3,\tilde{u}_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\right|p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\end{aligned}$$

is lower than

$$\begin{aligned}&\frac{4}{N^2}B\left(x_1,x_3,\left\|\vec{u}\right\|_1\right)^4\\&\quad\times\mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2}^{\alpha/2}>\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_1)|}{2B\left(x_1,x_3,\left\|\vec{u}\right\|_1\right)},D_{3,4}^{\alpha/2}>\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_3)|}{2B\left(x_1,x_3,\left\|\vec{u}\right\|_1\right)}\right|W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\Bigg],\end{aligned}$$

where $(D_{1,2},D_{3,4})$ is the distribution of $\|x_2-x_1\|$ and $\|x_4-x_3\|$ under $p_{2,1}$ (x_1 and x_3 being fixed), i.e.

for every measurable and positive function $g : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$

$$\mathbb{E}_{\|x_1-x_3\|}\left[g(D_{1,2}, D_{3,4})\right] = \int_{\mathbf{R}^2\times\mathbf{R}^2} g\left(\|x_2-x_1\|, \|x_4-x_3\|\right) p_{2,1}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4.$$

Note that, for $q>0$, the multivariate Markov inequality provides

$$\mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2} > d_1, D_{3,4} > d_3\right] \leq \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q D_{3,4}^q\right]}{d_1^q d_3^q}$$

while the univariate Markov inequalities provide

$$\begin{aligned}
\mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2} > d_1, D_{3,4} > d_3\right] &\leq \mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2} > d_1\right] \leq \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{d_1^q} \\
\mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2} > d_1, D_{3,4} > d_3\right] &\leq \mathbb{P}_{\|x_1-x_3\|}\left[D_{3,4} > d_3\right] \leq \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{3,4}^q\right]}{d_3^q}.
\end{aligned}$$

Hence an upper bound of $\mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2} > d_1, D_{3,4} > d_3\right]$ is given by $B_{\|x_1-x_3\|}(d_1,d_3)$, where

$$B_{\|x_1-x_3\|,q}(d_1,d_3) = \begin{cases} \mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q D_{3,4}^q\right]/d_1^q d_3^q & \text{if } d_1 > d_{1,b,q} \text{ and } d_3 > d_{3,b,q} \\ \mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]/d_1^q & \text{if } d_1 > d_{1,b,q} \text{ and } d_3 \leq d_{3,b,q} \\ \mathbb{E}_{\|x_1-x_3\|}\left[D_{3,4}^q\right]/d_3^q & \text{if } d_1 \leq d_{1,b,q} \text{ and } d_3 > d_{3,b,q} \\ 1 & \text{if } d_1 \leq d_{1,b,q} \text{ and } d_3 \leq d_{3,b,q} \end{cases},$$

with

$$\begin{aligned}
d_{1,b,q} &= \left(\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q D_{3,4}^q\right]/\mathbb{E}_{\|x_1-x_3\|}\left[D_{3,4}^q\right]\right)^{1/q} \\
d_{3,b,q} &= \left(\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q D_{3,4}^q\right]/\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]\right)^{1/q}.
\end{aligned}$$

Note that

$$0 < C_0 \leq \inf_{\|x_1-x_3\|>0} d_{1,b,q} \leq \sup_{\|x_1-x_3\|>0} d_{1,b,q} \leq C_1 < \infty.$$

It follows that

$$\begin{aligned}
\left|\psi_N^{\Psi\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right| &\leq \frac{4}{N^2}\mathbb{E}\Big[B\left(x_1,x_3,||\vec{U}||_1\right)^4 \\
&\times B_{\|x_1-x_3\|,q}\left(\left(\frac{N^{\alpha/4}|W^{(2\setminus 1)}(x_1)|}{2B\left(x_1,x_3,||\vec{U}||_1\right)}\right)^{2/\alpha}, \left(\frac{N^{\alpha/4}|W^{(2\setminus 1)}(x_3)|}{2B\left(x_1,x_3,||\vec{U}||_1\right)}\right)^{2/\alpha}\right)\Big| W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\Big]
\end{aligned}$$

where $\vec{U}$ has a standard multivariate Gaussian distribution independent of $(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3))$, and that

$$\begin{aligned}
\mathbb{E}\left[\left|\psi_N^{\Psi\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\right|\right] &\leq \frac{4}{N^2} \\
&\times \mathbb{E}\left[B\left(x_1,x_3,||\vec{U}||_1\right)^4 B_{\|x_1-x_3\|,q}\left(\left(\frac{N^{\alpha/4}|W^{(2\setminus 1)}(x_1)|}{2B\left(x_1,x_3,||\vec{U}||_1\right)}\right)^{2/\alpha}, \left(\frac{N^{\alpha/4}|W^{(2\setminus 1)}(x_3)|}{2B\left(x_1,x_3,||\vec{U}||_1\right)}\right)^{2/\alpha}\right)\right].
\end{aligned}$$

To deal with the right-hand side, let

$$\nu_N\left(y_1, y_3, \|\vec{u}\|_1\right) = C^4\left(\|\vec{u}\|_1 + |y_1| + |y_3|\right)^4$$
$$\times B_{\|x_1-x_3\|,q}\left(\left(\frac{\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}y_1}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)}\right)^{2/\alpha}, \left(\frac{\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2}y_3}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)}\right)^{2/\alpha}\right)\phi_{0,\Sigma^{(2,2)}_{x_1,x_3}}\left(y_1,y_3\right)$$

with

$$\phi_{0,\Sigma^{(2,2)}_{x_1,x_3}}\left(y_1,y_3\right) = \frac{1}{2\pi}\frac{1}{1-\kappa^2_{x_1,x_3}}\exp\left(-\frac{1}{2}\begin{pmatrix} y_1 & y_3 \end{pmatrix}\left(\Sigma^{(2,2)}_{x_1,x_3}\right)^{-1}\begin{pmatrix} y_1 \\ y_3 \end{pmatrix}'\right).$$

