Asymptotic properties of maximum composite likelihood estimators for max-stable Brown-Resnick random fields over a fixed-domain

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Abstract

Likelihood inference for max-stable random fields is in general impossible because their finite-dimensional probability density functions are unknown or cannot be computed efficiently. The weighted composite likelihood approach that utilizes lower dimensional marginal likelihoods (typically pairs or triples of sites that are not too distant) is rather favored. In this paper, we consider the family of spatial max-stable Brown-Resnick random fields associated with isotropic fractional Brownian fields. We assume that the sites are given by only one realization of a homogeneous Poisson point process restricted to $\mathbf{C} = (-1/2, 1/2]^2$ and that the random field is observed at these sites. As the intensity increases, we study the asymptotic properties of the composite likelihood estimators of the scale and Hurst parameters of the fractional Brownian fields using different weighting strategies: we exclude either pairs that are not edges of the Delaunay triangulation or triples that are not vertices of triangles.

Keywords: Brown-Resnick random fields, Composite likelihood estimators, Fixed-domain asymptotics, Gaussian random fields, Poisson random sampling, Delaunay triangulation.

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1 Introduction

Gaussian random fields are widely used to model spatial data because their finite-dimensional distributions are only characterized by the mean and covariance functions. In general it is assumed that these functions belong to some parametric models which leads to a parametric estimation problem. When extreme value phenomena are of interest and meaningful spatial patterns can be discerned, max-stable

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random field models are preferred to describe such phenomena. However, likelihood inference is challenging for such models because their corresponding finite-dimensional probability density functions are unknown or cannot be computed efficiently. In this paper we study composite likelihood estimators in a fixed-domain asymptotic framework for a widely used class of stationary max-stable random fields: the Brown-Resnick random fields. As a preliminary, we provide brief reviews of work on maximum likelihood estimators and on composite likelihood estimators for Gaussian random fields under fixed-domain asymptotics and present max-stable random fields with their canonical random tessellations.

1.1 Maximum likelihood estimators for Gaussian random fields under fixed-domain asymptotics

The fixed-domain asymptotic framework, also called infill asymptotics ([11, 33]), corresponds to the case where more and more data are observed in some fixed bounded sampling domain (usually a region of $\mathbf{R}^d$, $d \in \mathbf{N}_*$). Within this framework, the maximum likelihood estimators (MLE) of the covariance parameters of Gaussian random fields have been deeply studied in the last three decades.

It is noteworthy that two types of covariance parameters have to be distinguished: microergodic and non-microergodic parameters. A parameter is said to be microergodic if, for two different values of it, the two corresponding Gaussian measures are orthogonal [25, 33]). It is non-microergodic if, even for two different values of it, the two corresponding Gaussian measures are equivalent. Non-microergodic parameters cannot be estimated consistently under fixed-domain asymptotics. Moreover, there are no general results on the asymptotic properties of the microergodic MLE.

1.2 Maximum composite likelihood estimators for Gaussian random fields under fixed-domain asymptotics

From a theoretical point of view, the maximum likelihood method is the best approach for estimating the covariance parameters of a Gaussian random field. Nevertheless, the evaluation of the likelihood function under the Gaussian assumption requires a computational burden of order $O(n^3)$ for n observations (because of the inversion of the $n \times n$ covariance matrix), making this method computationally impractical for large datasets. The composite likelihood (CL) methods rather use objective functions based on the likelihood of lower dimensional marginal or conditional events [34]. These methods are generally appealing when dealing with large data sets or when it is difficult to specify the full likelihood, and provide estimation methods with a good balance between computational complexity and statistical efficiency.

There are few results under fixed domain asymptotics for maximum CL estimators (MCLE) for Gaussian random fields. They are all obtained in the case $d = 1$. The problem of estimating the covariance parameters of a Gaussian process with an exponential covariance function was studied by [1]. They showed that the weighted pairwise maximum likelihood estimator of the microergodic parameters can be consistent, but also inconsistent, according to the objective function; e.g. the weighted pairwise conditional maximum likelihood estimator is always consistent (and also asymptotically Gaussian). Meanwhile, [2] considered a Gaussian process whose covariance function is parametrized by variance, scale and smoothness parameters. They focused on CL objective functions based on the conditional log likelihood of the observations given the K (resp. L) observations corresponding to the left (resp. right) nearest neighbor observation points. They examined the case where only the variance parameter is unknown and the case where the variance and the spatial scale are jointly estimated. In the first case they proved that for small values of the smoothness parameter, the composite likelihood estimator converges at a sub-optimal rate and they showed that the asymptotic distribution is not Gaussian. For large values of the smoothness parameter, they proved that the estimator converges at the optimal rate.

1.3 Fixed-domain asymptotics for non-Gaussian random fields

To the best of our knowledge, there are a few papers that study MLE or MCLE for non-Gaussian random fields under fixed-domain asymptotics. For example, [28] proposed approximate maximum-likelihood estimation for diffusion processes ($d = 1$) and provided closed-form asymptotic expansion for transition density. But diffusion processes may not be generalized for $d \geq 2$.

Other papers rather considered variogram-based or power variation-based estimators. A random field of the form $g(X)$, where $g : \mathbf{R}\to\mathbf{R}$ is an unknown smooth function and X is a real-valued stationary Gaussian field on $\mathbf{R}^d$ ($d = 1$ or 2) whose covariance function obeys a power law at the origin, is considered in [5]. The authors addressed the question of the asymptotic properties of variogram-based estimators when $g(X)$ is observed instead of X under a fixed-domain framework. They established that the asymptotic distribution theory for nonaffine g is somewhat richer than in the Gaussian case (i.e. when g is an affine transformation). Although the variogram-based estimators are not MLE or MCLE, this study shows that their asymptotic properties can differ significantly from the Gaussian random field case.

A particular class of max-stable processes ($d = 1$) was studied in [30] : the class of simple Brown-Resnick max-stable processes whose spectral processes are continuous exponential martingales. The author developed the asymptotic theory for the realized power variations of these max-stable processes, i.e., sums of powers of absolute increments. Under a fixed-domain asymptotic setting, a biased central limit theorem was obtained, where the bias depends on the local times of the differences between the logarithms of the underlying spectral processes.

1.4 Max-stable random fields

Max-stable random fields appear as the only possible non-degenerate limits for normalized pointwise maxima of independent and identically distributed (i.i.d.) random fields with continuous sample paths (see e.g. [15]). The one-dimensional marginal distributions of max-stable fields belong to the parametric class of Generalized Extreme Value distributions. Since we are interested in the estimation of parameters characterizing the dependence structure, we restrict our attention to max-stable random fields $\eta = (\eta(x))_{x\in\mathcal{X}}$ on $\mathcal{X} \subset\mathbf{R}^d$ with standard unit Fréchet margins, that is, satisfying

$$\mathbb{P}\left[\eta(x) \leq z\right] = \exp\left(-z^{-1}\right), \qquad \text{for all } x \in \mathcal{X} \text{ and } z > 0.$$

The max-stability property has then the simple form

$$n^{-1} \bigvee_{i=1}^{n} \eta_i \overset{d}{=} \eta$$

where $(\eta_i)_{1\leq i\leq n}$ are i.i.d. copies of η, $\bigvee$ is the pointwise maximum, and $\overset{d}{=}$ denotes the equality of finite-dimensional distributions. Max-stable random fields are characterized by their spectral representation (see e.g., [14], [22]): any stochastically continuous max-stable process η can be written as

$$\eta(x) = \bigvee_{i\geq 1} U_i Y_i(x), \qquad x \in \mathcal{X}, \tag{1.1}$$

where $(U_i)_{i\geq 1}$ is the decreasing enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity measure $u^{-2}\mathrm{d}u$, $(Y_i)_{i\geq 1}$ are i.i.d. copies of a non-negative stochastic random field Y on $\mathcal{X}$ such that $\mathbb{E}[Y(x)] = 1$ for all $x \in \mathcal{X}$, the sequences $(U_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ are independent.

The spectral representation (1.1) makes it possible to construct a canonical tessellation of $\mathcal{X}$ as in [18]. We define the cell associated with each index $i \geq 1$ by $C_i = \{x \in \mathcal{X} : U_i Y_i(x) = \eta(x)\}$. It is a (possibly empty) random closed subset of $\mathcal{X}$ and each point $x \in \mathcal{X}$ belongs almost surely (a.s.) to a unique cell (the point process $\{U_i Y_i(x)\}_{i\geq 1}$ is a Poisson point process with intensity $u^{-2}\mathrm{d}u$ so that the maximum $\eta(x)$ is almost surely attained for a unique index i). It is noteworthy that the terms *cell* and *tessellation* are meant in a broader sense than in Stochastic Geometry where they originated. Here, a cell is a general (not necessarily convex or connected) random closed set and a tessellation is a random covering of $\mathcal{X}$ by closed sets with pairwise disjoint interiors.

Likelihood inference is challenging for max-stable random fields because their finite-dimensional probability density functions are unknown or cannot be computed efficiently. A composite-likelihood approach was proposed in [29], but the discussion on the asymptotic properties of the estimators was limited to cases where the data sites are fixed and where a large number of i.i.d. data replications is available.

1.5 Contributions of the paper

In this paper, we consider the class of spatial max-stable Brown-Resnick random fields ($d = 2$) associated with isotropic fractional Brownian random fields as defined in [27]. Because the sites where the random fields are observed are very rarely on grids in practice, we considered a random sampling scheme. We assume a Poisson stochastic spatial sampling scheme and use the Poisson-Delaunay triangulation to select the pairs and triples of sites with their associated marginal distributions that will be integrated into the CL objective functions (we exclude pairs that are not edges of the Delaunay triangulation or triples that are not vertices of triangles of this triangulation). Note that using the Delaunay triangulation is relatively natural here since we only use the distributions of pairs and triples. Moreover, the Delaunay triangulation appears to be the most "regular" triangulation in the sense that it is the one that maximises the minimum of the angles of the triangles.

We study for the first time the asymptotic properties of the MCLEs of the scale and Hurst parameters of the max-stable Brown-Resnick random fields under fixed domain asymptotics (for only one realization of a Poisson point process). Pairwise and triplewise CL objective functions (considering all pairs and triples) have been proposed for inference for max-stable processes, but the properties of the MCLE are only known when the sites are fixed and when there is a large number of independent observations over time of the max-stable random field (see, e.g., [3, 12, 24]). Note that the tapered CL estimators for max-stable random field excluding pairs that are at a too large distance apart have also been studied in [31] (here again with independent time observations), but this is the first time that a Delaunay triangulation is used to select the pairs and triples.

Understanding the asymptotic properties of the MCLEs under a fixed-domain asymptotic framework is very important in practice in order to know whether or not the limit distributions of these estimators are the same as in the case where there are several independent time observations. The statistician may only have one realization of the max-stable process at disposal, or may decide to approximate the realization of a max-stable process using pointwise maxima based on several observations of a process belonging to the domain of attraction of this max-stable process. Regardless, considering a fixed window is a major difficulty, as no mixing property can be directly exploited.

To derive the asymptotic properties of the MCLEs, we use results obtained in [8] and [9]. In [8] we considered sums of square increments of an isotropic fractional Brownian field based on the edges of the Delaunay triangles and provided asymptotic results using Malliavin calculus (see Theorem 1 in [8]). In [9] we considered sums of square increments of the pointwise maximum of two independent isotropic fractional Brownian fields and showed that the asymptotic behaviors of the sums depend on the local time at the level 0 of the difference between the two fractional Brownian fields and no more on the Gaussian limits of the sums of square increments of each isotropic fractional Brownian field (see Theorem 2 in [9]). In this paper we generalize this result to the max-stable Brown-Resnick random field which is built as the pointwise maximum of an infinite number of isotropic fractional Brownian fields (see Theorem 3). Using approximations of the pairwise and triplewise CL objective functions, we then derive the asymptotic properties of the MCLEs (see Theorem 6). The rates of convergence of the estimators as well as their limit distributions are not standard and are specific to the structure of the max-stable process. It is important to identify them in order to avoid using the Gaussian distributions typically obtained with data from multiple time observations.

It is noteworthy that we only consider isotropic fractional Brownian fields with Hurst index in $(0, 1/2)$ as in [8] and [9]. This is not a very restrictive constraint since almost all empirical studies that use the spatial Brown-Resnick random field obtain estimates in this interval (see e.g. [12, 13, 19, 20]).

Our paper is organized as follows. In Section 2 we present the family of stationary Brown-Resnick random fields introduced in [27], then we review some established concepts related to the Delaunay triangulation, and we end with the definition of the local time between two independent and identically distributed fractional Brownian random fields. In section 3, we first study the asymptotic distributions of the "normalized" increments of the logarithm of the Brown-Resnick random field based on pairs and triples of sites as the distances between sites tend to zero. We then provide asymptotic results for squared increment sums of the max-stable Brown-Resnick random field. In Section 4, we introduce the randomized sampling scheme and define the CL estimators of the scale and Hurst parameters. The asymptotic properties of the MCLEs for these parameters are then given. The proofs and some intermediate results are deferred to Section 5 and Section 6.

2 Preliminaries

2.1 Max-stable Brown-Resnick random fields

In this paper we are concerned with the class of max-stable random fields known as Brown-Resnick random fields introduced in [27]. This class of random fields is based on Gaussian random fields with linear stationary increments. Recall that a random process $(W(x))_{x\in\mathbf{R}^d}$ is said to have linear stationary increments if the law of $(W(x+x_0) - W(x_0))_{x\in\mathbf{R}^d}$ does not depend on the choice of $x_0 \in \mathbf{R}^d$. A prominent example is the isotropic fractional Brownian field where $W(0) = 0$ a.s. and semi-variogram given by

$$\gamma(x) = \frac{\mathrm{var}(W(x))}{2} = \frac{\sigma^2 \|x\|^\alpha}{2} \tag{2.1}$$

for some $\alpha \in (0, 2)$ and $\sigma^2 > 0$, where $\|x\|$ is the Euclidean norm of x. The parameter σ is called the scale parameter while α is called the range parameter ($H = \alpha/2$ is also known as the Hurst parameter and relates to the Hölder continuity exponent of W). It is noteworthy that W is a self-similar random field with linear stationary increments as presented in Definition 3.3.1 of [10] and it differs from the fractional Brownian sheet which is a self-similar random field with stationary rectangular increments (see e.g. Section 3.3.2 of [10]).

The Brown-Resnick random field is the max-stable random field η such that the random field Y given in the spectral representation (1.1) has the following form

$$Y\left(x\right)=\exp\left(W\left(x\right)-\gamma\left(x\right)\right),\qquad x\in\mathbf{R}^{2},\tag{2.2}$$

where W is an isotropic fractional Brownian field. With this choice, the max-stable random field η is stationary while W is not but has (linear) stationary increments (see [27]).

2.2 Delaunay triangulation

In this section, we recall some known results on Poisson-Delaunay triangulations.

Let P_N be a Poisson point process with intensity N in $\mathbf{R}^2$. The *Delaunay triangulation* $\mathrm{Del}(P_N)$ is the unique triangulation with vertices in P_N such that the circumball of each triangle contains no point of P in its interior, see e.g. p. 478 in [32]. Such a triangulation is the most regular one in the sense that it is the one which maximises the minimum of the angles of the triangles.

To define the mean behavior of the Delaunay triangulation (associated with a Poisson point process P_1 of intensity 1), the notion of typical cell is defined as follows. With each cell $C\in\mathrm{Del}(P_1)$, we associate the circumcenter $z(C)$ of C. Now, let $\mathbf{B}$ be a Borel subset in $\mathbf{R}^2$ with area $a(\mathbf{B})\in(0,\infty)$. The *cell intensity* β_2 of $\mathrm{Del}(P_1)$ is defined as the mean number of cells per unit area, i.e.

$$\beta_2=\frac{1}{a(\mathbf{B})}\mathbb{E}\left[\left|\{C\in\mathrm{Del}(P_1):z(C)\in\mathbf{B}\}\right|\right],$$

where $|\cdot|$ denotes the cardinality. It is well-known that $\beta_2=2$, see e.g. Theorem 10.2.9. in [32]. Then, we define the *typical cell* as a random triangle $\mathcal{C}$ with distribution given as follows: for any positive measurable and translation invariant function $g:\mathcal{K}_2\to\mathbf{R}$,

$$\mathbb{E}\left[g(\mathcal{C})\right]=\frac{1}{\beta_2 a(\mathbf{B})}\mathbb{E}\left[\sum_{C\in\mathrm{Del}(P_1):z(C)\in\mathbf{B}}g(C)\right],$$

where $\mathcal{K}_2$ denotes the set of convex compact subsets in $\mathbf{R}^2$, endowed with the Fell topology (see Section 12.2 in [32] for the definition). The distribution of $\mathcal{C}$ has the following integral representation (see e.g. Theorem 10.4.4. in [32]):

$$\mathbb{E}\left[g(\mathcal{C})\right]=\frac{1}{6}\int_0^\infty\int_{(\mathbf{S}^1)^3}r^3e^{-\pi r^2}a(\Delta(u_1,u_2,u_3))g(\Delta(ru_1,ru_2,ru_3))\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\sigma(\mathrm{d}u_3)\mathrm{d}r,\tag{2.3}$$

where $\mathbf{S}^1$ is the unit sphere of $\mathbf{R}^2$ and σ is the spherical Lebesgue measure on $\mathbf{S}^1$ with normalization $\sigma\left(\mathbf{S}^1\right)=2\pi$. In other words, $\mathcal{C}$ is equal in distribution to $R\Delta(U_1,U_2,U_3)$, where R and (U_1,U_2,U_3) are independent with probability density functions given respectively by $2\pi^2r^3e^{-\pi r^2}$ and $a(\Delta(u_1,u_2,u_3))/(12\pi^2)$.

