Experiments on growth series of braid groups

Jean Fromentin

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Words

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  - the empty word is denoted by ε ;
  - a word on the alphabet S is an S-word ;
  - the set of all S-words is denoted by S^* ;
  - for $u \in S^*$: $|u|$ its length and $\overline{u}$ the element of M it represents ;
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    and so $aba \equiv bab$.

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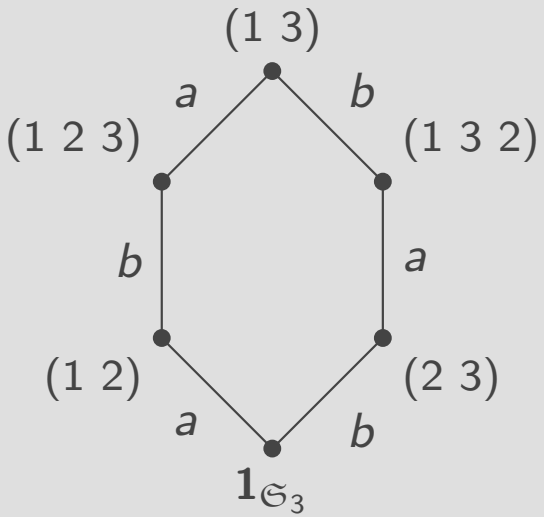

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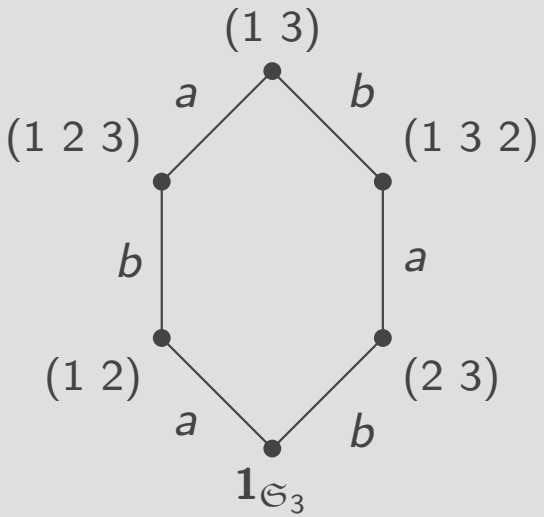

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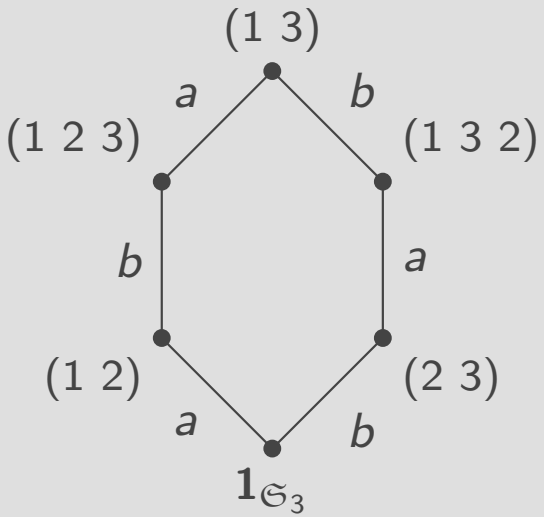

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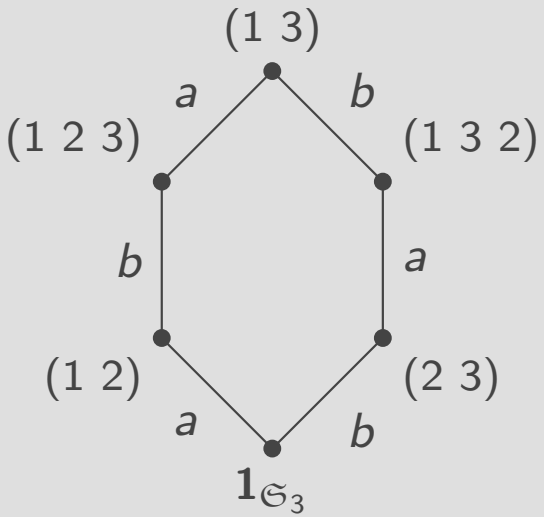