The quantity ν_N has been chosen in such a way that

$$\int_{\mathbf{R}^2}\nu_N\left(y_1,y_3,\|\vec{u}\|_1\right)\mathrm{d}y_1\mathrm{d}y_3$$
$$= \mathbb{E}\left[B\left(x_1,x_3,\|\vec{u}\|_1\right)^4 B_{\|x_1-x_3\|,q}\left(\left(\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_1)|}{2B\left(x_1,x_3,\|\vec{u}\|_1\right)}\right)^{2/\alpha}, \left(\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_3)|}{2B\left(x_1,x_3,\|\vec{u}\|_1\right)}\right)^{2/\alpha}\right)\right]$$

and that

$$\mathbb{E}\left[B\left(x_1,x_3,||\vec{U}||_1\right)^4 B_{\|x_1-x_3\|,q}\left(\left(\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_1)|}{2B\left(x_1,x_3,||\vec{U}||_1\right)}\right)^{2/\alpha}, \left(\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_3)|}{2B\left(x_1,x_3,||\vec{U}||_1\right)}\right)^{2/\alpha}\right)\right]$$
$$= \mathbb{E}\left[\int_{\mathbf{R}^2}\nu_N\left(y_1,y_3,||\vec{U}||_1\right)\mathrm{d}y_1\mathrm{d}y_3\right].$$

Let

$$\begin{aligned}
\mathcal{A}_{1,N} &= \left\{(y_1,y_3) : \frac{\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} \le d^{\alpha/2}_{1,b,q}, \frac{\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2}|y_3|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} \le d^{\alpha/2}_{3,b,q}\right\} \\
\mathcal{A}_{2,N} &= \left\{(y_1,y_3) : \frac{\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} > d^{\alpha/2}_{1,b,q}, \frac{\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2}|y_3|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} \le d^{\alpha/2}_{3,b,q}\right\} \\
\mathcal{A}_{3,N} &= \left\{(y_1,y_3) : \frac{\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} \le d^{\alpha/2}_{1,b,q}, \frac{\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2}|y_3|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} > d^{\alpha/2}_{3,b,q}\right\} \\
\mathcal{A}_{4,N} &= \left\{(y_1,y_3) : \frac{\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} > d^{\alpha/2}_{1,b,q}, \frac{\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2}|y_3|}{2C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)} > d^{\alpha/2}_{3,b,q}\right\}.
\end{aligned}$$

First, we deal with $\int_{\mathcal{A}_{1,N}}\nu_N\left(y_1,y_3,\|\vec{u}\|_1\right)\mathrm{d}y_1\mathrm{d}y_3$. For $(y_1,y_3)\in\mathcal{A}_{1,N}$, we have

$$\begin{aligned}
\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1| &\le 2d^{\alpha/2}_{1,b,q}C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right) \\
\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2}|y_3| &\le 2d^{\alpha/2}_{3,b,q}C\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2} - 2d^{\alpha/2}_{1,b,q}C\right)|y_1| &\le 2d^{\alpha/2}_{1,b,q}C\left(\|\vec{u}\|_1+|y_3|\right) \\
\left(\sqrt{2}N^{\alpha/4}\|x_3\|^{\alpha/2} - 2d^{\alpha/2}_{3,b,q}C\right)|y_3| &\le 2d^{\alpha/2}_{3,b,q}C\left(\|\vec{u}\|_1+|y_1|\right).
\end{aligned}$$

Therefore, for N large enough, we have

$$\left(\sqrt{2}N^{\alpha/4}\left\|x_1\right\|^{\alpha/2}-2d_{1,b,q}^{\alpha/2}C-\frac{2d_{3,b,q}^{\alpha/2}C}{\left(\sqrt{2}N^{\alpha/4}\left\|x_3\right\|^{\alpha/2}-2d_{3,b,q}^{\alpha/2}C\right)}\right)|y_1| \leq\left(2d_{1,b,q}^{\alpha/2}C+\frac{2d_{3,b,q}^{\alpha/2}C}{\left(\sqrt{2}N^{\alpha/4}\left\|x_3\right\|^{\alpha/2}-2d_{3,b,q}^{\alpha/2}C\right)}\right)\|\vec{u}\|_1$$

and

$$\left(\sqrt{2}N^{\alpha/4}\left\|x_3\right\|^{\alpha/2}-2d_{3,b,q}^{\alpha/2}C-\frac{2d_{1,b,q}^{\alpha/2}C}{\left(\sqrt{2}N^{\alpha/4}\left\|x_1\right\|^{\alpha/2}-2d_{1,b,q}^{\alpha/2}C\right)}\right)|y_3| \leq\left(2d_{3,b,q}^{\alpha/2}C+\frac{2d_{1,b,q}^{\alpha/2}C}{\left(\sqrt{2}N^{\alpha/4}\left\|x_1\right\|^{\alpha/2}-2d_{1,b,q}^{\alpha/2}C\right)}\right)\|\vec{u}\|_1 .$$

This implies that

$$|y_1| \leq C\frac{\|\vec{u}\|_1}{N^{\alpha/4}\left\|x_1\right\|^{\alpha/2}} \quad \text{and} \quad |y_3| \leq C\frac{\|\vec{u}\|_1}{N^{\alpha/4}\left\|x_3\right\|^{\alpha/2}}.$$