In a similar way, we can define the notion of typical edge. The *edge intensity* β_1 of $\mathrm{Del}(P_1)$ is defined as the mean number of edges per unit area and is equal to $\beta_1=3$ (see e.g. Theorem 10.2.9. in [32]). The distribution of the length of the *typical edge* is the same as the distribution of $D=R||U_1-U_2||$.

Its probability density function f_D satisfies the following equality

$$\begin{aligned}\mathbb{P}\left[D \leq \ell\right] &= \int_0^\ell f_D(d)\mathrm{d}d \\ &= \frac{\pi}{3}\int_0^\infty \int_{(\mathbf{S}^1)^2} r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,e_1))\mathbb{I}\left[r\left\|u_1-u_2\right\| \leq \ell\right]\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\mathrm{d}r, \end{aligned} \tag{2.4}$$

where $e_1 = (1,0)$ and $\ell > 0$. Following Eq. (2.3), a *typical couple* of (distinct) Delaunay edges with a common vertex can be defined as a 3-tuple of random variables (D_1, D_2, Θ), where $D_1, D_2 \geq 0$ and $\Theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, with distribution given by

$$\begin{aligned}\mathbb{P}[(D_1,D_2,\Theta)\in B] = \frac{1}{6}\int_0^\infty \int_{(\mathbf{S}^1)^3} & r^3 e^{-\pi r^2} a(\Delta(u_1,u_2,u_3)) \\ & \times \mathbb{I}[(r||u_3-u_2||, r||u_2-u_1||, \arcsin\left(\cos(\theta_{u_1,u_2}/2)\right)) \in B]\sigma(\mathrm{d}u_1)\sigma(\mathrm{d}u_2)\sigma(\mathrm{d}u_3)\mathrm{d}r,\end{aligned}$$

where θ_{u_1,u_2} is the measure of the angle (u_1,u_2) and where B is any Borel subset in $\mathbf{R}_+^2 \times [-\frac{\pi}{2}, \frac{\pi}{2})$. The random variables D_1, D_2 (resp. Θ) can be interpreted as the lengths of the two typical edges (resp. as the angle between the edges). In particular, the length of a typical edge is equal in distribution to $D = R||U_2 - U_1||$ with distribution given in Eq. (2.4).

Throughout the paper, we identify $\mathrm{Del}(P_1)$ to its skeleton and consider the window $\mathbf{C} := (-1/2, 1/2]^2$. When $x_1, x_2 \in P_N$ are Delaunay neighbors, we write $x_1 \sim x_2$ in $\mathrm{Del}(P_N)$. For a Borel subset $\mathbf{B}$ in $\mathbf{R}^2$, we denote by $E_{N,\mathbf{B}}$ the set of couples (x_1,x_2) such that the following conditions hold:

$$x_1 \sim x_2 \text{ in } \mathrm{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2,$$

where $\preceq$ denotes the lexicographic order. When $\mathbf{B} = \mathbf{C}$, we simply write $E_{N,\mathbf{C}} = E_N$. For a Borel subset $\mathbf{B}$ in $\mathbf{R}^2$, let $DT_{N,\mathbf{B}}$ be the set of triples (x_1,x_2,x_3) satisfying the following properties

$$\Delta(x_1,x_2,x_3) \in \mathrm{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2 \preceq x_3,$$

where $\Delta(x_1,x_2,x_3)$ is the convex hull of (x_1,x_2,x_3). When $\mathbf{B} = \mathbf{C}$, we simply write $DT_{N,\mathbf{C}} = DT_N$.

2.3 Local time

Let $W^{(1)}$ and $W^{(2)}$ be two independent and identically distributed fractional Brownian random fields in $\mathbf{R}^2$, with semi-variogram given by (2.1), and let

$$W^{(2\backslash 1)}(x) = W^{(2)}(x) - W^{(1)}(x), \quad x \in \mathbf{R}^2.$$

To measure the portion of the square $\mathbf{C} = (-1/2, 1/2]^2$ for which the two random fields coincide, the notion of local time is introduced as follows. Let $\nu^{(2\backslash 1)}$ be the *occupation measure* of $W^{(2\backslash 1)}$ over $\mathbf{C}$, i.e.

$$\nu^{(2\backslash 1)}(A) = \int_{\mathbf{C}} \mathbb{I}\left[W^{(2\backslash 1)}(x) \in A\right]\mathrm{d}x,$$

for any Borel measurable set $A \subset \mathbf{R}$. Observe that, for any $s,t \in [0,1]^2$,

$$\Delta(s,t) := \mathbb{E}\left[(W^{(2\backslash 1)}(s) - W^{(2\backslash 1)}(t))^2\right] = 2\sigma^2\left\|s-t\right\|^\alpha.$$

Because $\int_{\mathbf{C}}(\Delta(s,t))^{-1/2}\mathrm{d}s$ is finite for all $t \in \mathbf{C}$, it follows from Section 22 in [21] that the occupation measure $\nu^{(2\backslash 1)}$ admits a Lebesgue density. The *local time* at level ℓ is then defined as

$$L_{W^{(2\backslash 1)}}(\ell) := \frac{d\nu^{(2\backslash 1)}}{d\ell}(\ell).$$

An immediate consequence of the existence of the local time is the occupation time formula, which states that

$$\int_{\mathbf{C}} g(W^{(2\backslash 1)}(x))\mathrm{d}x = \int_{\mathbf{R}} g(\ell)\, L_{W^{(2\backslash 1)}}(\ell)\,\mathrm{d}\ell$$

for any Borel function g on $\mathbf{R}$. Adapting the proof of Lemma 1.1 in [26], we can easily show that, for any $\ell \in \mathbf{R}$,

$$L_{W^{(2\backslash 1)}}(\ell) = \lim_{\varepsilon\to 0}\int_{\mathbf{C}} \frac{1}{\sqrt{2\pi\varepsilon}}\exp\left(-\frac{1}{2\varepsilon}\left(W^{(2\backslash 1)}(x)-\ell\right)^2\right)\mathrm{d}x$$

or

$$L_{W^{(2\backslash 1)}}(\ell) = \frac{1}{2\pi}\lim_{M\to\infty}\int_{-[M,M]}\int_{\mathbf{R}} e^{\mathrm{i}\xi(W^{(2\backslash 1)}(x)-\ell)}\mathrm{d}x\mathrm{d}\xi, \tag{2.5}$$

where the limits hold in L^2.

3 Increments of a Brown-Resnick random field

Let $(\eta(x))_{x\in\mathbf{R}^2}$ be a max-stable Brown-Resnick random field defined as $\eta(x) = \bigvee_{i\geq 1} U_i Y_i(x)$ for any $x \in \mathbf{R}^2$, where $(U_i)_{i\geq 1}$ is the decreasing enumeration of points of a Poisson point process on $(0,+\infty)$ with intensity measure $u^{-2}\mathrm{d}u$, and $(Y_i)_{i\geq 1}$ are i.i.d. copies of the random field Y introduced in (2.2). The underlying process $(W(x))_{x\in\mathbf{R}^2}$ from (2.2) is an isotropic fractional Brownian field satisfying $W(0) = 0$ a.s. and $\gamma(x) = \mathrm{var}(W(x))/2 = \sigma^2\|x\|^\alpha/2$ for some $\alpha \in (0,1)$ and $\sigma^2 > 0$.

3.1 Asymptotic distributions of normalized increments

3.1.1 Single increments

Let $x_1, x_2 \in \mathbf{R}^2$ be two (distinct) sites and denote by $d_{1,2} = \|x_2 - x_1\|$ the distance between these sites. Let $z_1, z_2 \in \mathbf{R}_+$, $a = \sigma d^{\alpha/2}$, $u = \log(z_2/z_1)/a$ and $v(u) = a/2 + u$. It is well known that the joint probability distribution function of $(\eta(x_1), \eta(x_2))$ is given by (see e.g. [24])

$$F_{x_1,x_2}(z_1,z_2) = \mathbb{P}[\eta(x_1)\leq z_1, \eta(x_2)\leq z_2] = \exp(-V_{x_1,x_2}(z_1,z_2)), \tag{3.1}$$

where

$$V_{x_1,x_2}(z_1,z_2) = \frac{1}{z_1}\Phi(v(u)) + \frac{1}{z_2}\Phi(v(-u)), \qquad z_1, z_2 > 0.$$

Here Φ denotes the cumulative distribution function of the standard Gaussian distribution. The term V_{x_1,x_2} is referred to as the pairwise exponent function.

Let us now consider the "normalized" (linear) increments of the logarithm of the Brown-Resnick random field

$$U^{(\eta)}_{x_1,x_2} = \frac{1}{\sigma d_{1,2}^{\alpha/2}}\log(\eta(x_2)/\eta(x_1)). \tag{3.2}$$

The following proposition provides the conditional and marginal distributions of $U^{(\eta)}_{x_1,x_2}$ and allows us to deduce that it has asymptotically a standard Gaussian distribution as the distance d tends to 0. Such a result generalizes Proposition 3 in [30].

Proposition 1 *Let x_1, x_2 be two (distinct) points in $\mathbf{R}^2$ and $d_{1,2} = ||x_2 - x_1||$.*

(i) The conditional distribution of $U^{(\eta)}_{x_1,x_2}$ given $\eta(x_1) = \eta > 0$ is characterized by

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u \middle| \eta(x_1) = \eta\right] = \exp\left(-\frac{1}{\eta}\left[V_{x_1,x_2}(1, e^{\sigma d_{1,2}^{\alpha/2} u}) - 1\right]\right)\Phi(v(u)), \qquad u \in \mathbf{R},$$

and its unconditional distribution by

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u\right] = \frac{\Phi(v(u))}{V_{x_1,x_2}(1, e^{\sigma d_{1,2}^{\alpha/2} u})}, \qquad u \in \mathbf{R}.$$

(ii) Moreover,

$$\lim_{d_{1,2}\to 0} \mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u\right] = \Phi(u), \qquad u \in \mathbf{R}.$$

The fact that the asymptotic distribution of U (as $d_{1,2}$ tends to 0) is a standard Gaussian distribution is not a surprise since the probability that x_1 and x_2 belong to the same cell of the canonical tessellation of the max-stable random field tends to 1. Indeed, in a common cell, the values of the max-stable random field are generated by the same isotropic fractional Brownian random field.

3.1.2 Pairs of increments

Let us now consider three (distinct) sites $x_1, x_2, x_3 \in \mathbf{R}^2$ and denote by $d_{1,2} = \|x_2 - x_1\|$, $d_{1,3} = \|x_3 - x_1\|$, $d_{2,3} = \|x_3 - x_2\|$ the distances between two different sites. Let $z_1, z_2, z_3 \in \mathbf{R}_+$ and, for $i, j = 1, 2, 3$ such that $i \neq j$, let $a_{i,j} = \sigma d_{i,j}^{\alpha/2}$, $u_{i,j} = \log(z_j/z_i)/a_{i,j}$ and $v_{i,j}(u) = a_{i,j}/2 + u_{i,j}$. The joint probability distribution function of $(\eta(x_1), \eta(x_2), \eta(x_3))$ is given by (see e.g. [24])

$$\begin{aligned} F_{x_1,x_2,x_3}(z_1, z_2, z_3) &= \mathbb{P}[\eta(x_1) \leq z_1, \eta(x_2) \leq z_2, \eta(x_3) \leq z_3] \\ &= \exp(-V_{x_1,x_2,x_3}(z_1, z_2, z_3)), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} V_{x_1,x_2,x_3}(z_1, z_2, z_3) = \frac{1}{z_1}\Phi_2&\left(\begin{pmatrix} v_{1,2}(u_{1,2}) \\ v_{1,3}(u_{1,3}) \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\right) \\ &+ \frac{1}{z_2}\Phi_2\left(\begin{pmatrix} v_{1,2}(-u_{1,2}) \\ v_{2,3}(u_{2,3}) \end{pmatrix}; \begin{pmatrix} 1 & R_2 \\ R_2 & 1 \end{pmatrix}\right) \\ &+ \frac{1}{z_3}\Phi_2\left(\begin{pmatrix} v_{1,3}(-u_{1,3}) \\ v_{2,3}(-u_{2,3}) \end{pmatrix}; \begin{pmatrix} 1 & R_3 \\ R_3 & 1 \end{pmatrix}\right) \end{aligned} \tag{3.4}$$

with

$$R_1 = \frac{d_{1,2}^\alpha + d_{1,3}^\alpha - d_{2,3}^\alpha}{2(d_{1,2}d_{1,3})^{\alpha/2}}, \qquad R_2 = \frac{d_{1,2}^\alpha + d_{2,3}^\alpha - d_{1,3}^\alpha}{2(d_{1,2}d_{2,3})^{\alpha/2}}, \qquad R_3 = \frac{d_{1,3}^\alpha + d_{2,3}^\alpha - d_{1,2}^\alpha}{2(d_{1,3}d_{2,3})^{\alpha/2}}.$$

Here $\Phi_2(\bullet, \Sigma)$ denotes the bivariate cumulative distribution function of the centered Gaussian distribution with covariance matrix Σ.

The following proposition provides the conditional and marginal distributions of $\left(U^{(\eta)}_{x_1,x_2}, U^{(\eta)}_{x_1,x_3}\right)$ and allows us to deduce that it has asymptotically a bivariate Gaussian distribution as the distances $d_{1,2}$ and $d_{1,3}$ tend to 0 proportionally.

Proposition 2 *Let x_1, x_2, x_3 be three (distinct) points in $\mathbf{R}^2$ and $d_{i,j} = ||x_j - x_i||$, $i \neq j$.*

(i) The conditional distribution of $\left(U^{(\eta)}_{x_1,x_2}, U^{(\eta)}_{x_1,x_3}\right)$ given $\eta(x_1) = \eta > 0$ is characterized by

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u_2, U^{(\eta)}_{x_1,x_3} \leq u_3 \middle| \eta(x_1) = \eta\right] = \exp\left(-\frac{1}{\eta}\left[V_{x_1,x_2,x_3}(1, e^{\sigma d_{1,2}^{\alpha/2} u_2}, e^{\sigma d_{1,3}^{\alpha/2} u_3}) - 1\right]\right)$$
$$\times \Phi_2\left(\begin{pmatrix} v_{1,2}(u_2) \\ v_{1,3}(u_3) \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\right), \qquad u_1, u_2 \in \mathbf{R},$$

and its marginal distribution by

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u_2, U^{(\eta)}_{x_1,x_3} \leq u_3\right] = \frac{\Phi_2\left(\begin{pmatrix} v_{1,2}(u_2) \\ v_{1,3}(u_3) \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\right)}{V_{x_1,x_2,x_3}(1, e^{\sigma d_{1,2}^{\alpha/2} u_2}, e^{\sigma d_{1,3}^{\alpha/2} u_3})}, \qquad u_1, u_2 \in \mathbf{R}.$$

(ii) If $d_{i,j} = \delta \tilde{d}_{i,j}$ for some $\delta > 0$, where $\tilde{d}_{i,j}$, $i \neq j$, is fixed, then

$$\lim_{\delta \to 0} \mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u_2, U^{(\eta)}_{x_1,x_3} \leq u_3\right] = \Phi_2\left(\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}; \begin{pmatrix} 1 & \tilde{R}_1 \\ \tilde{R}_1 & 1 \end{pmatrix}\right), \qquad u_1, u_2 \in \mathbf{R},$$

where

$$\tilde{R}_1 = \frac{\tilde{d}_{1,2}^{\alpha} + \tilde{d}_{1,3}^{\alpha} - \tilde{d}_{2,3}^{\alpha}}{2\left(\tilde{d}_{1,2}\tilde{d}_{1,3}\right)^{\alpha/2}}.$$

The comment concerning the asymptotic distribution of $U^{(\eta)}_{x_1,x_2}$ also holds for $\left(U^{(\eta)}_{x_1,x_2}, U^{(\eta)}_{x_1,x_3}\right)$. The probability that x_1, x_2 and x_3 belong to the same cell of the canonical tessellation of the max-stable random field tends to 1 as δ tends to 0. Therefore the vector $\left(U^{(\eta)}_{x_1,x_2}, U^{(\eta)}_{x_1,x_3}\right)$ tends to have the same distribution as the vector of normalized linear increments of an isotropic fractional Brownian random field.

3.2 Asymptotic distributions of squared increment sums for the max-stable Brown-Resnick random field

Let us define, for $k \neq j \geq 1$,

$$Z_{k \setminus j}(x) = Z_k(x) - Z_j(x), \qquad x \in \mathbf{R}^2,$$

where

$$Z_i(x) = \log U_i + \log Y_i(x), \qquad x \in \mathbf{R}^2.$$

In the same spirit as [18], we build a random tessellation $(\mathbf{C}_{k,j})_{k>j\geq 1}$ of $\mathbf{C} := (-1/2, 1/2]^2$, with

$$\mathbf{C}_{k,j} = \left\{x \in \mathbf{C} : Z_k(x) \bigwedge Z_j(x) > \bigvee_{i \neq j,k} Z_i(x)\right\}. \tag{3.5}$$

If $\mathbf{C}_{k,j} \neq \varnothing$, we define for any Borel subset A of $\mathbf{R}$ the occupation measure of $Z_{k\setminus j}$ over $\mathbf{C}_{k,j}$ by

$$\nu^{(k\setminus j)}(A) = \int_{\mathbf{C}_{k,j}} \mathbb{I}\left[Z_{k\setminus j}(x) \in A\right] \mathrm{d}x.$$

The associated local time at level 0 is given by $L_{Z_{k\setminus j}}(0) := \frac{d\nu^{(k\setminus j)}}{d\ell}(0)$. If $\mathbf{C}_{k,j} = \varnothing$, we let $L_{Z_{k\setminus j}}(0) := 0$.