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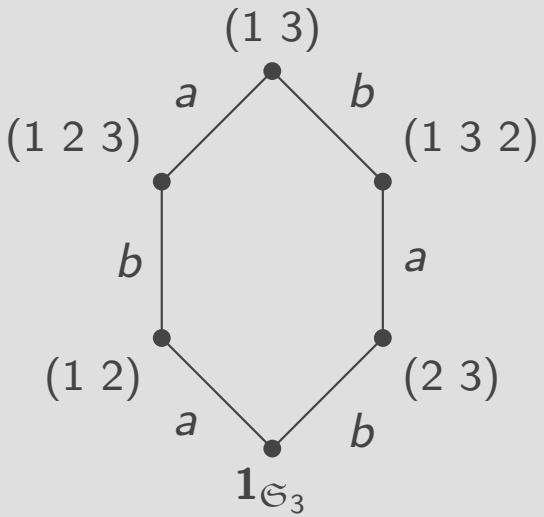

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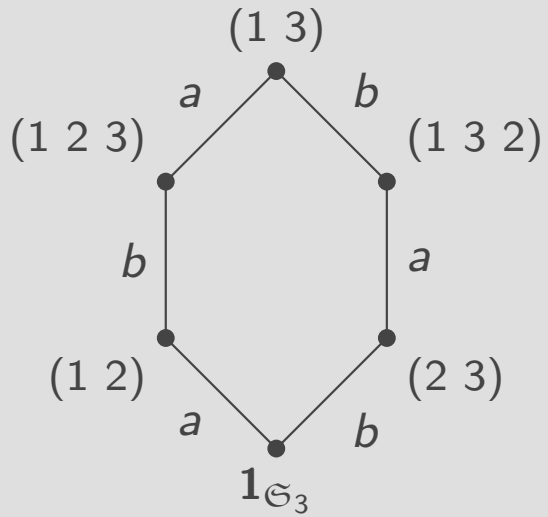

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and so $\mathcal{S}(\mathfrak{S}_3, S) = 1 + 2t + 2t^2 + t^3$.

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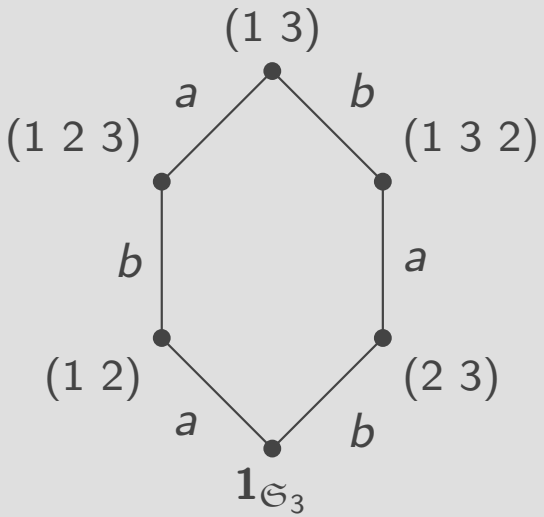

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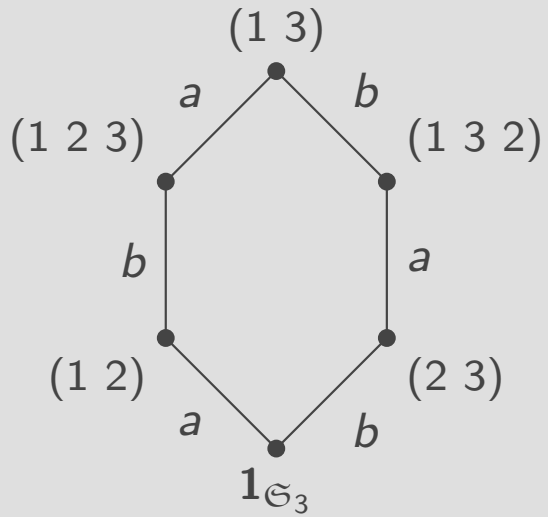