It follows that

$$\begin{aligned}\int_{\mathcal{A}_{1,N}}\nu_N\left(y_1,y_3,\|\vec{u}\|_1\right)\mathrm{d}y_1\mathrm{d}y_3 &\leq C\int_{\mathcal{A}_{1,N}}\left(\|\vec{u}\|_1+|y_1|+|y_3|\right)^4\frac{1}{1-\kappa_{x_1,x_3}^2}\mathrm{d}y_1\mathrm{d}y_3\\ &\leq C\frac{\|\vec{u}\|_1^6}{1-\kappa_{x_1,x_3}^2}\frac{1}{N^{\alpha/2}\left\|x_1\right\|^{\alpha/2}\left\|x_3\right\|^{\alpha/2}}.\end{aligned}$$

Let us now consider $\int_{\mathcal{A}_{2,N}}\nu_N\left(y_1,y_3,\|\vec{u}\|_1\right)\mathrm{d}y_1\mathrm{d}y_3$. For $(y_1,y_3)\in\mathcal{A}_{2,N}$, we have, for N large enough,

$$|y_1| > C\frac{\|\vec{u}\|_1}{N^{\alpha/4}\left\|x_1\right\|^{\alpha/2}} \quad \text{and} \quad |y_3| \leq C\frac{\|\vec{u}\|_1+|y_1|}{N^{\alpha/4}\left\|x_3\right\|^{\alpha/2}}.$$

Note that, for any $q>0$,

$$\sup_{\|x_1-x_3\|>0}\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]\leq C.$$

We have

$$\begin{aligned}
&\int_{\mathcal{A}_{2,N}} \nu_N\left(y_1, y_3, \|\vec{u}\|_1\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
&\qquad\le C \int_{\mathcal{A}_{2,N}} \left(\|\vec{u}\|_1 + |y_1| + |y_3|\right)^{4+2q/\alpha} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|\right)^{2q/\alpha}} \phi_{0,\Sigma_{x_1,x_3}^{(2,2)}}\left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
&\qquad\le C \int_{\mathcal{A}_{2,N}} \|\vec{u}\|_1^{4+2q/\alpha} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|\right)^{2q/\alpha}} \phi_{0,\Sigma_{x_1,x_3}^{(2,2)}}\left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
&\qquad\qquad + C \int_{\mathcal{A}_{2,N}} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}\right)^{2q/\alpha}} |y_1|^4 \phi_{0,\Sigma_{x_1,x_3}^{(2,2)}}\left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
&\qquad\qquad + C \int_{\mathcal{A}_{2,N}} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|\right)^{2q/\alpha}} |y_3|^{4+2q/\alpha} \phi_{0,\Sigma_{x_1,x_3}^{(2,2)}}\left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3.
\end{aligned}$$

This gives

$$\begin{aligned}
&\int_{\mathcal{A}_{2,N}} \nu_N\left(y_1, y_3, \|\vec{u}\|_1\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
&\quad\le C \frac{\|\vec{u}\|_1^{4+2q/\alpha}}{1-\kappa_{x_1,x_3}^2} \int_{|y_1|>C\|\vec{u}\|_1/(N^{\alpha/4}\|x_1\|^{\alpha/2})} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|\right)^{2q/\alpha}} \frac{\|\vec{u}\|_1 + |y_1|}{N^{\alpha/4}\|x_3\|^{\alpha/2}} \mathrm{d}y_1 \\
&\quad + C \frac{1}{1-\kappa_{x_1,x_3}^2} \int_{|y_1|>C\|\vec{u}\|_1/(N^{\alpha/4}\|x_1\|^{\alpha/2})} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]|y_1|^4}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}\right)^{2q/\alpha}} \frac{\|\vec{u}\|_1 + |y_1|}{N^{\alpha/4}\|x_3\|^{\alpha/2}} \phi_{0,1}\left(y_1\right) \mathrm{d}y_1 \\
&\quad + C \frac{1}{1-\kappa_{x_1,x_3}^2} \int_{|y_1|>C\|\vec{u}\|_1/(N^{\alpha/4}\|x_1\|^{\alpha/2})} \frac{\mathbb{E}_{\|x_1-x_3\|}\left[D_{1,2}^q\right]}{\left(\sqrt{2}N^{\alpha/4}\|x_1\|^{\alpha/2}|y_1|\right)^{2q/\alpha}} \left(\frac{\|\vec{u}\|_1 + |y_1|}{N^{\alpha/4}\|x_3\|^{\alpha/2}}\right)^4 \mathrm{d}y_1.
\end{aligned}$$

If $2q/\alpha > 2$ and $N^{\alpha/2}\|x_1\| \ge 1$, we have

$$\begin{aligned}
\int_{\mathcal{A}_{2,N}} \nu_N\left(y_1, y_3, \|\vec{u}\|_1\right) \mathrm{d}y_1 \mathrm{d}y_3 \le C \frac{\|\vec{u}\|_1^{5+2q/\alpha}}{1-\kappa_{x_1,x_3}^2} & \frac{1}{N^{\alpha/2}\|x_1\|^{\alpha/2}\|x_3\|^{\alpha/2}} \\
& + C \frac{1}{1-\kappa_{x_1,x_3}^2} \frac{1}{(N^{\alpha/4}\|x_1\|^{\alpha/2})^{2q/\alpha} N^{\alpha/4}\|x_3\|^{\alpha/2}} \\
& + C \frac{1}{1-\kappa_{x_1,x_3}^2} \frac{1}{N^{\alpha/4}\|x_1\|^{\alpha/2}(N^{\alpha/4}\|x_3\|^{\alpha/2})^4}.
\end{aligned}$$