Let $U^{(\eta)}_{x_1,x_2}$ be the (normalized) increment as defined in (3.2). We define two square increment sums as follows:

$$
\begin{aligned}
V^{(\eta)}_{2,N} &= \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \left((U^{(\eta)}_{x_1,x_2})^2 - 1 \right) \\
V^{(\eta)}_{3,N} &= \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \left(\begin{pmatrix} U^{(\eta)}_{x_1,x_2} & U^{(\eta)}_{x_1,x_3} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U^{(\eta)}_{x_1,x_2} \\ U^{(\eta)}_{x_1,x_3} \end{pmatrix} - 2 \right),
\end{aligned}
$$

where

$$
R_{x_1,x_2,x_3} := R_1 = \frac{d^{\alpha}_{1,2} + d^{\alpha}_{1,3} - d^{\alpha}_{2,3}}{2\left(d_{1,2} d_{1,3}\right)^{\alpha/2}}, \tag{3.6}
$$

with $d_{1,3} = \|x_3 - x_1\| > 0$ and $d_{2,3} = \|x_3 - x_2\| > 0$. The above terms naturally arise in the study of the asymptotic properties of the estimators presented in Section 4.

The following theorem characterizes the asymptotic distributions of $V^{(\eta)}_{2,N}$ and $V^{(\eta)}_{3,N}$.

Theorem 3 *Let $\alpha \in (0,1)$. Then, as $N \to \infty$,*

$$
\begin{aligned}
\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V^{(\eta)}_{2,N} &\xrightarrow{\mathbb{P}} c_{V_2} \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0) \\
\frac{\sqrt{2}}{2} N^{-(2-\alpha)/4} V^{(\eta)}_{3,N} &\xrightarrow{\mathbb{P}} c_{V_3} \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0).
\end{aligned}
$$

In the above theorem we have assumed that $\alpha \in (0,1)$ whereas, in general, $\alpha \in (0,2)$. Such an assumption is important and comes from fact that, in its proof, we applied two central limit theorems derived in [8] which only work for $\alpha < 1$.

Theorem 3 generalizes Theorem 2 in [9] to the max-stable Brown-Resnick random field which is built as the pointwise maximum of an infinite number of isotropic fractional Brownian fields. It can be noted that there is an a.s. finite number of local times $L_{Z_{k\setminus j}}(0)$, $j \geq 1$ and $k > j$, which are positive. This is related to the fact that there is an a.s. finite number of non-empty cells of the canonical tessellation in $\mathbf{C}$.

Remark 1 *Using the Slivnyak-Mecke formula (see e.g. Theorem 3.2.5 in [32]) and the same arguments as in the proof of Proposition 3 in [30], we can state that*

$$
\lim_{N\to\infty} N^{\alpha/4} \mathbb{E}\left[\frac{1}{N} \sum_{(x_1,x_2)\in E_N} \left((U^{(\eta)}_{x_1,x_2})^2 - 1 \right) \right] = 4\sigma \mathbb{E}\left[D^{\alpha/2} \right] \psi
$$

with

$$
\psi = \int_0^\infty u\varphi(u) \left[1/2 - \bar{\Phi}(u) - u\bar{\Phi}(u)\,\Phi(u)/\varphi(u) \right] du \simeq -0.094.
$$

As a consequence we deduce that c_{V_2} is negative.

4 The weighted CL approach

We now assume that the sites where the Brown-Resnick random field is observed are given by a realization of a homogeneous Poisson point process of intensity N in $\mathbf{R}^2$, denoted by P_N, which is independent of the Brown-Resnick random field. Recall that $\mathbf{C} = (-1/2, 1/2]^2$ denotes the square where we consider the sites for the observations of the max-stable random field.

4.1 The weighted CL objective functions and the CL estimators

The distribution of $(\eta(x_1), \eta(x_2))$ is absolutely continuous with respect to the Lebesgue measure on $(\mathbf{R}_+)^2$. Its density function satisfies

$$f_{x_1,x_2}(z_1, z_2) = \frac{\partial}{\partial z_1 \partial z_2} F_{x_1,x_2}(z_1, z_2), \qquad z_1, z_2 \in \mathbf{R}_+,$$

where $F_{x_1,x_2}(z_1, z_2)$ is given in (3.1). For any $\alpha \in (0, 2)$, $\sigma^2 > 0$ and $z_1, z_2 \in \mathbf{R}_+$, this joint density function is a differentiable function with respect to (α, σ). The (tapered) pairwise CL objective function is then defined as

$$\ell_{2,N}(\sigma, \alpha) = \sum_{(x_1,x_2) \in E_N} \log f_{x_1,x_2}(\eta(x_1), \eta(x_2)).$$

The distribution of $(\eta(x_1), \eta(x_2), \eta(x_3))$ is absolutely continuous with respect to the Lebesgue measure on $(\mathbf{R}_+)^3$. Its density function satisfies

$$f_{x_1,x_2,x_3}(z_1, z_2, z_3) = \frac{\partial}{\partial z_1 \partial z_2 \partial z_3} F_{x_1,x_2,x_3}(z_1, z_2, z_3), \qquad z_1, z_2, z_3 \in \mathbf{R}_+,$$

where $F_{x_1,x_2,x_3}(z_1, z_2, z_3)$ is given in (3.3). For any $\alpha \in (0, 2)$, $\sigma^2 > 0$ and $z_1, z_2, z_3 \in \mathbf{R}_+$, this joint density function is a differentiable function with respect to (α, σ). The (tapered) triplewise CL objective function is then defined as

$$\ell_{3,N}(\sigma, \alpha) = \sum_{(x_1,x_2,x_3) \in DT_N} \log f_{x_1,x_2,x_3}(\eta(x_1), \eta(x_2), \eta(x_3)).$$

Thereby, in the CL objective functions, we exclude pairs that are not edges of the Delaunay triangulation or triples that are not vertices of triangles of this triangulation. Restricting the CL objective functions to the most informative pairs and triples for the estimation of the parameters does not modify the approach that follows, but allows us to simplify the presentation and the proofs.

From Section 4.4 of [16], we know that, independently of α and σ, there exist families of positive functions $(l_{x_1,x_2})_{x_1,x_2 \in \mathbf{R}^2}$ and $(l_{x_1,x_2,x_3})_{x_1,x_2,x_3 \in \mathbf{R}^2}$ with $l_{x_1,x_2} : \mathbf{R}^2 \to \mathbf{R}$ and $l_{x_1,x_2,x_3} : \mathbf{R}^3 \to \mathbf{R}$ such that the following Lipschitz conditions hold: for any $\sigma_1, \sigma_2 > 0$ and $\alpha_1, \alpha_2 \in (0, 2)$

$$\left| \log \frac{f_{x_1,x_2}(z_1, z_2; (\sigma_2, \alpha_2))}{f_{x_1,x_2}(z_1, z_2; (\sigma_1, \alpha_1))} \right| \le l_{x_1,x_2}(z_1, z_2) \left(|\sigma_2 - \sigma_1| + |\alpha_2 - \alpha_1| \right)$$

and

$$\left| \log \frac{f_{x_1,x_2,x_3}(z_1, z_2, z_3; (\sigma_2, \alpha_2))}{f_{x_1,x_2,x_3}(z_1, z_2, z_3; (\sigma_1, \alpha_1))} \right| \le l_{x_1,x_2,x_3}(z_1, z_{2,z_3}) \left(|\sigma_2 - \sigma_1| + |\alpha_2 - \alpha_1| \right).$$

Let us denote by (σ_0, α_0) the true parameters. We assume that σ_0 belongs to a compact set S_σ of $\mathbf{R}_+$ and that α_0 belongs to a compact set S_α of $(0, 1)$. In what follows, we assume that one of the parameters is known and explain how we define the MCLEs for the other parameter.

When α_0 is assumed to be known, the pairwise and triplewise maximum (tapered) CL estimators of σ_0, denoted by $\hat{\sigma}_{j,N}$, are respectively defined as a solution of the maximization problems

$$\max_{\sigma \in \mathrm{S}_\sigma} \ell_{j,N}\left(\sigma, \alpha_0\right), \qquad j = 2, 3.$$

When σ_0 is assumed to be known, the pairwise and triplewise maximum (tapered) CL estimators of α_0, denoted by $\hat{\alpha}_{j,N}$, are respectively defined as a solution of the maximization problems

$$\max_{\alpha \in \mathrm{S}_\alpha} \ell_{j,N}\left(\sigma_0, \alpha\right), \qquad j = 2, 3.$$

Note that the solutions of these maximization problems become unique as $N \to \infty$. This can be viewed from the first-order optimality conditions and the asymptotic approximations of the score functions obtained in Propositions 4 and 5.

4.2 Asymptotic score functions

4.2.1 The pairwise case

The following proposition provides the asymptotic contributions of the pair $(\eta(x_1), \eta(x_2))$ to the pairwise score functions.

Proposition 4 *Let $u \in \mathbf{R}$ be fixed. Let $x_1, x_2 \in \mathbf{R}^2$ and $z_1, z_2 \in \mathbf{R}_+$ be such that $d^{-\alpha/2}\sigma^{-1}\log\left(z_2/z_1\right) = u$, where $d = \|x_2 - x_1\| > 0$. Then*

$$\begin{aligned}
\lim_{d\to 0} \frac{\partial}{\partial \sigma} \log f_{x_1,x_2}\left(z_1, z_2\right) &= \frac{1}{\sigma}\left(u^2 - 1\right), \\
\lim_{d\to 0} \frac{1}{\log d} \frac{\partial}{\partial \alpha} \log f_{x_1,x_2}\left(z_1, z_2\right) &= \frac{1}{2}\left(u^2 - 1\right).
\end{aligned}$$

The asymptotic score contributions of a pair are therefore proportional to $\left(u^2 - 1\right)$. Further, u will be replaced by the normalized increment of $\log(\eta)$ which has asymptotically a standard Gaussian distribution as stated in Proposition 1. This fact ensures that the asymptotic score contributions are asymptotically unbiased.

4.2.2 The triplewise case

The following proposition provides the asymptotic contributions of the triple $\left(\eta\left(x_1\right), \eta\left(x_2\right), \eta\left(x_3\right)\right)$ to the triplewise score functions.

Proposition 5 *Let $u_2, u_3 \in \mathbf{R}$ and $d_{i,j}$, $i \neq j$, be fixed. Let $x_1, x_2, x_3 \in \mathbf{R}^2$ and $z_1, z_2, z_3 \in \mathbf{R}_+$ be such that*

$$\delta^{-\alpha/2} d_{1,2}^{-\alpha/2} \sigma^{-1} \log\left(z_2/z_1\right) = u_2 \text{ and } \delta^{-\alpha/2} d_{1,3}^{-\alpha/2} \sigma^{-1} \log\left(z_3/z_1\right) = u_3,$$

with $\delta d_{1,2} = \|x_2 - x_1\|$, $\delta d_{1,3} = \|x_3 - x_1\|$, $\delta d_{2,3} = \|x_3 - x_2\|$ for some $\delta > 0$. Then

$$\begin{aligned}
\lim_{\delta\to 0} \frac{\partial}{\partial \sigma} \log f_{x_1,x_2,x_3}\left(z_1, z_2, z_3\right) &= \frac{1}{\sigma}\left(\begin{pmatrix} u_2 & u_3 \end{pmatrix} \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} - 2\right), \\
\lim_{\delta\to 0} \frac{1}{\log \delta} \frac{\partial}{\partial \alpha} \log f_{x_1,x_2,x_3}\left(z_1, z_2, z_3\right) &= \frac{1}{2}\left(\begin{pmatrix} u_2 & u_3 \end{pmatrix} \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} - 2\right).
\end{aligned}$$

The asymptotic contributions of a triple are therefore proportional to a quadratic function of (u_2, u_3). Further, u_2 and u_3 will be replaced by normalized increments of $\log(\eta)$ over a triangle of the Delaunay triangulation which have asymptotically a bivariate standard Gaussian distribution with correlation coefficient R_1, as stated in Proposition 2. We can also conclude that the asymptotic score contributions are asymptotically unbiased.

If we let $\delta^{-\alpha/2} d_{1,2}^{-\alpha/2} \sigma^{-1} \log(z_1/z_2) = \tilde{u}_1$ and $\delta^{-\alpha/2} d_{2,3}^{-\alpha/2} \sigma^{-1} \log(z_3/z_2) = \tilde{u}_3$ with fixed $\tilde{u}_1, \tilde{u}_3 \in \mathbf{R}$, we also get

$$\lim_{\delta \to 0} \frac{\partial}{\partial \sigma} \log f_{x_1,x_2,x_3}(z_1, z_2, z_3) = \frac{1}{\sigma} \left(\begin{pmatrix} \tilde{u}_1 & \tilde{u}_3 \end{pmatrix} \begin{pmatrix} 1 & R_2 \\ R_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_3 \end{pmatrix} - 2 \right).$$

In particular, an invariance property with respect to the choice of the order of the points x_1, x_2 and x_3 holds for the triplewise score functions. However, it will be necessary to order these points later.

4.3 Asymptotic properties of the MCLEs

We are now able to present the asymptotic properties of $\hat{\sigma}^2_{j,N}$ and $\hat{\alpha}_{j,N}$ for $j = 2, 3$.

Theorem 6 *Assume that σ_0 belongs to the interior of a compact set of $\mathbf{R}_+$, and that α_0 belongs to the interior of a compact set of $(0, 1)$. Recall that E_N denotes the set of edges (x_1, x_2) in the Delaunay triangulation, with $x_1 \in \mathbf{C}$. Then, as $N \to \infty$,*

$$\frac{\sqrt{3}}{3} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} \left(\hat{\sigma}^2_{2,N} - \sigma^2_0 \right) \xrightarrow{\mathbb{P}} c_{V_2} \sigma^2_0 \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0)$$

$$\frac{\sqrt{3}}{6} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} \log(N) \left(\hat{\alpha}_{2,N} - \alpha_0 \right) \xrightarrow{\mathbb{P}} -c_{V_2} \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0)$$

and

$$\frac{\sqrt{2}}{2} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} \left(\hat{\sigma}^2_{3,N} - \sigma^2_0 \right) \xrightarrow{\mathbb{P}} c_{V_3} \sigma^2_0 \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0)$$

$$\frac{\sqrt{2}}{4} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} \log(N) \left(\hat{\alpha}_{3,N} - \alpha_0 \right) \xrightarrow{\mathbb{P}} -c_{V_3} \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0).$$

Several important points have to be highlighted. First, the MCLEs of σ^2_0 and α_0 (when the other parameter is known) are consistent in our infill asymptotic setup. They have rates of convergence proportional to $N^{\alpha_0/4}$ for $\hat{\sigma}^2_{2,N}$ and $\log(N) N^{\alpha_0/4}$ for $\hat{\alpha}_{2,N}$ that differ from the expected rates of convergence $N^{1/2}$ and $\log(N) N^{1/2}$ as in [35] for the isotropic fractional Brownian field. Second the type of convergence is in probability. The random variables appearing in the limits in Theorem 6 are proportional to a sum of local times. However these local times have unknown distributions and they cannot be estimated from the data since the underlying random fields $(Y_i)_{i \geq 1}$ and the point process $(U_i)_{i \geq 1}$ are not observed. In particular, if the spatial data are only observed for a single date, the Gaussian approximation for the MCLEs given in [29] (when several independent replications over time of the spatial data are available) should not be used.

Second, let $W^{(1)}$ and $W^{(2)}$ be two independent and identically distributed fractional Brownian random fields with semi-variogram given by (2.1). According to Corollary 3.4 in [6], the Hausdorff dimension of $\{x \in \mathbf{R}^2 : W^{(1)}(x) = W^{(2)}(x)\}$ is $2 - \frac{\alpha}{2}$, with $\alpha \in (0,1)$. We think that such a result could be extended for a Brown-Resnick random field η. More precisely, the Hausdorff dimension of the boundaries of the cells associated with η should also be $2 - \frac{\alpha}{2}$. In the particular case where $\alpha = 2$, this result is true. Indeed, if so, the tessellation associated with η is a Laguerre tessellation [18] and therefore the boundaries of the cells are of dimension 1.

Third, note that Theorem 6 establishes that the estimators $\hat{\sigma}^2_{2,N}$ and $\hat{\sigma}^2_{3,N}$ share the same rate of convergence to σ_0^2 as well as the same limit in probability (up to a coefficient). A careful reading of the proofs shows that the local time terms are the same (up to a coefficient) and that a linear combination of the two estimators (which removes local time terms) would give an estimator with a classical rate of convergence and a Gaussian limit distribution. This study has been left for future work, as it would require fairly lengthy additional developments.