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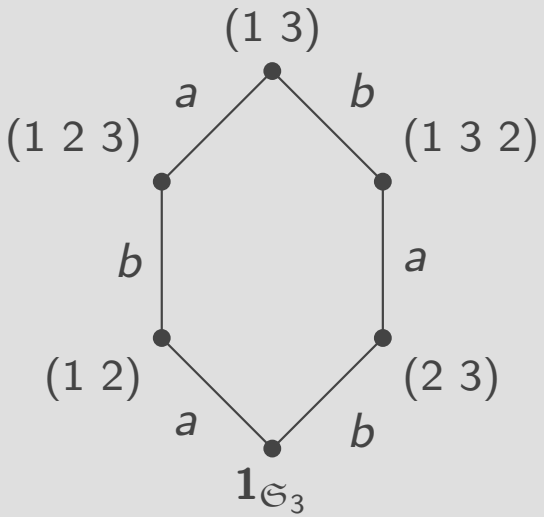

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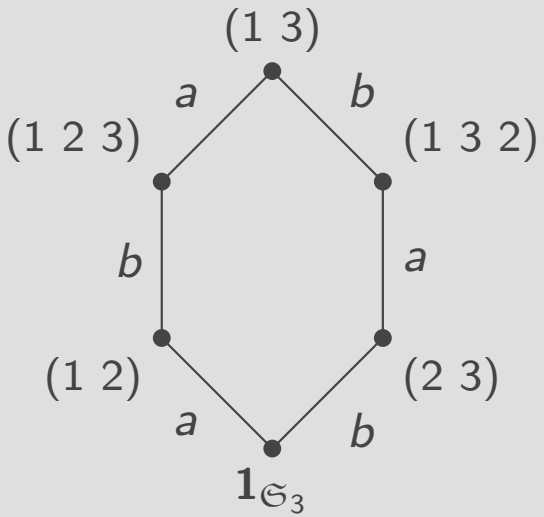

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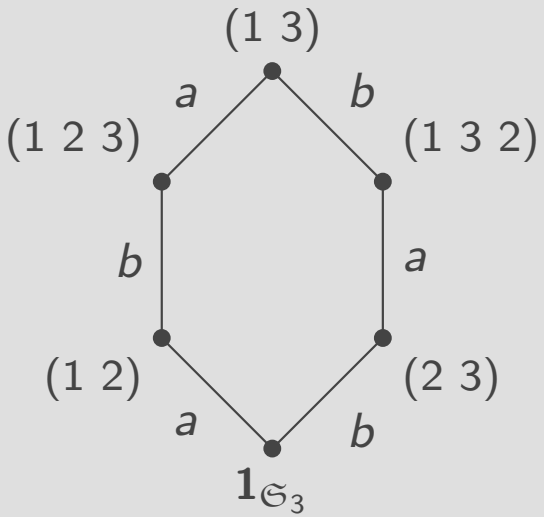

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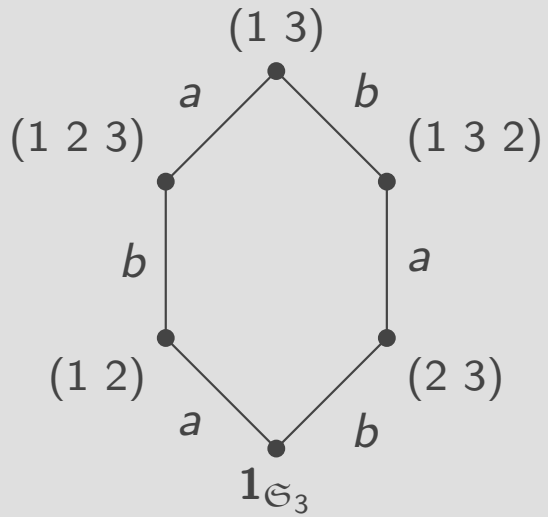