This implies

$$\int_{\mathcal{A}_{2,N}} \nu_N\left(y_1, y_3, \|\vec{u}\|_1\right) \mathrm{d}y_1 \mathrm{d}y_3 \le C \frac{1 + \|\vec{u}\|_1^{5+2q/\alpha}}{1-\kappa_{x_1,x_3}^2} \frac{1}{N^{\alpha/2}\|x_1\|^{\alpha/2}\|x_3\|^{\alpha/2}}.$$

In the same way, we deduce that

$$\int_{\mathcal{A}_{3,N}} \nu_N\left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \le C \frac{1 + \|\vec{u}\|_1^{5+2q/\alpha}}{1-\kappa_{x_1,x_3}^2} \frac{1}{N^{\alpha/2}\|x_1\|^{\alpha/2}\|x_3\|^{\alpha/2}}.$$

Let us consider the last part $\int_{\mathcal{A}_{4,N}} \nu_N \left(y_1, y_3, \|\vec{u}\|_1\right) \mathrm{d}y_1 \mathrm{d}y_3$. Note that, for any $q > 0$,

$$\sup_{\|x_1 - x_3\| > 0} \mathbb{E}_{\|x_1-x_3\|} \left[D_{1,2}^q D_{3,4}^q\right] \le C.$$

For $(y_1, y_3) \in \mathcal{A}_{4,N}$, we have, for N large enough,

$$|y_1| > C\frac{\|\vec{u}\|_1}{N^{\alpha/4} \|x_1\|^{\alpha/2}} \quad \text{and} \quad |y_3| > C\frac{\|\vec{u}\|_1}{N^{\alpha/4} \|x_3\|^{\alpha/2}}.$$

Then, if $2q/\alpha > 1$, $N^{\alpha/2} \|x_1\| \ge 1$, $N^{\alpha/2} \|x_3\| \ge 1$,

$$\begin{aligned}
&\int_{\mathcal{A}_{4,N}} \nu_N \left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
&\qquad \le \int_{\mathcal{A}_{4,N}} \left(\|\vec{u}\|_1 + |y_1| + |y_3|\right)^{4+4q/\alpha} \frac{\mathbb{E}_{\|x_1-x_3\|} \left[D_{1,2}^q D_{3,4}^q\right]}{\left(\sqrt{2}N^{\alpha/4} \|x_3\|^{\alpha/2} |y_3| \sqrt{2}N^{\alpha/4} \|x_1\|^{\alpha/2} |y_1|\right)^{2q/\alpha}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \phi_{0,\Sigma^{(2,2)}_{x_1,x_3}} \left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3.
\end{aligned}$$

This gives

$$\begin{aligned}
&\int_{\mathcal{A}_{4,N}} \nu_N \left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
\le\;& C\frac{\|\vec{u}\|_1^{4+4q/\alpha}}{1-\kappa^2_{x_1,x_3}} \int_{\mathcal{A}_{4,N}} \frac{\mathbb{E}_{\|x_1-x_3\|} \left[D_{1,2}^q D_{3,4}^q\right]}{\left(\sqrt{2}N^{\alpha/4} \|x_3\|^{\alpha/2} |y_3| \sqrt{2}N^{\alpha/4} \|x_1\|^{\alpha/2} |y_1|\right)^{2q/\alpha}} \mathrm{d}y_1 \mathrm{d}y_3 \\
&+C \int_{\mathcal{A}_{4,N}} \frac{\mathbb{E}_{\|x_1-x_3\|} \left[D_{1,2}^q D_{3,4}^q\right] |y_1|^{4+4q/\alpha}}{\left(\sqrt{2}N^{\alpha/4} \|x_3\|^{\alpha/2} |y_3| \sqrt{2}N^{\alpha/4} \|x_1\|^{\alpha/2} |y_1|\right)^{2q/\alpha}} \phi_{0,\Sigma^{(2,2)}_{x_1,x_3}} \left(y_1, y_3\right) \mathrm{d}y_1 \mathrm{d}y_3 \\
\le\;& C\frac{\|\vec{u}\|_1^{4+4q/\alpha}}{1-\kappa^2_{x_1,x_3}} \frac{1}{N^{\alpha/2} \|x_1\|^{\alpha/2} \|x_3\|^{\alpha/2}} + C\frac{1}{(N^{\alpha/4} \|x_1\|^{\alpha/2})^{2q/\alpha} N^{\alpha/4} \|x_3\|^{\alpha/2}} \\
\le\;& C\frac{1+\|\vec{u}\|_1^{4+4q/\alpha}}{1-\kappa^2_{x_1,x_3}} \frac{1}{N^{\alpha/2} \|x_1\|^{\alpha/2} \|x_3\|^{\alpha/2}}.
\end{aligned}$$

We can therefore deduce, by adapting the proof of Lemma 3 that

$$\begin{aligned}
\mathbb{E}\left[\left|\psi_N^{\Psi\Psi}\left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3)\right)\right|\right] &\le C\frac{1}{1-\kappa^2_{x_1,x_3}} \frac{1}{N^{2+\alpha/2} \|x_1\|^{\alpha/2} \|x_3\|^{\alpha/2}} \\
&\le \frac{C}{N^{2+\alpha/2}} \frac{1}{\|x_1 - x_3\|^{\alpha}}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\left|R^{(1),\Psi\Psi}_{N,\delta}\right| &\le N^{\alpha/2+2} \int_{\mathbf{C}^2} \mathbb{I}\left[\|x_1 - x_3\| \le \delta\right] \mathbb{E}\left[\left|\psi_N^{\Psi\Psi}\left(W^{(2\backslash 1)}(x_1), W^{(2\backslash 1)}(x_3)\right)\right|\right] \mathrm{d}x_1 \mathrm{d}x_3 \\
&\le C\int_{\mathbf{C}^2} \mathbb{I}\left[\|x_1 - x_3\| \le \delta\right] \frac{1}{\|x_3 - x_1\|^{\alpha}} \mathrm{d}x_1 \mathrm{d}x_3 \\
&\le c\delta^{2-\alpha}.
\end{aligned}$$

and that

$$\limsup_{N\to\infty}\left|R_{N,\delta}^{(1),\Psi\Psi}\right| \leq c\delta^{2-\alpha}.$$