Fourth, the problem of joint parameter estimation of $\left(\sigma_0^2, \alpha_0\right)$ is considerably more difficult with a random sampling scheme. The study on a deterministic and regular grid seems a reasonable objective. It is expected that the respective rates of convergence will be modified into $N^{\alpha_0/4}/\log(N)$ and $N^{\alpha_0/4}$ as it may be intuited from [4] in the case of a fractional Gaussian process ($d = 1$).

5 Proofs

5.1 Proof of Proposition 1

(i). Since $U^{(\eta)}_{x_1,x_2} = \sigma^{-1} d^{-\alpha/2} \log(\eta(x_2)/\eta(x_1))$ with $d = ||x_2 - x_1||$, we have, for any real number u and for any $z_1 > 0$,

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u | \eta(x_1) = z_1\right] = \mathbb{P}\left[\eta(x_2) \leq z_2 | \eta(x_1) = z_1\right],$$

with $z_2 = z_1 e^{u\sigma d^{\alpha/2}}$. Because $(\eta(x))_{x\in\mathbf{R}^2}$ is a stationary random field, we also have

$$\mathbb{P}\left[\eta(x_2) \leq z_2 | \eta(x_1) = z_1\right] = \mathbb{P}\left[\eta(x_2 - x_1) \leq z_1 | \eta(0) = z_1\right].$$

According to Proposition 4.2 in [17], we know that

$$\begin{aligned}
&\mathbb{P}\left[\eta(x_2 - x_1) \leq z_1 | \eta(0) = z_1\right] \\
&\qquad = \exp\left(-\mathbb{E}\left[\left(\frac{Y(x_2 - x_1)}{z_2} - \frac{Y(0)}{z_1}\right)_+\right]\right) \mathbb{E}\left[\mathbb{I}\left[\frac{Y(x_2 - x_1)}{z_2} \leq \frac{Y(0)}{z_1}\right] Y(0)\right].
\end{aligned}$$

We provide below explicit representations of the terms of the right-hand side. First, because $Y(0) = 1$ a.s., we have

$$\begin{aligned}
\mathbb{E}\left[\mathbb{I}\left[\frac{Y(x_2 - x_1)}{z_2} \leq \frac{Y(0)}{z_1}\right] Y(0)\right] &= \mathbb{P}\left[\frac{Y(x_2 - x_1)}{z_2} \leq \frac{1}{z_1}\right] \\
&= \mathbb{P}\left[e^{W(x_2 - x_1) - \gamma(x_2 - x_1)} \leq \frac{z_2}{z_1}\right] \\
&= \mathbb{P}\left[\frac{W(x_2 - x_1)}{\sqrt{2\gamma(x_2 - x_1)}} \leq \frac{1}{\sqrt{2\gamma(x_2 - x_1)}} \log\left(\frac{z_2}{z_1}\right) + \sqrt{\frac{\gamma(x_2 - x_1)}{2}}\right],
\end{aligned}$$

and where $\gamma(x)$ is given in (2.1). Since $W(x_2 - x_1) \sim \mathcal{N}(0, 2\gamma(x_2 - x_1))$ and since $2\gamma(x_2 - x_1) = \sigma^2 ||x_2 - x_1||^\alpha = \sigma^2 d^\alpha$, we deduce that

$$\mathbb{E}\left[\mathbb{I}\left[\frac{Y(x_2 - x_1)}{z_2} \leq \frac{Y(0)}{z_1}\right] Y(0)\right] = \Phi\left(\sigma^{-1} d^{-\alpha/2} \log\left(\frac{z_2}{z_1}\right) + \frac{1}{2}\sigma d^{\alpha/2}\right)$$
$$= \Phi\left(u + \frac{1}{2}\sigma d^{\alpha/2}\right),$$

with $u = \sigma^{-1} d^{-\alpha/2} \log(z_2/z_1)$. Moreover,

$$\mathbb{E}\left[\left(\frac{Y(x_2 - x_1)}{z_2} - \frac{Y(0)}{z_1}\right)_+\right] = \mathbb{E}\left[\max\left\{\frac{Y(x_2 - x_1)}{z_2} - \frac{1}{z_1}, 0\right\}\right]$$
$$= \mathbb{E}\left[\max\left\{\frac{Y(x_2 - x_1)}{z_2}, \frac{Y(0)}{z_1}\right\} - \frac{1}{z_1}\right]$$
$$= V_{0, x_2 - x_1}(z_1, z_2) - \frac{1}{z_1}$$
$$= V_{x_1, x_2}(z_1, z_2) - \frac{1}{z_1}$$
$$= \frac{1}{z_1}\left(V_{x_1, x_2}(1, e^{u\sigma d^{\alpha/2}}) - 1\right),$$

where the last line comes from the fact that V_{x_1,x_2} is a homogeneous function of order -1. We conclude that

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u | \eta(x_1) = z_1\right] = \Phi(v(u)) \exp\left(-\frac{1}{z_1}\left(V_{x_1,x_2}(1, e^{u\sigma d^{\alpha/2}}) - 1\right)\right)$$

which gives the conditional distribution of $U^{(\eta)}_{x_1,x_2}$. To obtain the marginal distribution of $U^{(\eta)}_{x_1,x_2}$, we write

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u\right] = \mathbb{E}\left[\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u | \eta(x_1)\right]\right].$$

Since $1/\eta(x_1)$ has a standard exponential distribution, we get

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u\right] = \frac{\Phi\left(u + \frac{1}{2}\sigma d^{\alpha/2}\right)}{V_{x_1,x_2}(1, e^{u\sigma d^{\alpha/2}})}.$$

(ii). It follows from *(i)* that

$$\lim_{d \to 0} \mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u\right] = \frac{\Phi(u)}{\Phi(u) + \Phi(-u)} = \Phi(u).$$

5.2 Proof of Proposition 2

(i). We follow the same lines as in the proof of Proposition 1. For any real numbers u_1, u_2 and for any $z_1 > 0$, we have

$$\mathbb{P}\left[U^{(\eta)}_{x_1,x_2} \leq u_2, U^{(\eta)}_{x_1,x_3} \leq u_3 | \eta(x_1) = z_1\right] = \mathbb{P}\left[\eta(x_2) \leq z_2, \eta(x_3) \leq z_3 | \eta(x_1) = z_1\right],$$

with $z_2 = z_1 e^{u_2 \sigma d_{1,2}^{\alpha/2}}$ and $z_3 = z_1 e^{u_3 \sigma d_{1,3}^{\alpha/2}}$. Because $(\eta(x))_{x \in \mathbf{R}^2}$ is a stationary random field, we also have

$$\mathbb{P}\left[\eta(x_2) \leq z_2, \eta(x_3) \leq z_3 | \eta(x_1) = z_1\right] = \mathbb{P}\left[\eta(x_2 - x_1) \leq z_2, \eta(x_3 - x_1) \leq z_3 | \eta(0) = z_1\right].$$

According to Proposition 4.2 in [17], we know that

$$
\begin{aligned}
&\mathbb{P}\left[\eta\left(x_2-x_1\right)\leq z_2,\eta\left(x_3-x_1\right)\leq z_3|\eta\left(0\right)=z_1\right]\\
&\qquad\qquad=\exp\left(-\mathbb{E}\left[\left(\max_{i=2,3}\frac{Y(x_i-x_1)}{z_i}-\frac{1}{z_1}\right)_+\right]\right)\mathbb{P}\left[\max_{i=2,3}\frac{Y(x_i-x_1)}{z_i}\leq\frac{1}{z_1}\right].
\end{aligned}
$$

We provide below an explicit representation of the term of the right-hand side. First, we write

$$
\mathbb{P}\left[\max_{i=2,3}\frac{Y(x_i-x_1)}{z_i}\leq\frac{1}{z_1}\right]=\mathbb{P}\left[\frac{W(x_2-x_1)}{\sqrt{2\gamma(x_2-x_1)}}\leq v_2,\frac{W(x_3-x_1)}{\sqrt{2\gamma(x_3-x_1)}}\leq v_3\right],
$$

where

$$
v_2=\frac{1}{\sqrt{2\gamma(x_2-x_1)}}\log\left(\frac{z_2}{z_1}\right)+\sqrt{\frac{\gamma(x_2-x_1)}{2}}=u_2+\frac{1}{2}\sigma d_{1,2}^{\alpha/2}
$$

and

$$
v_3=\frac{1}{\sqrt{2\gamma(x_3-x_1)}}\log\left(\frac{z_3}{z_1}\right)+\sqrt{\frac{\gamma(x_3-x_1)}{2}}=u_3+\frac{1}{2}\sigma d_{1,3}^{\alpha/2},
$$

with $\gamma(x)$ given in (2.1).

Since the distribution of the random vector $\left(\frac{W(x_2-x_1)}{\sqrt{2\gamma(x_2-x_1)}},\frac{W(x_3-x_1)}{\sqrt{2\gamma(x_3-x_1)}}\right)$ is a centered Gaussian distribution with covariance matrix $\begin{pmatrix}1 & R_1\\ R_1 & 1\end{pmatrix}$, we have

$$
\mathbb{P}\left[\max_{i=2,3}\frac{Y(x_i-x_1)}{z_i}\leq\frac{1}{z_1}\right]=\Phi_2\left(\begin{pmatrix}u_2+\frac{1}{2}\sigma d_{1,2}^{\alpha/2}\\ u_3+\frac{1}{2}\sigma d_{1,3}^{\alpha/2}\end{pmatrix},\begin{pmatrix}1 & R_1\\ R_1 & 1\end{pmatrix}\right).
$$

Moreover, following the same lines as in the proof of Proposition 1, we have

$$
\begin{aligned}
\mathbb{E}\left[\left(\max_{i=2,3}\frac{Y(x_i-x_1)}{z_i}-\frac{1}{z_1}\right)_+\right] &= \frac{1}{z_1}\left(V_{0,x_2-x_1,x_3-x_1}(1,e^{u_2\sigma d_{1,2}^{\alpha/2}},e^{u_3\sigma d_{1,3}^{\alpha/2}})-1\right)\\
&= \frac{1}{z_1}\left(V_{x_1,x_2,x_3}(1,e^{u_2\sigma d_{1,2}^{\alpha/2}},e^{u_3\sigma d_{1,3}^{\alpha/2}})-1\right).
\end{aligned}
$$

An explicit formula of the exponent measure $V_{0,x_2-x_1,x_3-x_1}(1,e^{u_2\sigma d_{1,2}^{\alpha/2}},e^{u_3\sigma d_{1,3}^{\alpha/2}})$ is given in Eq. (3.4). The above computations imply

$$
\begin{aligned}
&\mathbb{P}\left[U^{(\eta)}_{x_1,x_2}\leq u_2,U^{(\eta)}_{x_1,x_3}\leq u_3|\eta\left(x_1\right)=z_1\right]\\
&\quad=\Phi_2\left(\begin{pmatrix}u_2+\frac{1}{2}\sigma d_{1,2}^{\alpha/2}\\ u_3+\frac{1}{2}\sigma d_{1,3}^{\alpha/2}\end{pmatrix};\begin{pmatrix}1 & R_1\\ R_1 & 1\end{pmatrix}\right)\times\exp\left(-\frac{1}{z_1}\left(V_{x_1,x_2,x_3}\left(1,e^{\sigma d_{1,2}^{\alpha/2}u_2},e^{\sigma d_{1,3}^{\alpha/2}u_3}\right)-1\right)\right).
\end{aligned}
$$

This gives the conditional distribution of $(U^{(\eta)}_{x_1,x_2},U^{(\eta)}_{x_1,x_3})$. Using the fact that $1/\eta\left(x_1\right)$ has a standard exponential distribution, we obtain the marginal distribution of $(U^{(\eta)}_{x_1,x_2},U^{(\eta)}_{x_1,x_3})$:

$$
\mathbb{P}\left[U^{(\eta)}_{x_1,x_2}\leq u_2,U^{(\eta)}_{x_1,x_3}\leq u_3\right]=\frac{\Phi_2\left(\begin{pmatrix}v_{1,2}\left(u_2\right)\\ v_{1,3}\left(u_3\right)\end{pmatrix};\begin{pmatrix}1 & R_1\\ R_1 & 1\end{pmatrix}\right)}{V_{x_1,x_2,x_3}(1,e^{\sigma d_{1,2}^{\alpha/2}u_2},e^{\sigma d_{1,3}^{\alpha/2}u_3})}.
$$

(ii). If $d_{1,2}=\delta\tilde{d}_{1,2}$, $d_{1,3}=\delta\tilde{d}_{1,3}$, $d_{2,3}=\delta\tilde{d}_{2,3}$ then

$$\lim_{\delta\to 0}\mathbb{P}\left[U^{(\eta)}_{x_1,x_2}\le u_2, U^{(\eta)}_{x_1,x_3}\le u_3\right]=\frac{\Phi_2\left(\begin{pmatrix}u_2\\u_3\end{pmatrix};\begin{pmatrix}1&\tilde{R}_1\\\tilde{R}_1&1\end{pmatrix}\right)}{\lim_{\delta\to 0}V_{0,x_2-x_1,x_3-x_1}(1,e^{u_2\sigma\delta^{\alpha/2}\tilde{d}_{1,2}^{\alpha/2}},e^{u_3\sigma\delta^{\alpha/2}\tilde{d}_{1,3}^{\alpha/2}})}.$$

Moreover, we have

$$\begin{aligned}\lim_{\delta\to 0}V_{0,x_2-x_1,x_3-x_1}(1,e^{u_2\sigma\delta^{\alpha/2}\tilde{d}_{1,2}^{\alpha/2}},e^{u_3\sigma\delta^{\alpha/2}\tilde{d}_{1,3}^{\alpha/2}})&=\Phi_2\left(\begin{pmatrix}u_2\\u_3\end{pmatrix};\begin{pmatrix}1&\tilde{R}_1\\\tilde{R}_1&1\end{pmatrix}\right)\\&+\Phi_2\left(\begin{pmatrix}-u_2\\\frac{\tilde{d}_{1,3}^{\alpha/2}u_3-\tilde{d}_{1,2}^{\alpha/2}u_2}{\tilde{d}_{2,3}^{\alpha/2}}\end{pmatrix};\begin{pmatrix}1&\tilde{R}_2\\\tilde{R}_2&1\end{pmatrix}\right)+\Phi_2\left(\begin{pmatrix}-u_3\\-\frac{\tilde{d}_{1,3}^{\alpha/2}u_3-\tilde{d}_{1,2}^{\alpha/2}u_2}{\tilde{d}_{2,3}^{\alpha/2}}\end{pmatrix};\begin{pmatrix}1&\tilde{R}_3\\\tilde{R}_3&1\end{pmatrix}\right),\end{aligned}$$

where $\tilde{R}_2$ and $\tilde{R}_3$ are defined in the same spirit as R_2 and R_3 by replacing $d_{i,j}$ by $\tilde{d}_{i,j}$. With standard computations, we can show that the last term equals 1. This concludes the proof of Proposition 2.

5.3 Proof of Theorem 3

5.3.1 Proof for $V^{(\eta)}_{2,N}$

Let $H_2(u):=u^2-1$ be the Hermite polynomial of degree 2 so that

$$V^{(\eta)}_{2,N}=\frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N}H_2\left(U^{(\eta)}_{x_1,x_2}\right).$$

To deal with the right-hand side, we write

$$H_2(U^{(\eta)}_{x_1,x_2})=H_2\left(U^{(\eta)}_{x_1,x_2}+\frac{\gamma(x_2)-\gamma(x_1)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right)-2U^{(\eta)}_{x_1,x_2}\frac{\gamma(x_2)-\gamma(x_1)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}-\frac{(\gamma(x_2)-\gamma(x_1))^2}{\sigma^2\left\|x_2-x_1\right\|^{\alpha}},\qquad(5.1)$$

where $\gamma(x)$ is as in (2.1). The sums over $(x_1,x_2)\in E_N$ of the two last terms are negligeable compared to $N^{(2-\alpha)/4}$. Indeed, according to Lemma 7 *(ii)*, we know that

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\frac{(\gamma(x_2)-\gamma(x_1))^2}{\sigma^2\left\|x_2-x_1\right\|^{\alpha}}=O_{\mathbb{P}}\left(N^{-1+\alpha/2}\right)$$

and therefore

$$N^{-(2-\alpha)/4}\frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N}\frac{(\gamma(x_2)-\gamma(x_1))^2}{\sigma^2\left\|x_2-x_1\right\|^{\alpha}}=O_{\mathbb{P}}\left(N^{3\alpha/4-1}\right)\xrightarrow{\mathbb{P}}0.$$

Moreover, since

$$\begin{aligned}\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\left|U^{(\eta)}_{x_1,x_2}\right|\frac{|\gamma(x_2)-\gamma(x_1)|}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right]&=\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\mathbb{E}_{x_1,x_2}\left[\left|U^{(\eta)}_{x_1,x_2}\right|\right]\frac{|\gamma(x_2)-\gamma(x_1)|}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right]\\&\le C\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\frac{|\gamma(x_2)-\gamma(x_1)|}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right],\end{aligned}$$

we deduce from Lemma 7 *(ii)* that

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N} U^{(\eta)}_{x_1,x_2}\frac{\gamma\left(x_2\right)-\gamma\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}} = O_{L_1}\left(N^{-1+\alpha/4}\right)$$

and consequently that

$$N^{-(2-\alpha)/4}\frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N} U^{(\eta)}_{x_1,x_2}\frac{\gamma\left(x_2\right)-\gamma\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}} = O_{L_1}\left(N^{\alpha/2-1}\right)\xrightarrow{L_1} 0.$$

We deal below with the first term of the right-hand side of (5.1). To do it, we write

$$H_2\left(U^{(\eta)}_{x_1,x_2}+\frac{\gamma\left(x_2\right)-\gamma\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right) = H_2\left(\frac{1}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\left[\log\left(\frac{\eta(x_2)}{\eta(x_1)}\right)+\left(\gamma\left(x_2\right)-\gamma\left(x_1\right)\right)\right]\right).$$

Because

$$\log\eta(x) = \bigvee_{i\geq 1} Z_i\left(x\right) = \bigvee_{i\geq 1}\left(\log U_i + W_i\left(x\right)\right)-\gamma\left(x\right),$$

we have

$$\begin{aligned}
&H_2\left(U^{(\eta)}_{x_1,x_2}+\frac{\gamma\left(x_2\right)-\gamma\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right)\\
&\qquad= \sum_{j\geq 1}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right),\bigvee_{i\geq 1} Z_i\left(x_2\right)=Z_j\left(x_2\right)\right] H_2\left(U^{(W_j)}_{x_1,x_2}\right)\\
&\qquad+\sum_{j\geq 1}\sum_{k\neq j}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right),\bigvee_{i\geq 1} Z_i\left(x_2\right)=Z_k\left(x_2\right)\right] H_2\left(U^{(W_k)}_{x_1,x_2}+\frac{Z_{k\setminus j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right),
\end{aligned}$$

where, for any distinct points $x_1, x_2 \in \mathbf{R}^2$ and for any fractional Brownian field with semi-variogram as in (2.1),

$$U^{(W)}_{x_1,x_2} = \frac{1}{\sigma||x_2-x_1||^{\alpha/2}}\left(W\left(x_2\right)-W\left(x_1\right)\right)$$

denotes the normalized increment between x_1 and x_2 w.r.t. W. This gives

$$\begin{aligned}
H_2\left(U^{(\eta)}_{x_1,x_2}+\frac{\gamma\left(x_2\right)-\gamma\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right) &= \sum_{j\geq 1}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right)\right] H_2\left(U^{(W_j)}_{x_1,x_2}\right)\\
&+\sum_{j\geq 1}\sum_{k\neq j}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right),\bigvee_{i\geq 1} Z_i\left(x_2\right)=Z_k\left(x_2\right)\right]\\
&\times\left(H_2\left(U^{(W_k)}_{x_1,x_2}+\frac{Z_{k\setminus j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right)-H_2\left(U^{(W_j)}_{x_1,x_2}\right)\right).
\end{aligned}$$

We re-write the term appearing in the double sums over $j \geq 1$, $k \neq j$, by considering two cases: the first one is when x_1 and x_2 belongs to the same cell, namely $\mathbf{C}_{k,j}$ (see Eq. (3.5) for its definition) and the second one deals with the complement. To do it, we write

$$\begin{aligned}
&\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right), \bigvee_{i\geq 1} Z_i\left(x_2\right)=Z_k\left(x_2\right)\right]\mathbb{I}\left[x_1,x_2\in\mathbf{C}_{k,j}\right]\\
&\qquad=\mathbb{I}\left[Z_j\left(x_1\right)\geq Z_k\left(x_1\right), Z_k\left(x_2\right)\geq Z_j\left(x_2\right)\right]\mathbb{I}\left[x_1,x_2\in\mathbf{C}_{k,j}\right]\\
&\qquad=\mathbb{I}\left[\frac{Z_{k\backslash j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\leq 0, \frac{Z_{k\backslash j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\geq U^{(W_j)}_{x_1,x_2}-U^{(W_k)}_{x_1,x_2}\right]\mathbb{I}\left[x_1,x_2\in\mathbf{C}_{k,j}\right],
\end{aligned}$$

where the last line comes from the fact that

$$Z_k\left(x_2\right)\geq Z_j\left(x_2\right)\Longleftrightarrow\frac{Z_{k\backslash j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\geq U^{(W_j)}_{x_1,x_2}-U^{(W_k)}_{x_1,x_2}.$$

Now, for any function measurable function $f:\mathbf{R}\to\mathbf{R}$, we define a function ψ_f in the same way as in Section 3 of [9] as

$$\Psi_f\left(x,y,w\right)=\left(f(y+w)-f(x)\right)\mathbb{I}\left[x-y\leq w\leq 0\right]+\left(f(x-w)-f(y)\right)\mathbb{I}\left[0\leq w\leq x-y\right]. \tag{5.2}$$

Taking $f=H_2$, we get

$$\begin{aligned}
H_2\left(U^{(\eta)}_{x_1,x_2}+\frac{\gamma\left(x_2\right)-\gamma\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right)&=\sum_{j\geq 1}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right)\right]H_2\left(U^{(W_j)}_{x_1,x_2}\right)\\
&+\sum_{j\geq 1}\sum_{k>j}\mathbb{I}\left[x_1,x_2\in\mathbf{C}_{k,j}\right]\Psi_{H_2}(U^{(W_j)}_{x_1,x_2},U^{(W_k)}_{x_1,x_2},Z_{k\backslash j}(x_1)/(\sigma d^{\alpha/2}_{1,2}))\\
&+\sum_{j\geq 1}\sum_{k\neq j}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i\left(x_1\right)=Z_j\left(x_1\right), \bigvee_{i\geq 1} Z_i\left(x_2\right)=Z_k\left(x_2\right), x_1 \text{ or } x_2\notin\mathbf{C}_{k,j}\right]\\
&\qquad\times\left(H_2\left(U^{(W_k)}_{x_1,x_2}+\frac{Z_{k\backslash j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right)-H_2\left(U^{(W_j)}_{x_1,x_2}\right)\right).
\end{aligned}$$

Three cases appear in the above expression and are discussed below. The first one is when only one trajectory (the j-th) is considered, the second one deals with the case of two trajectories (the j-th and the k-th) and the last one concerns an arbitrarily large number of trajectories. For the first one, we consider the canonical tessellation of the max-stable random field η, i.e. the collection of cells $C_j=\{x\in\mathbf{R}^2:Z_j(x)=\log\eta(x)\}$, $j\geq 1$. Notice that the set $\mathcal{I}_{\mathbf{C}}=\{j\geq 1:\mathbf{C}\cap C_j\neq\emptyset\}$ is a.s finite. Adapting the proof of Theorem 1 in [8], we can easily show that, for any $j\in\mathcal{I}_{\mathbf{C}}$,

$$\frac{1}{\sqrt{|E_N|}}\sum_{(x_1,x_2)\in E_N}\mathbb{I}\left[x_1\in\mathbf{C}\cap C_j\right]H_2\left(U^{(W_j)}_{x_1,x_2}\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^2_{V_2}|\mathbf{C}\cap C_j|\right),$$

for some $\sigma_{V_2}^2 > 0$. This gives

$$\frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \sum_{j\geq 1} \mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1)\right] H_2\left(U_{x_1,x_2}^{(W_j)}\right)$$
$$= \sum_{j\geq 1}\left[\frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \mathbb{I}\left[x_1 \in \mathbf{C}\cap C_j\right] H_2\left(U_{x_1,x_2}^{(W_j)}\right)\right] \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_{V_2}^2\right).$$

To deal with the second term, we consider the tessellation $(\mathbf{C}_{k,j})_{k\neq j\geq 1}$, where $\mathbf{C}_{k,j} \subset \mathbf{C}$ is defined in Eq. (3.5). Notice also that the set $\mathcal{J}_{\mathbf{C}} = \{(k,j) : \mathbf{C}_{k,j} \neq \emptyset\}$ is a.s. finite. Adapting the proof of Proposition 1 in [9], we can prove that, for any $(k,j) \in \mathcal{J}_{\mathbf{C}}$,

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \Psi_{H_2}(U_{x_1,x_2}^{(W_j)}, U_{x_1,x_2}^{(W_k)}, Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2}))\mathbb{I}\left[x_1, x_2 \in \mathbf{C}_{k,j}\right]$$
$$\xrightarrow{\mathbb{P}} c_{V_2} L_{Z_{k\setminus j}}(0),$$

where $d_{1,2} := ||x_2 - x_1||$ and c_{V_2} is a constant. Taking the sums over j and k, this shows that

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \sum_{j\geq 1}\sum_{k>j} \Psi_{H_2}(U_{x_1,x_2}^{(j)}, U_{x_1,x_2}^{(k)}, Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2}))\mathbb{I}\left[x_1, x_2 \in \mathbf{C}_{k,j}\right]$$
$$\xrightarrow{\mathbb{P}} c_{V_2} \sum_{j\geq 1}\sum_{k>j} L_{Z_{k\setminus j}}(0)$$

as N goes to infinity.

For the third case, we notice that the event

$$\left\{\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_2) = Z_k(x_2), x_1 \text{ or } x_2 \notin \mathbf{C}_{k,j}\right\}$$

has a small probability as $N \to \infty$. Indeed, since x_1 and x_2 are closer and closer as $N \to \infty$, it is more likely that the second largest path in x_1 is the one that maximises in x_2. If x_1 (e.g.) does not belong to $\mathbf{C}_{k,j}$, then there is another path, say the l-th, which takes the second largest value in x_1; such a configuration is less likely than if it was the one that maximises in x_2, i.e. the k-th. Using the above observations, we can show that

$$N^{-(2-\alpha)/4} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \sum_{j\geq 1}\sum_{k\neq j} \mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_2) = Z_k(x_2)\right]$$
$$\times\, \mathbb{I}\left[x_1 \text{ or } x_2 \notin \mathbf{C}_{k,j}\right]\left(H_2\left(U_{x_1,x_2}^{(W_k)} + \frac{Z_{k\setminus j}(x_1)}{\sigma \left\|x_2 - x_1\right\|^{\alpha/2}}\right) - H_2\left(U_{x_1,x_2}^{(W_j)}\right)\right)$$

converges to 0 in probability. Therefore

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} H_2(U_{x_1,x_2}^{(\eta)}) \xrightarrow{\mathbb{P}} c_{V_2} \sum_{j\geq 1}\sum_{k>j} L_{Z_{k\setminus j}}(0).$$

This together with (5.1) concludes the proof of Theorem 3 for $V_{2,N}^{(\eta)}$.

5.3.2 Proof for $V_{3,N}^{(\eta)}$

We proceed in the same spirit as the proof for $V_{2,N}^{(\eta)}$. First, we write

$$\begin{aligned}
V_{3,N}^{(\eta)} &= \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \left(\begin{pmatrix} U_{x_1,x_2}^{(\eta)} & U_{x_1,x_3}^{(\eta)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1,x_2,x_3} \\ R_{x_1,x_2,x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1,x_2}^{(\eta)} \\ U_{x_1,x_3}^{(\eta)} \end{pmatrix} - 2 \right) \\
&= \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} \frac{1}{1-R_{x_1,x_2,x_3}^2} \left(H_2(U_{x_1,x_2}^{(\eta)}) + H_2(U_{x_1,x_3}^{(\eta)}) - 2R_{x_1,x_2,x_3} U_{x_1,x_2}^{(\eta)} U_{x_1,x_3}^{(\eta)} \right) \\
&= \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1,x_2,x_3)\in DT_N} H_2\left(U_{x_1,x_2}^{(\eta)}, U_{x_1,x_3}^{(\eta)}; R_{x_1,x_2,x_3} \right),
\end{aligned}$$

where R_{x_1,x_2,x_3} is given in Eq. (3.6). For any $(x_1,x_2,x_3)\in DT_N$, we have

$$\begin{aligned}
H_2(U_{x_1,x_2}^{(\eta)}) = \sum_{j\geq 1} \mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1) \right] H_2\left(U_{x_1,x_2}^{(W_j)} \right) \\
&+ \sum_{j\geq 1}\sum_{k>j} \mathbb{I}\left[x_1,x_2,x_3 \in \mathbf{C}_{k,j}\right] \Psi_{H_2}(U_{x_1,x_2}^{(W_j)}, U_{x_1,x_2}^{(W_k)}, Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2})) \\
&+ \sum_{j\geq 1}\sum_{k>j} \mathbb{I}\left[x_1,x_2 \in \mathbf{C}_{k,j}, x_3 \notin \mathbf{C}_{k,j}\right] \Psi_{H_2}(U_{x_1,x_2}^{(W_j)}, U_{x_1,x_2}^{(W_k)}, Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2})) \\
&+ \sum_{j\geq 1}\sum_{k\neq j} \mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_2) = Z_k(x_2), x_1 \text{ or } x_2 \notin \mathbf{C}_{k,j} \right] \\
&\times \left(H_2\left(U_{x_1,x_2}^{(W_k)} + \frac{Z_{k\setminus j}(x_1)}{\sigma\|x_2-x_1\|^{\alpha/2}} \right) - H_2\left(U_{x_1,x_2}^{(W_j)} \right) \right) \\
&- 2U_{x_1,x_2}^{(\eta)} \frac{\gamma(x_2)-\gamma(x_1)}{\sigma\|x_2-x_1\|^{\alpha/2}} - \frac{(\gamma(x_2)-\gamma(x_1))^2}{\sigma^2\|x_2-x_1\|^{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
U_{x_1,x_2}^{(\eta)} &= \sum_{j\geq 1} \mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1) \right] U_{x_1,x_2}^{(W_j)} \\
&+ \sum_{j\geq 1}\sum_{k>j} \mathbb{I}\left[x_1,x_2,x_3 \in \mathbf{C}_{k,j}\right] \Psi_I(U_{x_1,x_2}^{(W_j)}, U_{x_1,x_2}^{(W_k)}, Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2})) \\
&+ \sum_{j\geq 1}\sum_{k>j} \mathbb{I}\left[x_1,x_2 \in \mathbf{C}_{k,j}, x_3 \notin \mathbf{C}_{k,j}\right] \Psi_I(U_{x_1,x_2}^{(W_j)}, U_{x_1,x_2}^{(W_k)}, Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2})) \\
&+ \sum_{j\geq 1}\sum_{k\neq j} \mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_2) = Z_k(x_2), x_1 \text{ or } x_2 \notin \mathbf{C}_{k,j} \right] \frac{Z_{k\setminus j}(x_1)}{\sigma\|x_2-x_1\|^{\alpha/2}} \\
&- \frac{\gamma(x_2)-\gamma(x_1)}{\sigma\|x_2-x_1\|^{\alpha/2}}.
\end{aligned}$$

In what follows, for any $-1<R<1$, we denote in the same way as in Section 3 of [9] by Ω the following function

$$\begin{aligned}\Omega(u_1,v_1,u_2,v_2,w_1,w_2;R) = \frac{1}{1-R^2}&\left[\Psi_{H_2}\left(u_1,v_1,w_1\right)+\Psi_{H_2}\left(u_2,v_2,w_2\right)\right]\\ &- 2\frac{R}{1-R^2}\Psi_I\left(u_1,v_1,w_1\right)\Psi_I\left(u_2,v_2,w_2\right)\\ &-2\frac{R}{1-R^2}\left[u_1\Psi_I\left(u_2,v_2,w_2\right)+u_2\Psi_I\left(u_1,v_1,w_1\right)\right]\mathbb{I}[w_1<0]\\ &-2\frac{R}{1-R^2}\left[v_1\Psi_I\left(u_2,v_2,w_2\right)+v_2\Psi_I\left(u_1,v_1,w_1\right)\right]\mathbb{I}[w_1>0],\end{aligned}$$

with $I\left(u\right)=u$ for all $u\in\mathbf{R}$. This gives

$$\begin{aligned}&H_2\left(U^{(\eta)}_{x_1,x_2},U^{(\eta)}_{x_1,x_3};R_{x_1,x_2,x_3}\right)\\ &\quad=\sum_{j\geq1}\mathbb{I}\left[\bigvee_{i\geq1}Z_i\left(x_1\right)=Z_j\left(x_1\right)\right]H_2\left(U^{(W_j)}_{x_1,x_2},U^{(W_j)}_{x_1,x_2};R_{x_1,x_2,x_3}\right)\\ &\quad+\sum_{j\geq1}\sum_{k>j}\mathbb{I}\left[x_1,x_2,x_3\in\mathbf{C}_{k,j}\right]\\ &\quad\times\Omega(U^{(W_j)}_{x_1,x_2},U^{(W_j)}_{x_1,x_3},U^{(W_k)}_{x_1,x_2},U^{(W_k)}_{x_1,x_3},Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2}),Z_{k\setminus j}(x_1)/(\sigma d_{1,3}^{\alpha/2});R_{x_1,x_2,x_3})\\ &\quad+\frac{1}{1-R^2_{x_1,x_2,x_3}}\left(N^{(1)}_{x_1,x_2,x_3}+N^{(2)}_{x_1,x_2,x_3}+N^{(3)}_{x_1,x_2,x_3}\right),\end{aligned}$$