In the same way, we can show that

$$\limsup_{N\to\infty}\left|R_{N,\delta}^{(1),\Psi F}\right| \leq C\delta^{2-\alpha}, \quad \limsup_{N\to\infty}\left|R_{N,\delta}^{(1),F\Psi}\right| \leq C\delta^{2-\alpha} \quad \limsup_{N\to\infty}\left|R_{N,\delta}^{(1),FF}\right| \leq C\delta^{2-\alpha}.$$

Step 2: proof of (4.9).
First note

$$\psi_N\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right) = \tilde{\psi}_N^{\Psi\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right) - \tilde{\psi}_N^{\Psi F}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right) - \tilde{\psi}_N^{F\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)$$

with

$$\begin{aligned}
&\tilde{\psi}_N^{\Psi\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\\
&= \int_{(\mathbf{R}^2)^2}\int_{(\mathbf{R}^2)^2} \Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\\
&\quad\times \Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\setminus 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\\
&\quad\times \phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\\
&\quad - F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_1)\right)F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2} p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\psi}_N^{\Psi F}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\\
&= F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2}\int_{(\mathbf{R}^2)^2} \Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\setminus 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\\
&\quad\times \phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\\
&\quad - F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_1)\right)F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2} p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\psi}_N^{F\Psi}\left(W^{(2\setminus 1)}(x_1), W^{(2\setminus 1)}(x_3)\right)\\
&= F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_1)\right)\int_{(\mathbf{R}^2)^2}\int_{(\mathbf{R}^2)^2} \Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\setminus 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\\
&\quad\times \phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\\
&\quad - F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_1)\right)F_{H_2}\left(N^{\alpha/4}W^{(2\setminus 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2} p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4.
\end{aligned}$$

Hence we define

$$
\begin{aligned}
R_{N,\delta}^{(2),\Psi\Psi} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|>\delta\right]\mathbb{E}\left[\tilde{\psi}_N^{\Psi\Psi}\left(W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3\\
R_{N,\delta}^{(2),\Psi F} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|>\delta\right]\mathbb{E}\left[\tilde{\psi}_N^{\Psi F}\left(W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3\\
R_{N,\delta}^{(2),F\Psi} &= N^{\alpha/2+2}\int_{\mathbf{C}^2}\mathbb{I}\left[\|x_1-x_3\|>\delta\right]\mathbb{E}\left[\tilde{\psi}_N^{F\Psi}\left(W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\right)\right]\mathrm{d}x_1\mathrm{d}x_3.
\end{aligned}
$$

Note that if $\|x_1-x_3\|>\delta$, $d_{1,2}\to 0$ and $d_{3,4}\to 0$, then, uniformly in x_1 and x_3,

$$
\mu_{x_1,x_2,x_3,x_4}^{|x_1,x_3}\to 0_{4\times 1}\qquad\text{and}\qquad \Sigma_{x_1,x_2,x_3,x_4}^{|x_1,x_3}\to I_4.
$$

If $\|x_1-x_3\|>\delta$, there exists a positive constant K_δ such that

$$
\det\Sigma_{x_1,x_2,x_3,x_4}^{|x_1,x_3}>K_\delta^{-1}\qquad\text{and}\qquad \max_{i,j}|\Sigma_{x_1,x_2,x_3,x_4}^{|x_1,x_3}(i,j)|<K_\delta
$$

with a probability tending to 1 as $N\to\infty$. By Proposition 4, there exist positive constants ς_δ and c_{K_δ} such that

$$
\begin{aligned}
\Big|\int_{(\mathbf{R}^2)^2}&\Psi_{H_2}\left(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}\right)\\
&\times\Psi_{H_2}\left(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2}\right)\\
&\times\phi_{\mu_{x_1,x_2,x_3,x_4}^{|x_1,x_3},\Sigma_{x_1,x_2,x_3,x_4}^{|x_1,x_3}}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\\
&-\int_{(\mathbf{R}^2)^2}\Psi_{H_2}\left(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}\right)\\
&\qquad\times\Psi_{H_2}\left(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2}\right)\\
&\qquad\times\phi_{0,I_4}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4\Big|
\end{aligned}
$$

is lower than

$$
\begin{aligned}
\varsigma_\delta\left(|\eta_{x_1,x_2,x_3,x_4}|+\upsilon_{x_1,x_2,x_3,x_4}\right)&\left(1+\frac{\left|W^{(2\backslash 1)}(x_1)\right|}{\|x_1\|^{\alpha/2}}+\frac{\left|W^{(2\backslash 1)}(x_3)\right|}{\|x_3\|^{\alpha/2}}\right)\\
&\times\int_{(\mathbf{R}^2)^2}\Big|\Psi_{H_2}\left(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}\right)\\
&\quad\times\Psi_{H_2}\left(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2}\right)\Big|\\
&\qquad\times\left(1+\|\vec{u}\|^2\right)\exp\left(-c_{K_\delta}\|\vec{u}\|^2\right)\mathrm{d}\vec{u}.
\end{aligned}
$$