where

$$\begin{aligned}N^{(1)}_{x_1,x_2,x_3}=\sum_{j\geq1}\sum_{k\neq j}\mathbb{I}&\left[\bigvee_{i\geq1}Z_i\left(x_1\right)=Z_j\left(x_1\right),\bigvee_{i\geq1}Z_i\left(x_2\right)=Z_k\left(x_2\right),x_1\text{ or }x_2\notin\mathbf{C}_{k,j}\right]\\ &\times\left(H_2\left(U^{(W_k)}_{x_1,x_2}+\frac{Z_{k\setminus j}\left(x_1\right)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\right)-H_2\left(U^{(W_j)}_{x_1,x_2}\right)\right)\\ &+\sum_{j\geq1}\sum_{k\neq j}\mathbb{I}\left[\bigvee_{i\geq1}Z_i\left(x_1\right)=Z_j\left(x_1\right),\bigvee_{i\geq1}Z_i\left(x_3\right)=Z_k\left(x_3\right),x_1\text{ or }x_3\notin\mathbf{C}_{k,j}\right]\\ &\times\left(H_2\left(U^{(W_k)}_{x_1,x_3}+\frac{Z_{k\setminus j}\left(x_1\right)}{\sigma\left\|x_3-x_1\right\|^{\alpha/2}}\right)-H_2\left(U^{(W_j)}_{x_1,x_3}\right)\right)\\ &+\sum_{j\geq1}\sum_{k>j}\mathbb{I}\left[x_1,x_2\in\mathbf{C}_{k,j},x_3\notin\mathbf{C}_{k,j}\right]\Psi_{H_2}(U^{(W_j)}_{x_1,x_2},U^{(W_k)}_{x_1,x_2},Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2}))\\ &+\sum_{j\geq1}\sum_{k>j}\mathbb{I}\left[x_1,x_3\in\mathbf{C}_{k,j},x_2\notin\mathbf{C}_{k,j}\right]\Psi_{H_2}(U^{(W_j)}_{x_1,x_3},U^{(W_k)}_{x_1,x_3},Z_{k\setminus j}(x_1)/(\sigma d_{1,3}^{\alpha/2}))\\ &+\sum_{j\geq1}\sum_{k>j}\mathbb{I}\left[\bigvee_{i\geq1}Z_i\left(x_1\right)=Z_j\left(x_1\right),x_1,x_2\in\mathbf{C}_{k,j},x_3\notin\mathbf{C}_{k,j}\right]\\ &\times\Psi_I(U^{(W_j)}_{x_1,x_2},U^{(W_k)}_{x_1,x_2},Z_{k\setminus j}(x_1)/(\sigma d_{1,2}^{\alpha/2}))U^{(W_j)}_{x_1,x_3}\end{aligned}$$

and

$$N^{(2)}_{x_1,x_2,x_3} = -2\left(U^{(\eta)}_{x_1,x_2} - R_{x_1,x_2,x_3}U^{(\eta)}_{x_1,x_3}\right)\frac{\gamma(x_2)-\gamma(x_1)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}} - \frac{(\gamma(x_2)-\gamma(x_1))^2}{\sigma^2\left\|x_2-x_1\right\|^{\alpha}}$$
$$-2\left(U^{(\eta)}_{x_1,x_3} - R_{x_1,x_2,x_3}U^{(\eta)}_{x_1,x_2}\right)\frac{\gamma(x_3)-\gamma(x_1)}{\sigma\left\|x_3-x_1\right\|^{\alpha/2}} - \frac{(\gamma(x_3)-\gamma(x_1))^2}{\sigma^2\left\|x_3-x_1\right\|^{\alpha}}$$
$$+2R_{x_1,x_2,x_3}\frac{(\gamma(x_2)-\gamma(x_1))}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\frac{(\gamma(x_3)-\gamma(x_1))}{\sigma\left\|x_3-x_1\right\|^{\alpha/2}}$$

and

$$N^{(3)}_{x_1,x_2,x_3} = \sum_{j\geq 1}\sum_{k\neq j}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_2) = Z_k(x_2), x_1 \text{ or } x_2 \notin \mathbf{C}_{k,j}\right]$$
$$\times\frac{U^{(W_j)}_{x_1,x_2}}{\sigma\left\|x_3-x_1\right\|^{\alpha/2}} Z_{k\setminus j}(x_1)$$
$$+\sum_{j\geq 1}\sum_{k\neq j}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_3) = Z_k(x_3), x_1 \text{ or } x_3 \notin \mathbf{C}_{k,j}\right]$$
$$\times\frac{U^{(W_j)}_{x_1,x_3}}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}} Z_{k\setminus j}(x_1)$$
$$+\sum_{j\geq 1}\sum_{k\neq j}\sum_{k'\neq j}\mathbb{I}\left[\begin{array}{c}\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1), \bigvee_{i\geq 1} Z_i(x_2) = Z_k(x_2), \bigvee_{i\geq 1} Z_i(x_3) = Z_{k'}(x_3), \\ x_1 \text{ or } x_2 \notin \mathbf{C}_{k,j}, x_1 \text{ or } x_3 \notin \mathbf{C}_{k',j}\end{array}\right]$$
$$\times\frac{Z_{k\setminus j}(x_1)}{\sigma\left\|x_2-x_1\right\|^{\alpha/2}}\frac{Z_{k'\setminus j}(x_1)}{\sigma\left\|x_3-x_1\right\|^{\alpha/2}}$$
$$+\sum_{j\geq 1}\sum_{k\neq j}\sum_{k'\neq j}\mathbb{I}\left[x_1,x_2\in\mathbf{C}_{k,j}, x_3\notin\mathbf{C}_{k,j}\right]\mathbb{I}\left[x_1,x_3\in\mathbf{C}_{k',j}, x_2\notin\mathbf{C}_{k',j}\right]$$
$$\times\Psi_I(U^{(W_j)}_{x_1,x_2}, U^{(W_k)}_{x_1,x_2}, W^{(k\setminus j)}(x_1)/(\sigma d^{\alpha/2}_{1,2}))\Psi_I(U^{(W_j)}_{x_1,x_3}, U^{(W_k)}_{x_1,x_3}, W^{(k\setminus j)}(x_1)/(\sigma d^{\alpha/2}_{1,3})).$$

Proceeding in the same spirit as the proof for $V^{(\eta)}_{2,N}$ and adapting the proof of Theorem 1 in [8], we can show that, for $\alpha\in(0,1)$, as $N\to\infty$,

$$\frac{1}{\sqrt{|DT_N|}}\sum_{(x_1,x_2,x_3)\in DT_N}\sum_{j\geq 1}\mathbb{I}\left[\bigvee_{i\geq 1} Z_i(x_1) = Z_j(x_1)\right]H_2\left(U^{(W_j)}_{x_1,x_2}, U^{(W_j)}_{x_1,x_2}; R_{x_1,x_2,x_3}\right)\xrightarrow{\mathcal{D}}\mathcal{N}\left(0,\sigma^2_{V_3}\right),$$

for some $\sigma^2_{V_3} > 0$. Using the same type of arguments as in the proof for $V^{(\eta)}_{2,N}$ and adapting the proof of Proposition 1 in [9], we can also show that, for $\alpha\in(0,1)$, as $N\to\infty$,

$$\frac{\sqrt{2}}{2}N^{-(2-\alpha)/4}\frac{1}{\sqrt{|DT_N|}}\sum_{(x_1,x_2,x_3)\in DT_N}\sum_{j\geq 1}\sum_{k>j}\mathbb{I}\left[x_1,x_2,x_3\in\mathbf{C}_{k,j}\right]$$
$$\times\Omega(U^{(W_j)}_{x_1,x_2}, U^{(W_j)}_{x_1,x_3}, U^{(W_k)}_{x_1,x_2}, U^{(W_k)}_{x_1,x_3}, Z_{k\setminus j}(x_1)/(\sigma d^{\alpha/2}_{1,2}), Z_{k\setminus j}(x_1)/(\sigma d^{\alpha/2}_{1,3}); R_{x_1,x_2,x_3})$$
$$\xrightarrow{\mathbb{P}} c_{V_3}\sum_{j\geq 1}\sum_{k>j} L_{Z_{k\setminus j}}(0).$$

Finally, we use the same type of arguments as in the proof for $V_{2,N}^{(\eta)}$ to derive that

$$\frac{\sqrt{2}}{2}N^{-(2-\alpha)/4}\frac{1}{\sqrt{|DT_N|}}\sum_{(x_1,x_2,x_3)\in DT_N}\left[N_{x_1,x_2,x_3}^{(1)}+N_{x_1,x_2,x_3}^{(2)}+N_{x_1,x_2,x_3}^{(3)}\right]\xrightarrow{\mathbb{P}}0$$

and to conclude.

5.4 Proof of Proposition 4

Recall that $a=\sigma d^{\alpha/2}$, $u=\log(z_2/z_1)/a$ and $v\left(u\right)=a/2+u$ (see Section 3.1.1). Since

$$\mathbb{P}\left[\eta\left(x_1\right)\le z_1,\eta\left(x_2\right)\le z_2\right]=\exp(-V_{x_1,x_2}(z_1,z_2)),$$

we have

$$f_{x_1,x_2}(z_1,z_2)=\exp(-V_{x_1,x_2}(z_1,z_2))\times\left(\frac{\partial V_{x_1,x_2}(z_1,z_2)}{\partial z_1}\frac{\partial V_{x_1,x_2}(z_1,z_2)}{\partial z_2}-\frac{\partial^2 V_{x_1,x_2}(z_1,z_2)}{\partial z_1\partial z_2}\right).$$

The derivatives appearing in the above expression can be interpreted as follows: the second derivative of the exponent measure deals with the case where x_1 and x_2 are in the same cell of the canonical tessellation, whereas the product of the first derivatives deals with the opposite case. As d tends to 0, the probability that x_1 and x_2 are in the same cell of the canonical tessellation tends to 1, and $\partial^2 V_{x_1,x_2}(z_1,z_2)/\partial z_1\partial z_2$ becomes the dominant term as observed in Remark 2.

The exponent measure can be expressed as

$$V_{x_1,x_2}(z_1,z_2)=\frac{1}{z_1}\left(\Phi\left(v\left(u\right)\right)+e^{-au}\Phi\left(v\left(-u\right)\right)\right).$$

Taking the derivative with respect to z_1, z_2 and using the facts that $\varphi'(x)=-x\varphi(x)$, where φ is the density function of the standard Gaussian distribution, and that $v(u)=a-v(-u)$, we get

$$\begin{aligned}\frac{\partial V_{x_1,x_2}(z_1,z_2)}{\partial z_1}&=\frac{1}{z_1^2}A(u),\\ \frac{\partial V_{x_1,x_2}(z_1,z_2)}{\partial z_2}&=\frac{1}{z_2^2}B(u),\\ \frac{\partial^2 V_{x_1,x_2}(z_1,z_2)}{\partial z_1\partial z_2}&=-\frac{1}{z_1^3}C(u),\end{aligned}$$

where

$$\begin{aligned}A\left(u\right)&=\Phi\left(v\left(u\right)\right)+\frac{1}{a}\varphi\left(v\left(u\right)\right)-\frac{1}{a}e^{-au}\varphi\left(v\left(-u\right)\right),\\ B\left(u\right)&=\Phi\left(v\left(-u\right)\right)+\frac{1}{a}\varphi\left(v\left(-u\right)\right)-\frac{1}{a}e^{au}\varphi\left(v\left(u\right)\right),\\ C\left(u\right)&=\frac{1}{a^2}e^{-au}\left(v\left(-u\right)\varphi\left(v\left(u\right)\right)+e^{-au}v\left(u\right)\varphi\left(v\left(-u\right)\right)\right).\end{aligned}$$

Therefore

$$f_{x_1,x_2}(z_1,z_2)=\exp\left(-\frac{1}{z_1}\left\{\Phi\left(v\left(u\right)\right)+e^{-au}\Phi\left(v\left(-u\right)\right)\right\}\right)\times\left(\frac{1}{z_1^4}e^{-2au}A\left(u\right)B\left(u\right)+\frac{1}{z_1^3}C\left(u\right)\right).$$

Hence we have

$$\log f_{x_1,x_2}(z_1,z_2) = -\frac{1}{z_1}\left\{\Phi\left(v\left(u\right)\right) + e^{-au}\Phi\left(v\left(-u\right)\right)\right\} + \log\left(e^{-2au}A\left(u\right)B\left(u\right) + z_1 C\left(u\right)\right) - 4\log z_1.$$

Proof for the derivative with respect to σ. To deal with the derivative with respect to σ, we use the following equalities

$$\frac{\partial v\left(u\right)}{\partial\sigma} = \frac{d^{\alpha/2}}{2} - \frac{1}{d^{\alpha/2}\sigma^2}\log\left(\frac{z_2}{z_1}\right) = \frac{1}{\sigma}v\left(-u\right), \quad \frac{\partial v\left(-u\right)}{\partial\sigma} = \frac{1}{\sigma}v\left(u\right), \quad \frac{\partial e^{au}}{\partial\sigma} = 0$$
$$\text{and } \frac{\partial 1/a}{\partial\sigma} = -\frac{1}{\sigma}\frac{1}{a}. \tag{5.3}$$

This gives

$$\frac{\partial\log f_{x_1,x_2}(z_1,z_2)}{\partial\sigma} = -\frac{1}{z_1}\frac{1}{\sigma}\left\{\varphi\left(v\left(u\right)\right)v\left(-u\right) + e^{-au}\varphi\left(v\left(-u\right)\right)v\left(u\right)\right\}$$
$$+\frac{1}{e^{-2au}A\left(u\right)B\left(u\right)+z_1C\left(u\right)} \times \left(e^{-2au}\left(\frac{\partial A\left(u\right)}{\partial\sigma}B\left(u\right) + A\left(u\right)\frac{\partial B\left(u\right)}{\partial\sigma}\right) + z_1\frac{\partial C\left(u\right)}{\partial\sigma}\right) \tag{5.4}$$

with

$$\frac{\partial A\left(u\right)}{\partial\sigma} = \frac{1}{\sigma}\left(\varphi\left(v\left(u\right)\right)v\left(-u\right)\left(1-\frac{1}{a}v\left(u\right)\right) + \frac{1}{a}e^{-au}\varphi\left(v\left(-u\right)\right)v\left(-u\right)v\left(u\right)\right)$$
$$-\frac{1}{\sigma}\left(\frac{1}{a}\varphi\left(v\left(u\right)\right) - \frac{1}{a}e^{-au}\varphi\left(v\left(-u\right)\right)\right),$$

$$\frac{\partial B\left(u\right)}{\partial\sigma} = \frac{1}{\sigma}\left(\varphi\left(v\left(-u\right)\right)v\left(u\right)\left(1-\frac{1}{a}v\left(-u\right)\right) + \frac{1}{a}e^{au}\varphi\left(v\left(u\right)\right)v\left(u\right)v\left(-u\right)\right)$$
$$-\frac{1}{\sigma}\left(\frac{1}{a}\varphi\left(v\left(-u\right)\right) - \frac{1}{a}e^{au}\varphi\left(v\left(u\right)\right)\right)$$

$$\frac{\partial C\left(u\right)}{\partial\sigma} = \frac{1}{\sigma}\frac{1}{a^2}e^{-au}v\left(u\right)\varphi\left(v\left(u\right)\right)\left(1-v^2\left(-u\right)\right)$$
$$+\frac{1}{\sigma}\frac{1}{a^2}e^{-2au}v\left(-u\right)\varphi\left(v\left(-u\right)\right)\left(1-v^2\left(u\right)\right)$$
$$-2\frac{1}{\sigma}\frac{1}{a^2}e^{-au}\left(v\left(-u\right)\varphi\left(v\left(u\right)\right) + e^{-au}v\left(u\right)\varphi\left(v\left(-u\right)\right)\right).$$

Here, for two functions $c\left(\cdot\right)$ and $b\left(\cdot\right)$ of a, we write $c = o(b)$ (resp. $c = O(b)$) if $c\left(a\right)/b\left(a\right) \underset{a\to 0}{\to} 0$ (resp. $\left|c\left(a\right)/b\left(a\right)\right| \leq C$ for some positive constant C). Moreover, we write $b \underset{a\to 0}{\sim} c$ if $b\left(a\right)/c\left(a\right) \underset{a\to 0}{\to} 1$.

As $d \to 0$ or equivalently as $a \to 0$, we have (using the fact that $\varphi(-u) = \varphi(u)$)

$$\begin{aligned}\Phi\left(v\left(u\right)\right) &= \Phi\left(u\right) + \frac{a}{2}\varphi\left(u\right) + o(a), \qquad \Phi\left(v\left(-u\right)\right) = \Phi\left(-u\right) + \frac{a}{2}\varphi\left(u\right) + o(a)\\ \varphi\left(v\left(u\right)\right) &= \varphi\left(u\right)\left(1-\frac{a}{2}u + o(a)\right), \qquad \varphi\left(v\left(-u\right)\right) = \varphi\left(u\right)\left(1+\frac{a}{2}u+o(a)\right).\end{aligned}$$

We deal below with each term of Eq. (5.4). First, we notice that

$$\begin{aligned}\varphi\left(v(u)\right)v(-u) &= \varphi\left(u\right)\Big(1-\frac{a}{2}u+o(a)\Big)\Big(\frac{a}{2}-u\Big)+o(a)\\ &= -u\varphi\left(u\right)\Big(1-\frac{a}{2}\left(u+u^{-1}\right)+o(a)\Big)+o(a)\end{aligned}$$

and that

$$\begin{aligned}e^{-au}\varphi\left(v(-u)\right)v(u) &= (1-au+o(a))\,\varphi\left(u\right)\Big(1+\frac{a}{2}u+o(a)\Big)\Big(\frac{a}{2}+u\Big)\\ &= u\varphi\left(u\right)\Big(1+\frac{a}{2}\left(-u+u^{-1}\right)+o(a)\Big)+o(a).\end{aligned}$$

Therefore

$$\varphi\left(v(u)\right)v(-u)+e^{-au}\varphi\left(v(-u)\right)v\left(u\right)=a\varphi(u)+o(a)$$

This implies that

$$\varphi\left(v(u)\right)v(-u)+e^{-au}\varphi\left(v(-u)\right)v\left(u\right)\underset{a\to0}{\to}0.$$

Secondly, we have

$$\begin{aligned}A(u) &= \frac{1}{a}\Big(a\Phi\left(u\right)+\varphi\left(u\right)\Big(1-\frac{a}{2}u+o(a)\Big)-(1-au+o(a))\,\varphi\left(u\right)\Big(1+\frac{a}{2}u+o(a)\Big)+o(a)\Big)\\ &= \Phi\left(u\right)+o(1),\end{aligned}$$

$$\begin{aligned}B(u) &= \frac{1}{a}\Big(a\Phi\left(-u\right)+\varphi\left(u\right)\Big(1+\frac{a}{2}u+o(a)\Big)-(1+au+o(a))\,\varphi\left(u\right)\Big(1-\frac{a}{2}u+o(a)\Big)+o(a)\Big)\\ &= \Phi\left(-u\right)+o(1),\end{aligned}$$

$$\begin{aligned}C(u) = \frac{1}{a^2}&\left(1-au+o(a)\right)\\ &\times\Big(\Big(\frac{a}{2}-u\Big)\varphi\left(u\right)\Big(1-\frac{a}{2}u+o(a)\Big)+(1-au+o(a))\Big(\frac{a}{2}+u\Big)\varphi\left(u\right)\Big(1+\frac{a}{2}u+o(a)\Big)\Big),\end{aligned}$$

which gives

$$\begin{aligned}C(u) = \frac{1}{a^2}&\left(1-au+o(a)\right)\\ &\times\Big(-u\varphi\left(u\right)\Big(1-\frac{a}{2}u+o(a)\Big)+\frac{a}{2}\varphi\left(u\right)+o(a)+u\varphi\left(u\right)\Big(1-\frac{a}{2}u+o(a)\Big)+\frac{a}{2}\varphi\left(u\right)+o(a)\Big)\\ &= \frac{1}{a}\varphi\left(u\right)+O(1).\end{aligned}$$

The above computations give that

$$e^{-2au}A(u)B(u)+z_1C(u)\underset{a\to0}{\sim}\frac{z_1\varphi(u)}{a}.$$

Thirdly, we have

$$\sigma\frac{\partial A(u)}{\partial\sigma} = \varphi\left(u\right)\left(1-\frac{a}{2}u+o(a)\right)\left(\frac{a}{2}-u\right)\left(\frac{1}{2}-\frac{u}{a}\right)$$
$$+\frac{1}{a}\left(1-au+o(a)\right)\varphi\left(u\right)\left(1+\frac{a}{2}u+o(a)\right)\left(\left(\frac{a}{2}\right)^{2}-u^{2}\right)$$
$$-\frac{1}{a}\left(\varphi\left(u\right)\left(1-\frac{a}{2}u+o(a)\right)-\left(1-au+o(a)\right)\varphi\left(u\right)\left(1+\frac{a}{2}u+o(a)\right)\right).$$

It can easily be shown that the last term is $O(1)$ and therefore that $\sigma\frac{\partial A(u)}{\partial\sigma} = O(1)$. In a similar way, we also have $\sigma\frac{\partial B(u)}{\partial\sigma} = O(1)$. Moreover,

$$\begin{aligned}\sigma\frac{\partial C(u)}{\partial\sigma} &= \frac{1}{a^2}e^{-au}\left[v(u)\varphi\left(v\left(u\right)\right)\left(1-v^{2}\left(-u\right)\right)+e^{-au}v(-u)\varphi\left(v(-u)\right)\left(1-v^{2}\left(u\right)\right)\right]\\ &\quad -2\frac{1}{a^2}e^{-au}\left(v(-u)\varphi\left(v(u)\right)+e^{-au}v(u)\varphi\left(v(-u)\right)\right)\end{aligned}$$

and after some simple calculations

$$\sigma\frac{\partial C(u)}{\partial\sigma} = \frac{1}{a}\varphi\left(u\right)\left(\left(u^{2}-1\right)+o(1)\right).$$

Therefore

$$e^{-2au}\left(\frac{\partial A(u)}{\partial\sigma}B(u)+A(u)\frac{\partial B\left(u\right)}{\partial\sigma}\right)+z_1\frac{\partial C(u)}{\partial\sigma}\underset{a\to0}{\sim}\frac{1}{\sigma a}z_1\varphi(u)(u^2-1).$$

Finally, we obtain from Eq. (5.4) that

$$\frac{\partial\log f_{x_1,x_2}\left(z_1,z_2\right)}{\partial\sigma}\underset{a\to0}{\to}\frac{u^2-1}{\sigma}.$$

This concludes the proof.

Proof for the derivative with respect to α. To deal with the derivative with respect to α, we use the following equalities

$$\frac{\partial a}{\partial\alpha} = \frac{\log d}{2}a,\quad \frac{\partial 1/a}{\partial\alpha} = -\frac{\log d}{2}\frac{1}{a},\quad \frac{\partial v\left(u\right)}{\partial\alpha} = \frac{\log d}{2}a-\frac{\log d}{2}u = \frac{\log d}{2}v\left(-u\right),$$
$$\text{and}\quad \frac{\partial v\left(-u\right)}{\partial\alpha} = \frac{\log d}{2}v\left(u\right).$$

Notice that the above equalities are the same as in Eq. (5.3) except that the derivative over σ (resp. the term $\frac{1}{\sigma}$) is replaced by the derivative over α (resp. by the term $\frac{\log d}{2}$). Therefore $\frac{1}{\sigma}\frac{\partial}{\partial\alpha}\log f_{x_1,x_2}(z_1,z_2) = \frac{\log d}{2}\frac{\partial}{\partial\sigma}\log f_{x_1,x_2}(z_1,z_2)$. It follows from the same lines as above that

$$\lim_{d\to0}\frac{1}{\log d}\frac{\partial}{\partial\alpha}\log f_{x_1,x_2}(z_1,z_2) = \frac{\sigma}{2}\lim_{d\to0}\frac{\partial}{\partial\sigma}\log f_{x_1,x_2}(z_1,z_2) = \frac{1}{2}(u^2-1).$$

Remark 2 *In the above computations, we have proved that the term $\frac{\partial}{\partial\sigma}\log f_{x_1,x_2}(z_1,z_2)$ can be approximated by*

$$\frac{\partial}{\partial\sigma}\log\left(-\frac{\partial^2 V_{x_1,x_2}(z_1,z_2)}{\partial z_1\partial z_2}\right),$$

as $d\to0$.

5.5 Proof of Proposition 5

Proof for the derivative with respect to σ. In the same spirit as in the proof of Proposition 4, the likelihood can be approximated by only considering the term involving the third derivative of the exponent measure corresponding to the event where all three points are in the same cell of the canonical tessellation. More precisely, similarly to Remark 2, we have

$$\frac{\partial}{\partial \sigma} \log f_{x_1,x_2,x_3}(z_1,z_2,z_3) \underset{\delta \to 0}{\sim} \frac{\partial}{\partial \sigma} \log \left(-\frac{\partial^3 V_{x_1,x_2,x_3}(z_1,z_2,z_3)}{\partial z_1 \partial z_2 \partial z_3} \right).$$

No further details are given for these equivalences, but they can be obtained using the same arguments as in the proof of Proposition 4.

From Appendix B.4 in [16], we have

$$\begin{aligned}
&\frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} V_{x_1,x_2,x_3}(z_1,z_2,z_3) = -\frac{1}{z_1^2 z_2 z_3} \\
&\times \varphi_2 \left(\left(\begin{array}{c} a_{1,2} v_{1,2}\left(u_{1,2}\right) \\ a_{1,3} v_{1,3}\left(u_{1,3}\right) \end{array} \right); \left(\begin{array}{cc} (a_{1,2})^2 & \left[(a_{1,2})^2 + (a_{1,3})^2 - (a_{2,3})^2\right]/2 \\ \left[(a_{1,2})^2 + (a_{1,3})^2 - (a_{2,3})^2\right]/2 & (a_{1,3})^2 \end{array} \right) \right),
\end{aligned}$$

where $\varphi_2\left(\cdot;\Sigma\right)$ is the probability density function of a centered bivariate Gaussian distribution with covariance matrix Σ and, for $i,j = 1,2,3$ such that $i \neq j$, $a_{i,j} = \sigma \delta^{\alpha/2} d_{i,j}^{\alpha/2}$, $u_{i,j} = \log(z_j/z_i)/a_{i,j}$ and $v_{i,j}\left(u\right) = a_{i,j}/2 + u_{i,j}$. We deduce that

$$\begin{aligned}
&\frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} V_{x_1,x_2,x_3}(z_1,z_2,z_3) \\
&\qquad = -\frac{1}{z_1^2 z_2 z_3} \frac{1}{a_{1,2} a_{1,3}} \varphi_2 \left(\left(\begin{array}{c} v_{1,2}\left(u_{1,2}\right) \\ v_{1,3}\left(u_{1,3}\right) \end{array} \right); \left(\begin{array}{cc} 1 & R_1 \\ R_1 & 1 \end{array} \right) \right) \\
&\qquad = -\frac{1}{z_1^4} e^{-a_{1,2} u_{1,2}} e^{-a_{1,3} u_{1,3}} \frac{1}{a_{1,2} a_{1,3}} \varphi_2 \left(\left(\begin{array}{c} v_{1,2}\left(u_{1,2}\right) \\ v_{1,3}\left(u_{1,3}\right) \end{array} \right); \left(\begin{array}{cc} 1 & R_1 \\ R_1 & 1 \end{array} \right) \right).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\log \left(-\frac{\partial^3}{\partial z_1 \partial z_2 \partial z_3} V_{x_1,x_2,x_3}(z_1,z_2,z_3) \right) \\
&\quad = -4 \log\left(z_1\right) - a_{1,2} u_{1,2} - a_{1,3} u_{1,3} - \log\left(a_{1,2} a_{1,3}\right) - \frac{1}{2} \log\left(1 - R_1^2\right) - \log(2\pi) \\
&\qquad - \frac{1}{2} \left(\begin{array}{cc} v_{1,2}\left(u_{1,2}\right) & v_{1,3}\left(u_{1,3}\right) \end{array} \right) \left(\begin{array}{cc} 1 & R_1 \\ R_1 & 1 \end{array} \right)^{-1} \left(\begin{array}{c} v_{1,2}\left(u_{1,2}\right) \\ v_{1,3}\left(u_{1,3}\right) \end{array} \right).
\end{aligned}$$

Since

$$\frac{\partial}{\partial \sigma} v_{i,j}\left(u_{i,j}\right) = \frac{1}{\sigma} v_{i,j}\left(-u_{i,j}\right) \quad \text{and} \quad \frac{\partial}{\partial \sigma} a_{i,j} = \frac{1}{\sigma} a_{i,j},$$

we deduce that

$$\sigma\frac{\partial}{\partial\sigma}\log\left(-\frac{\partial^3}{\partial z_1\partial z_2\partial z_3}V_{x_1,x_2,x_3}(z_1,z_2,z_3)\right)$$
$$= -a_{1,2}u_{1,2} - a_{1,3}u_{1,3} - 2 - \frac{1}{1-R_1^2}\left(v_{1,2}\left(u_{1,2}\right) - R_1 v_{1,3}\left(u_{1,3}\right)\right)v_{1,2}\left(-u_{1,2}\right)$$
$$- \frac{1}{1-R_1^2}\left(v_{1,3}\left(u_{1,3}\right) - R_1 v_{1,2}\left(u_{1,2}\right)\right)v_{1,3}\left(-u_{1,3}\right).$$

As $\delta \to 0$, we get

$$\sigma\frac{\partial}{\partial\sigma}\log\left(-\frac{\partial^3 V_{x_1,x_2,x_3}(z_1,z_2,z_3)}{\partial z_1\partial z_2\partial z_3}\right) \to -2 + \frac{1}{1-R_1^2}\left[\left(u_{1,2} - R_1 u_{1,3}\right)u_{1,2} + \left(u_{1,3} - R_1 u_{1,2}\right)u_{1,3}\right]$$
$$= -2 + \frac{1}{1-R_1^2}\left(\left(u_{1,2}\right)^2 - 2R_1 u_{1,2}u_{1,3} + \left(u_{1,3}\right)^2\right).$$

Proof for the derivative with respect to α. The result is derived in the same way as for σ noting that

$$\frac{\partial}{\partial\alpha}v_{i,j}\left(u_{i,j}\right) = \frac{\log\delta d_{i,j}}{2}v_{i,j}\left(-u_{i,j}\right) \quad \text{and} \quad \frac{\partial}{\partial\alpha}a_{i,j} = \frac{\log\delta d_{i,j}}{2}a_{i,j}.$$

5.6 Proof of Theorem 6

Adapting the proof of Proposition 4, we can show that for any pairs (x_1, x_2),

$$\begin{aligned}\frac{\partial\log f_{x_1,x_2}\left(z_1,z_2\right)}{\partial\sigma} &= \frac{1}{\sigma}\left[u^2 - 1 + \theta\left(a,u\right)\right]\\ \frac{\partial\log f_{x_1,x_2}\left(z_1,z_2\right)}{\partial\alpha} &= \frac{\log\|x_2-x_1\|}{2}\left[u^2 - 1 + \theta\left(a,u\right)\right],\end{aligned}$$

where

$$|\theta\left(a,u\right)| \le C\max_{k=0,\ldots,k_0}\left(|u| + a\right)^k a,$$

$a = \sigma\|x_2 - x_1\|^{\alpha/2}$, $u = \log(z_2/z_1)/a$, $k_0 \in \mathbf{N}_*$. We also have

$$|\theta\left(a,u\right)| \le C\sum_{k=0}^{k_0}\sum_{j=0}^{k}|u|^j a^{k+1-j}.$$

For triples (x_1, x_2, x_3), we have

$$\begin{aligned}\frac{\partial\log f_{x_1,x_2,x_3}\left(z_1,z_2,z_3\right)}{\partial\sigma} &= \frac{1}{\sigma}\left(\begin{pmatrix} u_2 & u_3\end{pmatrix}\begin{pmatrix}1 & R_1\\ R_1 & 1\end{pmatrix}^{-1}\begin{pmatrix}u_2\\ u_3\end{pmatrix} - 2 + \theta_\alpha\left(\vec{a},u_2,u_3\right)\right)\\ \frac{\partial\log f_{x_1,x_2,x_3}\left(z_1,z_2,z_3\right)}{\partial\alpha} &= \frac{\log\delta}{2}\left(\begin{pmatrix} u_2 & u_3\end{pmatrix}\begin{pmatrix}1 & R_1\\ R_1 & 1\end{pmatrix}^{-1}\begin{pmatrix}u_2\\ u_3\end{pmatrix} - 2 + \theta_\sigma\left(\vec{a},u_2,u_3\right)\right),\end{aligned}$$

where

$$\begin{aligned}|\theta_\alpha\left(\vec{a},u\right)| &\le C\max_{k=0,\ldots,k_0}\left(|u_2|\vee|u_3| + \|\vec{a}\|\right)^k\|\vec{a}\|,\\ |\theta_\sigma\left(\vec{a},u\right)| &\le C\max_{k=0,\ldots,k_0}\left(|u_2|\vee|u_3| + \|\vec{a}\|\right)^k\|\vec{a}\|,\end{aligned}$$

with $k_0 \in \mathbf{N}_*$, $\vec{a} = (a_{1,2}, a_{1,3}, a_{2,3})$, $a_{i,j} = \sigma\delta^{\alpha/2} \|x_j - x_i\|^{\alpha/2}$, $u_2 = \log(z_2/z_1)/a_{12}$ and $u_3 = \log(z_3/z_1)/a_{13}$.