Moreover, if $d_{1,2}\le\gamma d_{1,3}^{2+\varepsilon}$ and $d_{3,4}\le\gamma d_{1,3}^{2+\varepsilon}$ with $\gamma,\varepsilon>0$,

$$
\upsilon_{x_1,x_2,x_3,x_4}\le C|\rho_{x_1,x_2}|\vee|\rho_{x_3,x_4}|\le C\frac{d_{1,2}^{\alpha/2}}{\|x_1\|^{\alpha/2}}\vee\frac{d_{3,4}^{\alpha/2}}{\|x_3\|^{\alpha/2}}.
$$

and using the same approach as in the proof of Lemma 6 (i) in [3]

$$
|\eta_{x_1,x_2,x_3,x_4}|\le C\left(d_{1,2}d_{3,4}\right)^{1-\alpha/2}.
$$

Then, using the same type of arguments as in Part 1 and the fact that $0<\alpha<1$, we obtain that

$$\Big| \int_{(\mathbf{R}^2)^2\times(\mathbf{R}^2)^2} \Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}) \times \Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2}) \times \phi_{\mu^{|x_1,x_3}_{x_1,x_2,x_3,x_4},\Sigma^{|x_1,x_3}_{x_1,x_2,x_3,x_4}}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4 - \int_{(\mathbf{R}^2)^2\times(\mathbf{R}^2)^2} \Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}) \times \Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2}) \times \phi_{0,I_4}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4 \Big|$$

is lower than

$$\varsigma_\delta \left(1+\frac{\left|W^{(2\backslash 1)}(x_1)\right|}{\|x_1\|^{\alpha/2}}+\frac{\left|W^{(2\backslash 1)}(x_3)\right|}{\|x_3\|^{\alpha/2}}\right) \times \int_{(\mathbf{R}^2)^2} \frac{1}{N^2}\left(u_1^2+u_2^2\right)\left(u_3^2+u_4^2\right)\mathbb{E}_{\|x_1-x_3\|}\Big[\left(\frac{D_{1,2}^{\alpha/2}}{N^{\alpha/4}\|x_1\|^{\alpha/2}}+\frac{D_{3,4}^{\alpha/2}}{N^{\alpha/4}\|x_3\|^{\alpha/2}}\right) \times \mathbb{I}\left[D_{1,2}^{\alpha/2}>\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_1)|}{|u_1-u_2|}, D_{3,4}^{\alpha/2}>\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_3)|}{|u_3-u_4|}\right]\Big| W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\Big] \times \mathbb{P}_{\|x_1-x_3\|}\left[D_{1,2}^{\alpha/2}>\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_1)|}{|u_1-u_2|}, D_{3,4}^{\alpha/2}>\frac{N^{\alpha/4}|W^{(2\backslash 1)}(x_3)|}{|u_3-u_4|}\,\middle|\, W^{(2\backslash 1)}(x_1),W^{(2\backslash 1)}(x_3)\right] \times \left(1+\|\vec{u}\|^2\right)\exp\left(-c_{K_\delta}\|\vec{u}\|^2\right)\mathrm{d}\vec{u}$$

and therefore lower than

$$C\left(1+\frac{\left|W^{(2\backslash 1)}(x_1)\right|}{\|x_1\|^{\alpha/2}}+\frac{\left|W^{(2\backslash 1)}(x_3)\right|}{\|x_3\|^{\alpha/2}}\right)\left(\frac{1}{\|x_1\|^{\alpha/2}}+\frac{1}{\|x_3\|^{\alpha/2}}\right)\times o\left(\frac{1}{N^{2+\alpha/2}}\right).$$

Now note that

$$\int_{(\mathbf{R}^2)^2\times(\mathbf{R}^2)^2} \Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\backslash 1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2}) \times \Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\backslash 1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2}) \times \phi_{0,I_4}(u_1,u_2,u_3,u_4)\,\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4 - F_{H_2}\left(N^{\alpha/4}W^{(2\backslash 1)}(x_1)\right)F_{H_2}\left(N^{\alpha/4}W^{(2\backslash 1)}(x_3)\right)\int_{(\mathbf{R}^2)^2} p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4$$

is equal to

$$\begin{aligned}\int_{(\mathbf{R}^2)^2\times(\mathbf{R}^2)^2} &\Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\setminus1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\\ &\times\Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\setminus1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\\ &\times\phi_{0,I_4}\left(u_1,u_2,u_3,u_4\right)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\\ &-\int_{(\mathbf{R}^2)^2\times\mathbf{R}_+^2}\Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\setminus1)}(x_1)/\left(d_1\right)^{\alpha/2})\\ &\qquad\times\Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\setminus1)}(x_3)/\left(d_2\right)^{\alpha/2})\\ &\qquad\times\phi_{0,I_4}\left(u_1,u_2,u_3,u_4\right)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 f_D\left(d_1\right)f_D\left(d_2\right)\mathrm{d}d_1\mathrm{d}d_2\\ &\qquad\times\int_{(\mathbf{R}^2)^2}p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4.\end{aligned}$$

If x_1 and x_3 are fixed such that $\|x_1-x_3\|>\delta$, it follows by the ergodicity of the Delaunay triangulation that

$$\begin{aligned}\Big|\int_{(\mathbf{R}^2)^2\times(\mathbf{R}^2)^2} &\Psi_{H_2}(u_1,u_2,N^{\alpha/4}W^{(2\setminus1)}(x_1)/(d_{1,2}N^{1/2})^{\alpha/2})\\ &\times\Psi_{H_2}(u_3,u_4,N^{\alpha/4}W^{(2\setminus1)}(x_3)/(d_{3,4}N^{1/2})^{\alpha/2})\\ &\times\phi_{0,I_4}\left(u_1,u_2,u_3,u_4\right)\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\mathrm{d}u_4 p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\\ -F_{H_2}\left(N^{\alpha/4}W^{(2\setminus1)}(x_1)\right)&F_{H_2}\left(N^{\alpha/4}W^{(2\setminus1)}(x_3)\right)\int_{(\mathbf{R}^2)^2}p_{2,N}(x_1,x_2,x_3,x_4)\mathrm{d}x_2\mathrm{d}x_4\Big|\\ &=o\left(\frac{1}{N^{2+\alpha/2}}\right)\end{aligned}$$

uniformly in (x_1,x_3).