5.6.1 Proofs for $\hat{\sigma}_{2,N}$ and $\hat{\alpha}_{2,N}$

(i). When α is assumed to be known, the maximum (tapered) CL estimator of σ, $\hat{\sigma}_{2,N}$, satisfies

$$\frac{\partial \ell_{2,N}\left(\hat{\sigma}_{2,N}, \alpha\right)}{\partial \sigma} = \sum_{(x_1,x_2)\in E_N} \left.\frac{\partial \log f_{x_1,x_2}\left(\eta\left(x_1\right), \eta\left(x_2\right)\right)}{\partial \sigma}\right|_{\sigma=\hat{\sigma}_{2,N}} = 0.$$

Therefore we have

$$\sum_{(x_1,x_2)\in E_N} \left(\frac{\sigma^2}{(\hat{\sigma}_{2,N})^2}(U^{(\eta)}_{x_1,x_2})^2 - 1\right) + \sum_{(x_1,x_2)\in E_N} \theta\left(\hat{\sigma}_{2,N}\left\|x_2 - x_1\right\|^{\alpha/2}, \frac{\sigma}{\hat{\sigma}_{2,N}} U^{(\eta)}_{x_1,x_2}\right) = 0$$

with

$$U^{(\eta)}_{x_1,x_2} = \frac{1}{\sigma d_{1,2}^{\alpha/2}} \log\left(\frac{\eta(x_2)}{\eta(x_1)}\right).$$

We have equivalently

$$\begin{aligned}
&\left((\hat{\sigma}_{2,N})^2 - \sigma^2\right)\left[\frac{1}{(\hat{\sigma}_{2,N})^2}\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}(U^{(\eta)}_{x_1,x_2})^2\right] \\
&\quad = \frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\left((U^{(\eta)}_{x_1,x_2})^2 - 1\right) + \frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\theta\left(\hat{\sigma}_{2,N}\left\|x_2 - x_1\right\|^{\alpha/2}, \frac{\sigma}{\hat{\sigma}_{2,N}}U^{(\eta)}_{x_1,x_2}\right).
\end{aligned}$$

Note that

$$\left|\theta\left(\hat{\sigma}_{2,N}\left\|x_2 - x_1\right\|^{\alpha/2}, \frac{\sigma}{\hat{\sigma}_{2,N}}U^{(\eta)}_{x_1,x_2}\right)\right| \le C\sum_{k=0}^{k_0}\sum_{j=0}^{k}|U^{(\eta)}_{x_1,x_2}|^j\left\|x_2 - x_1\right\|^{(k+1-j)\alpha/2}$$

and

$$\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}|U^{(\eta)}_{x_1,x_2}|^j\left\|x_2 - x_1\right\|^{(k+1-j)\alpha/2}\right] \le C\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\left\|x_2 - x_1\right\|^{(k+1-j)\alpha/2}\right]$$

according to Proposition 8. Moreover, it follows from Lemma 7, that as $N \to \infty$

$$\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\left\|x_2 - x_1\right\|^{(k+1-j)\alpha/2}\right] \sim N^{1-(k+1-j)\alpha/2}.$$

We can deduce that

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\left|\theta\left(\hat{\sigma}_{2,N}\left\|x_2 - x_1\right\|^{\alpha/2}, \frac{\sigma}{\hat{\sigma}_{2,N}}U^{(\eta)}_{x_1,x_2}\right)\right| \xrightarrow{\mathbb{P}} 0.$$

From Theorem 3, we have

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}(U^{(\eta)}_{x_1,x_2})^2 \xrightarrow{\mathbb{P}} 1.$$

Since σ^2 belongs to a compact set, this shows that $(\hat{\sigma}_{2,N})^2 \xrightarrow{\mathbb{P}} \sigma^2$. By Theorem 3, we have

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V_{2,N}^{(\eta)} \xrightarrow{\mathbb{P}} c_{V_2} \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0).$$

Moreover, by using Proposition 8, we can also prove that

$$N^{-(2-\alpha)/4} \frac{1}{\sqrt{|E_N|}} \sum_{(x_1,x_2)\in E_N} \left| \theta \left(\hat{\sigma}_{2,N} \|x_2 - x_1\|^{\alpha/2}, \frac{\sigma}{\hat{\sigma}_{2,N}} U_{x_1,x_2}^{(\eta)} \right) \right| \xrightarrow{\mathbb{P}} 0.$$

Since

$$\begin{aligned}
&\frac{\sqrt{3}}{3} \sqrt{|E_N|} N^{-(2-\alpha)/4} \left((\hat{\sigma}_{2,N})^2 - \sigma^2 \right) \\
&\qquad = \left[\frac{1}{(\hat{\sigma}_{2,N})^2} \frac{1}{|E_N|} \sum_{(x_1,x_2)\in E_N} (U_{x_1,x_2}^{(\eta)})^2 \right]^{-1} \frac{\sqrt{3}}{3} \sqrt{|E_N|} N^{-(2-\alpha)/4} \\
&\qquad\qquad \times \Bigg[\frac{1}{|E_N|} \sum_{(x_1,x_2)\in E_N} \left((U_{x_1,x_2}^{(\eta)})^2 - 1 \right) \\
&\qquad\qquad\qquad + \frac{1}{|E_N|} \sum_{(x_1,x_2)\in E_N} \left| \theta \left(\hat{\sigma}_{2,N} \|x_2 - x_1\|^{\alpha/2}, \frac{\sigma}{\hat{\sigma}_{2,N}} U_{x_1,x_2}^{(\eta)} \right) \right| \Bigg],
\end{aligned}$$

it follows that

$$\frac{\sqrt{3}}{3} \sqrt{|E_N|} N^{-(2-\alpha)/4} \left((\hat{\sigma}_{2,N})^2 - \sigma^2 \right) \xrightarrow{\mathbb{P}} \sigma^2 c_{V_2} \sum_{j\geq 1} \sum_{k>j} L_{Z_{k\setminus j}}(0).$$

(ii). When σ is assumed to be known, the maximum (tapered) CL estimators of α, $\hat{\alpha}_{2,N}$, satisfies

$$\frac{\partial \ell_{2,N}(\sigma, \hat{\alpha}_{2,N})}{\partial \alpha} = \sum_{(x_1,x_2)\in E_N} \left. \frac{\partial \log f_{x_1,x_2}(\eta(x_1), \eta(x_2))}{\partial \alpha} \right|_{\alpha = \hat{\alpha}_{2,N}} = 0.$$

Therefore we have

$$\begin{aligned}
&\sum_{(x_1,x_2)\in E_N} \log d_{1,2} \left(e^{-(\hat{\alpha}_{2,N} - \alpha)\log d_{1,2}} (U_{x_1,x_2}^{(\eta)})^2 - 1 \right) \\
&\qquad\qquad + \sum_{(x_1,x_2)\in E_N} \log d_{1,2} \theta \left(\sigma \|x_2 - x_1\|^{\alpha/2}, e^{-(\hat{\alpha}_{2,N} - \alpha)\log d_{1,2}/2} U_{x_1,x_2}^{(\eta)} \right) = 0.
\end{aligned}$$

Given a realization of P_1, let x_1 be a point chosen with respect to the uniform distribution in $P_1 \cap \mathbf{C}_N$ and let x_2 be a neighbor of x_1 in the Delaunay triangulation which is chosen randomly, with the property that $x_1 \preceq x_2$. Then $d_{1,2}^{(N)} = N^{1/2} d_{1,2}$ has the same distribution as the typical Delaunay edge. Since $d_{1,2}^{(N)} = O_{\mathbb{P}}(1)$ and $\max_{(x_1,x_2)\in E_N} d_{1,2}^{(N)} = O_{\mathbb{P}}(\log N)$ (see e.g. [23] or [7]), we deduce that

$$\log d_{1,2} = -\frac{1}{2} \log N + \log d_{1,2}^{(N)} = -\frac{1}{2} \log N + O_{\mathbb{P}}(\log \log N).$$

Hence, we have

$$
\begin{aligned}
&\sum_{(x_1,x_2)\in E_N} \log d_{1,2} \left(e^{-(\hat{\alpha}_{2,N}-\alpha)\log d_{1,2}}(U^{(\eta)}_{x_1,x_2})^2 - 1\right) \\
&\qquad\qquad = -\frac{1}{2}\log N(1+o_{\mathbb{P}}(1)) \sum_{(x_1,x_2)\in E_N} \left(e^{(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))/2}(U^{(\eta)}_{x_1,x_2})^2 - 1\right) \\
&\qquad\qquad = -\frac{1}{2}\log N|E_N|(1+o_{\mathbb{P}}(1)) \left(e^{(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))/2}\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}(U^{(\eta)}_{x_1,x_2})^2 - 1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{(x_1,x_2)\in E_N} \log d_{1,2}\theta\left(\sigma\left\|x_2-x_1\right\|^{\alpha/2}, e^{-(\hat{\alpha}_{2,N}-\alpha)\log d_{1,2}/2}U^{(\eta)}_{x_1,x_2}\right) \\
&\qquad\qquad = -\frac{1}{2}\log N(1+o_{\mathbb{P}}(1)) \sum_{(x_1,x_2)\in E_N} \theta\left(\sigma\left\|x_2-x_1\right\|^{\alpha/2}, e^{(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))/4}U^{(\eta)}_{x_1,x_2}\right).
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&e^{(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))/2}\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}(U^{(\eta)}_{x_1,x_2})^2 - 1 \\
&\qquad = (1+o_{\mathbb{P}}(1))\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\theta\left(\sigma\left\|x_2-x_1\right\|^{\alpha/2}, e^{(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))/4}U^{(\eta)}_{x_1,x_2}\right). \qquad (5.5)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|\theta\left(\sigma\left\|x_2-x_1\right\|^{\alpha/2}, e^{(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))/4}U^{(\eta)}_{x_1,x_2}\right)\right| \\
&\qquad\qquad \le Ce^{k_0|\hat{\alpha}_{2,N}-\alpha|\log N(1+o_{\mathbb{P}}(1))/4}\sum_{k=0}^{k_0}\sum_{j=0}^{k}|U^{(\eta)}_{x_1,x_2}|^j\left\|x_2-x_1\right\|^{(k+1-j)\alpha/2},
\end{aligned}
$$

and (according to Proposition 8)

$$
\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\sum_{k=0}^{k_0}\sum_{j=0}^{k}|U^{(\eta)}_{x_1,x_2}|^j\left\|x_2-x_1\right\|^{(k+1-j)\alpha/2} \xrightarrow{\mathbb{P}} 0,
$$

and

$$
\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}(U^{(\eta)}_{x_1,x_2})^2 \xrightarrow{\mathbb{P}} 1,
$$

we deduce that

$$
|\hat{\alpha}_{2,N}-\alpha|\log N \xrightarrow{\mathbb{P}} 0.
$$

From Eq. (5.5), we have

$$
\begin{aligned}
&\left(1+\frac{1}{2}(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))\right)\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}(U^{(\eta)}_{x_1,x_2})^2 - 1 \\
&\qquad\qquad = (1+o_{\mathbb{P}}(1))\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\theta\left(\sigma\left\|x_2-x_1\right\|^{\alpha/2}, U^{(\eta)}_{x_1,x_2}\right)
\end{aligned}
$$

and

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\theta\left(\sigma\left\|x_2-x_1\right\|^{\alpha/2},U_{x_1,x_2}^{(\eta)}\right)=O_{\mathbb{P}}\left(N^{-\alpha/2}\right).$$

It follows that

$$\frac{1}{2}(\hat{\alpha}_{2,N}-\alpha)\log N(1+o_{\mathbb{P}}(1))=-\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\left((U_{x_1,x_2}^{(\eta)})^2-1\right)+O_{\mathbb{P}}\left(N^{-\alpha/2}\right).$$

In the same spirit as we did for $(\hat{\sigma}_{2,N})^2$, we conclude that

$$\frac{\sqrt{3}}{6}\sqrt{|E_N|}N^{-(2-\alpha)/4}\log(N)\left(\hat{\alpha}_{2,N}-\alpha\right)\xrightarrow{\mathbb{P}}-c_{V_2}\sum_{j\geq 1}\sum_{k>j}L_{Z_{k\setminus j}}(0).$$

5.6.2 Proof for $\hat{\sigma}_{3,N}$ and $\hat{\alpha}_{3,N}$

We use the same type of reasoning as in the proofs for $\hat{\sigma}_{2,N}$ and $\hat{\alpha}_{2,N}$.

6 Technical lemmas

6.1 Some additional bounds

Lemma 7 *(i) There exists a positive constant C such that, for any $N\geq 0$,*

$$\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\left\|x_2-x_1\right\|^{\alpha}\right]\leq CN^{1-\alpha/2}.$$

(ii) As $N\to\infty$, we have

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\frac{\left(\gamma\left(x_2\right)-\gamma\left(x_1\right)\right)^2}{\left\|x_2-x_1\right\|^{\alpha}}=O_{\mathbb{P}}\left(N^{-1+\alpha/2}\right).$$

Proof of Lemma 7. *(i).* For any distinct points $x_1,x_2,x_3,x_4\in\mathbf{R}^2$ such that $x_1\preceq x_2$ and $x_3\preceq x_4$, and any $N>0$, let

$$p_{2,N}(x_1,x_2,x_3,x_4)=\mathbb{P}\left[x_1\sim x_2,\ x_3\sim x_4\text{ in }\mathrm{Del}(P_N\cup\{x_1,x_2,x_3,x_4\})\text{ and }x_1\preceq x_2,\ x_3\preceq x_4\right].$$

It follows from the Slivnyak-Mecke formula that

$$\begin{aligned}\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N}\left\|x_2-x_1\right\|^{\alpha}\right]&=N^2\int_{\mathbf{C}_1\times\mathbf{R}^2}\left\|x_2-x_1\right\|^{\alpha}p_{2,N}\left(x_1,x_2\right)\mathrm{d}x_1\mathrm{d}x_2\\&=N^{-\alpha/2}\int_{\mathbf{C}_N\times\mathbf{R}^2}\left\|x_2-x_1\right\|^{\alpha}p_{2,1}\left(x_1,x_2\right)\mathrm{d}x_1\mathrm{d}x_2\\&=N^{1-\alpha/2}\int_{\mathbf{R}^2}\left\|x_2\right\|^{\alpha}p_{2,1}\left(0,x_2\right)\mathrm{d}x_2.\end{aligned}$$

According to Lemma 7 in [8], we know that for any $x_2\in\mathbf{R}^2$,

$$p_{2,1}\left(0,x_2\right)\leq C||x_2||^2e^{-\frac{\pi}{4}||x_2||^2}.$$

Therefore

$$\mathbb{E}\left[\sum_{(x_1,x_2)\in E_N} \|x_2 - x_1\|^{\alpha}\right] \le CN^{1-\alpha/2}.$$

(ii). Let

$$\Gamma_N = \frac{1}{N}\sum_{(x_1,x_2)\in E_N} \frac{(\gamma(x_2)-\gamma(x_1))^2}{\|x_2-x_1\|^{\alpha}}.$$

We have

$$\begin{aligned}\mathbb{E}[\Gamma_N] &= CN^{-\alpha/2-1}\int_{\mathbf{C}_N\times\mathbf{R}^2} \frac{(\|x_2\|^{\alpha}-\|x_1\|^{\alpha})^2}{\|x_2-x_1\|^{\alpha}} p_{2,1}(x_1,x_2)\,\mathrm{d}x_1\mathrm{d}x_2\\ &= CN^{-\alpha/2-1}\int_{\mathbf{C}_N}\left(\int_{\mathbf{R}^2}\frac{(\|y_1+y_2\|^{\alpha}-\|y_1\|^{\alpha})^2}{\|y_2\|^{\alpha}} p_{2,1}(y_1,y_1+y_2)\,\mathrm{d}y_2\right)\mathrm{d}y_1.\end{aligned}$$

Note that

- according to Lemma 7 in [8], for any $y_1, y_2 \in \mathbf{R}^2$

$$p_{2,1}(y_1,y_1+y_2) \le C||y_2||^2 e^{-\frac{\pi}{4}||y_2||^2},$$

- when y_1 is fixed,

$$\frac{(\|y_1+y_2\|^{\alpha}-\|y_1\|^{\alpha})^2}{\|y_2\|^{\alpha}} \underset{\|y_2\|\to 0}{\sim} C\|y_2\|^{2-\alpha},$$

  which is locally integrable in 0.

Thus

$$\int_{\mathbf{R}^2}\frac{(\|y_1+y_2\|^{\alpha}-\|y_1\|^{\alpha})^2}{\|y_2\|^{\alpha}} p_{2,1}(y_1,y_1+y_2)\,\mathrm{d}y_2$$

is well defined for any y_1.

Moreover, for $\|y_1\|$ large enough, we have

$$\frac{(\|y_1+y_2\|^{\alpha}-\|y_1\|^{\alpha})^2}{\|y_2\|^{\alpha}} \le C\|y_1\|^{2\alpha-2}$$

and, for any $\varepsilon > 0$,

$$\int_{\mathbf{C}_N\setminus B(0,\varepsilon)} \|y_1\|^{2\alpha-2}\,\mathrm{d}y_1 = O(N^{\alpha}).$$

We can easily show that $\mathbb{E}[\Gamma_N] = O\left(N^{-1+\alpha/2}\right)$ and consequently, thanks to Markov's inequality, that $\Gamma_N = O_{\mathbb{P}}\left(N^{-1+\alpha/2}\right)$. Because $|E_N|/N \xrightarrow{\mathbb{P}} 1$, we deduce that

$$\frac{1}{|E_N|}\sum_{(x_1,x_2)\in E_N}\frac{(\gamma(x_2)-\gamma(x_1))^2}{\|x_2-x_1\|^{\alpha}} = O_{\mathbb{P}}\left(N^{-1+\alpha/2}\right).$$

□

6.2 Uniform bounds for the moments of $|U_{x_1,x_2}^{(\eta)}|^p$

Proposition 8 *For any $p \geq 1$, there exists a positive constant $C_p < \infty$ such that*

$$\sup_{x_1,x_2 \in \mathbf{R}^2} \mathbb{E}\left[|U_{x_1,x_2}^{(\eta)}|^p\right] \leq C_p.$$

Proof of Proposition 8. Let $m_p = \mathbb{E}\left[|X|^p\right]$, where X has standard Gaussian distribution. By Proposition 1 and Proposition 3 in [30], there exists $\delta > 0$ such that

$$\sup_{\|x_2 - x_1\| \leq \delta} \mathbb{E}\left[|U_{x_1,x_2}^{(\eta)}|^p\right] \leq 2m_p.$$

Moreover

$$|U_{x_1,x_2}^{(\eta)}|^p \leq 2^{p-1} \frac{|\log \eta(x_1)|^p + |\log \eta(x_2)|^p}{\sigma^p \|x_2 - x_1\|^{p\alpha/2}}.$$

Because $\log \eta(x)$ has a Gumbel distribution for any $x \in \mathbf{R}^2$, we deduce that

$$\sup_{\|x_2 - x_1\| > \delta} \mathbb{E}\left[|U_{x_1,x_2}^{(\eta)}|^p\right] \leq c_p' \delta^{-p\alpha/2}.$$

□

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