For any $\delta>0$, we deduce that $\lim_{N\to\infty}|R_{N,\delta}^{(2),\Psi\Psi}|=0$. The same conclusion holds for $R_{N,\delta}^{(2),\Psi F}$ and $R_{N,\delta}^{(2),F\Psi}$.

Part 3. Proof of the fact that F_{H_2} belongs to the Schwartz space

Let us remark that

$$F_{H_2}(z)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{\mathbf{R}^2\times\mathbf{R}_+}e^{-\frac{x^2+y^2}{2}}f_D(d)\left((y+z_d)^2-x^2\right)\mathbb{I}\left[x-y\leq z_d\right]\mathrm{d}x\mathrm{d}y\mathrm{d}d & \text{if } z\leq 0\\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbf{R}^2\times\mathbf{R}_+}e^{-\frac{x^2+y^2}{2}}f_D(d)\left((x-z_d)^2-y^2\right)\mathbb{I}\left[z_d\leq x-y\right]\mathrm{d}x\mathrm{d}y\mathrm{d}d & \text{if } z\geq 0,\end{cases}$$

where $z_d=\frac{z}{d^{\alpha/2}}$. Notice that the function F_{H_2} is even and infinitely differentiable. Moreover, for any $z\geq 0$,

$$F_{H_2}(z)=\frac{1}{2\pi}\int_{\mathbf{R}_+}\int_{\mathbf{R}}\left(\int_{[y,\infty)}\left(x'^2-y^2\right)e^{-\frac{(x'+z_d)^2}{2}}\mathrm{d}x'\right)e^{-\frac{y^2}{2}}f_D(d)\mathrm{d}y\mathrm{d}d.$$

According to (2.4), we know that $f_D(d)\leq\kappa d^3e^{-\pi d^2}$ for some positive constant κ. Applying the Lebesgue's dominated convergence theorem, we can show that $e^{\frac{1}{2}z}F_{H_2}(z)$ converges to 0 as $z\to\infty$.

By induction, we also have $e^{\frac{1}{2}z}F_{H_2}^{(k)}(z) \underset{z\to\infty}{\to} 0$ for any $k \geq 1$. Such a property implies that F_{H_2} belongs to the Schwartz space.

4.1.2 Proof for $V_{3,N}^{(2/1)}$

We do not give the proof since it is more technical but it relies on a simple adaptation of the proof for $V_{2,N}^{(2/1)}$.

4.2 Proof of Theorem 2

4.2.1 Proof for $V_{2,N}^{(W_\vee)}$

By Proposition 1, Eq. (3.3) and Eq. (3.6), the result is immediate.

4.2.2 Proof for $V_{3,N}^{(W_\vee)}$

First, we prove the decomposition of $V_{3,N}^{(W_\vee)}$ given in Eq. (3.5). We have

$$V_{3,N}^{(W_\vee)} = \frac{1}{\sqrt{|DT_N|}} \times \sum_{(x_1,x_2,x_3)\in DT_N} \left(\begin{pmatrix} U_{x_1,x_2}^{(W_\vee)} & U_{x_1,x_3}^{(W_\vee)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1,x_2}^{(W_\vee)} \\ U_{x_1,x_3}^{(W_\vee)} \end{pmatrix} - 2 \right).$$

This gives

$$V_{3,N}^{(W_\vee)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \frac{1}{1-R_{x_1,x_2,x_3}^2} \times \left([(U_{x_1,x_2}^{(W_\vee)})^2 - 1] + [(U_{x_1,x_3}^{(W_\vee)})^2 - 1] - 2R_{x_1,x_2,x_3} U_{x_1,x_2}^{(W_\vee)} U_{x_1,x_3}^{(W_\vee)} \right),$$

and consequently

$$V_{3,N}^{(W_\vee)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} H_2(U_{x_1,x_2}^{(W_\vee)}, U_{x_1,x_3}^{(W_\vee)}; R_{x_1,x_2,x_3}),$$

where, for any $R \in (0,1)$,

$$H_2(u_1,u_2;R) = \frac{1}{1-R^2} \left[H_2(u_1) + H_2(u_2) - 2Ru_1u_2 \right].$$

According to Eq. (3.2), applied to $f = H_2$ and $f = I$ respectively, we have, for $i = 2,3$,

$$H_2(U_{x_1,x_i}^{(W_\vee)}) = H_2(U_{x_1,x_i}^{(1)})\mathbb{I}[W^{(2\setminus 1)}(x_1) < 0] + H_2(U_{x_1,x_i}^{(2)})\mathbb{I}[W^{(2\setminus 1)}(x_1) > 0] + \Psi_{H_2}(U_{x_1,x_i}^{(1)}, U_{x_1,x_i}^{(2)}, W^{(2\setminus 1)}(x_1)/(\sigma d_{1,i}^{\alpha/2}))$$

and

$$U_{x_1,x_i}^{(W_\vee)} = U_{x_1,x_i}^{(1)}\mathbb{I}[W^{(2\setminus 1)}(x_1) < 0] + U_{x_1,x_i}^{(2)}\mathbb{I}[W^{(2\setminus 1)}(x_1) > 0] + \Psi_I(U_{x_1,x_i}^{(1)}, U_{x_1,x_i}^{(2)}, W^{(2\setminus 1)}(x_1)/(\sigma d_{1,i}^{\alpha/2})).$$

Therefore we have

$$
\begin{aligned}
H_2(U^{(W_\vee)}_{x_1,x_2}, U^{(W_\vee)}_{x_1,x_3}; R_{x_1,x_2,x_3}) = {} & H_2(U^{(1)}_{x_1,x_2}, U^{(1)}_{x_1,x_3}; R_{x_1,x_2,x_3})\mathbb{I}[W^{(2\backslash 1)}(x_1) < 0] \\
& + H_2(U^{(2)}_{x_1,x_2}, U^{(2)}_{x_1,x_3}; R_{x_1,x_2,x_3})\mathbb{I}[W^{(2\backslash 1)}(x_1) > 0] \\
& + \Omega(U^{(1)}_{x_1,x_2}, U^{(2)}_{x_1,x_2}, U^{(1)}_{x_1,x_3}, U^{(2)}_{x_1,x_3}, W^{(2\backslash 1)}(x_1)/(\sigma d_{1,2}^{\alpha/2}), W^{(2\backslash 1)}(x_1)/(\sigma d_{1,3}^{\alpha/2}); R_{x_1,x_2,x_3}),
\end{aligned}
$$

where $\Omega(u_1, v_1, u_2, v_2, w_1, w_2; R)$ is defined in Eq. (3.4). This proves Eq. (3.5).

This together with Eq. (3.7) and Proposition 1 concludes the proof of Theorem 2.

A Appendix

A.1 Asymptotic expansion for $1 - \kappa^2_{x_1,x_3}$

Lemma 3 *Let $x_1 \in \mathbf{R}^2 \setminus \{0\}$ be fixed. For any point $x_3 \in \mathbf{R}^2 \setminus \{0\}$, let*

$$\kappa_{x_1,x_3} = \frac{\|x_1\|^\alpha + \|x_3\|^\alpha - \|x_1 - x_3\|^\alpha}{2\|x_1\|^{\alpha/2}\|x_3\|^{\alpha/2}}.$$

Then

$$1 - \kappa^2_{x_1,x_3} \underset{d_{1,3}\to 0}{\sim} 2(1 - \kappa_{x_1,x_3}) \underset{d_{1,3}\to 0}{\sim} \frac{d_{1,3}^\alpha}{\|x_1\|^\alpha},$$

where $d_{1,3} := \|x_3 - x_1\|$.

Proof of Lemma 3.

Let $x_3 = x_1 + ru$ be such that $\|x_1 - x_3\| = r > 0$ with $u \in \mathbf{S}^1$. We have

$$
\begin{aligned}
\|x_3\| &= \|x_1 + ru\| \\
&= \left(\|x_1\|^2 + 2\langle u, x_1\rangle + r^2\right)^{1/2} \\
&= \|x_1\|\left(1 + \frac{\langle u, x_1\rangle}{\|x_1\|} r + o(r)\right),
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{R}^2$. Therefore, using the fact that $0 < \alpha < 1$,

$$
\begin{aligned}
\kappa_{x_1,x_3} &= \frac{\|x_1\|^\alpha + \|x_1\|^\alpha (1 + \langle u, x_1\rangle r/\|x_1\| + o(r))^\alpha - r^\alpha}{2\|x_1\|^{\alpha/2}\|x_1\|^{\alpha/2}(1 + \langle u, x_1\rangle r/\|x_1\| + o(r))^{\alpha/2}} \\
&= \frac{2\|x_1\|^\alpha - r^\alpha + o(r^\alpha)}{2\|x_1\|^\alpha (1 + \alpha\langle u, x_1\rangle r/(2\|x_1\|) + o(r))} \\
&= (1 - r^\alpha/(2\|x_1\|^\alpha) + o(r^\alpha))(1 - \alpha\langle u, x_1\rangle r/(2\|x_1\|) + o(r)) \\
&= 1 - \frac{1}{2\|x_1\|^\alpha} r^\alpha + o(r^\alpha).
\end{aligned}
$$

We deduce that

$$1 - \kappa^2_{x_1,x_3} = (1 - \kappa_{x_1,x_3})(1 + \kappa_{x_1,x_3}) \underset{d_{1,3}\to 0}{\sim} 2(1 - \kappa_{x_1,x_3}) \underset{d_{1,3}\to 0}{\sim} \frac{d_{1,3}^\alpha}{\|x_1\|^\alpha}.$$

□

A.2 Generalization of Proposition 3.1 in [7]

The following proposition is a broader version of Proposition 3.1 in [7]. We omit its proof, as it follows the same reasoning.

Proposition 4 *Let $Z \sim \mathcal{N}_d(0, \Sigma)$ and $Z' \sim \mathcal{N}_d(\mu', \Sigma')$, where $\mu' \in \mathbf{R}^d$ and $\Sigma, \Sigma' \in \mathbf{R}^{d\times d}$ are positive definite matrices. Assume that there exists a constant $K > 0$ with*

$$\max_{1\leq i,j\leq d}\{|\Sigma_{i,j}| + |\Sigma'_{i,j}|\} < K, \qquad \min\{\det\Sigma, \det\Sigma'\} > K^{-1}.$$

Let $G : \mathbf{R}^d \to \mathbf{R}$ be a function with polynomial growth. Then there exist constants $c_K, C_K > 0$ such that

$$\begin{aligned}&|\mathbb{E}[G(Z)] - \mathbb{E}[G(Z')]| \\ &\qquad \leq C_K \left[\max_{1\leq i,j\leq d}\{|\Sigma_{i,j} - \Sigma'_{i,j}|\} + \max_{1\leq i\leq d}|\mu'_i|\right] \int_{\mathbf{R}^d} |G(y)| \left(1 + \|y\|^2\right) \exp\left(-c_K \|y\|^2\right) \mathrm{d}y.\end{aligned}$$

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