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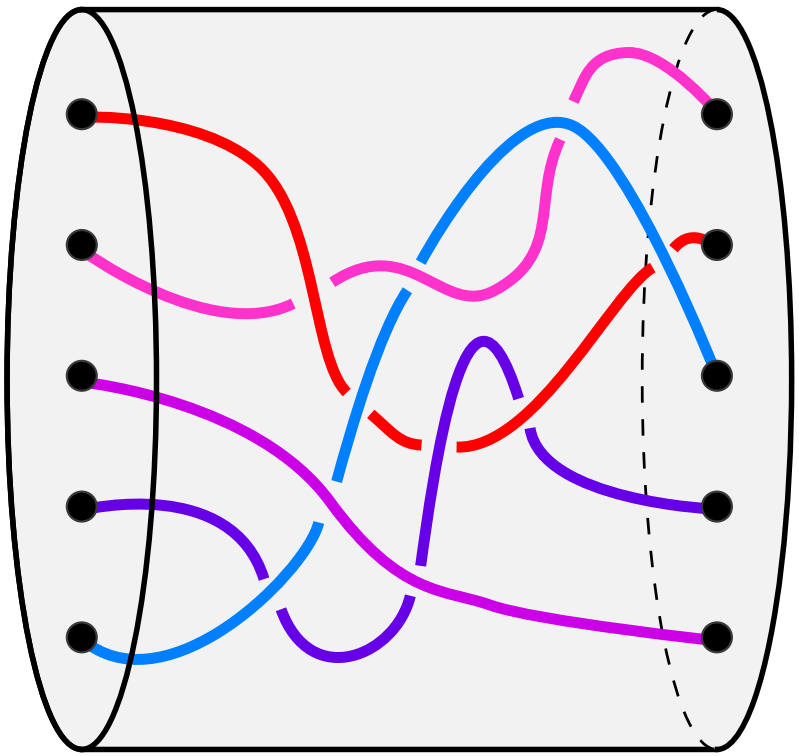

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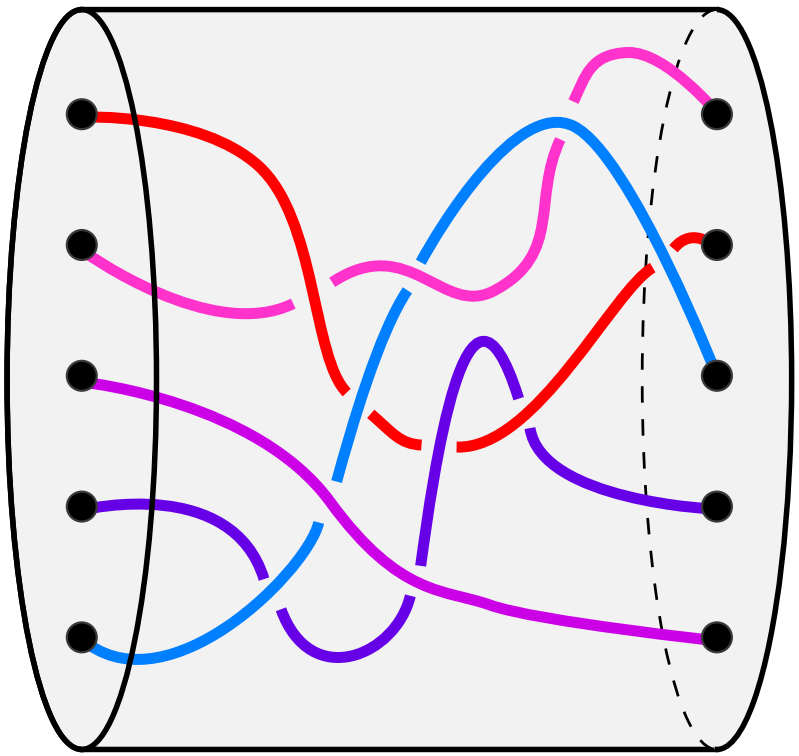

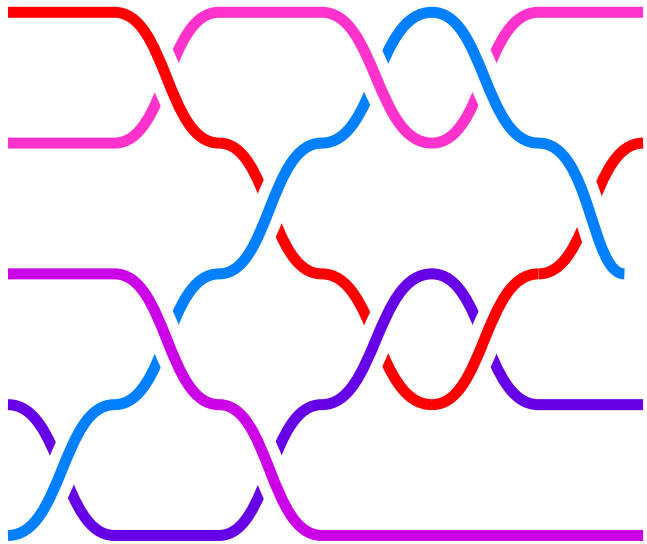

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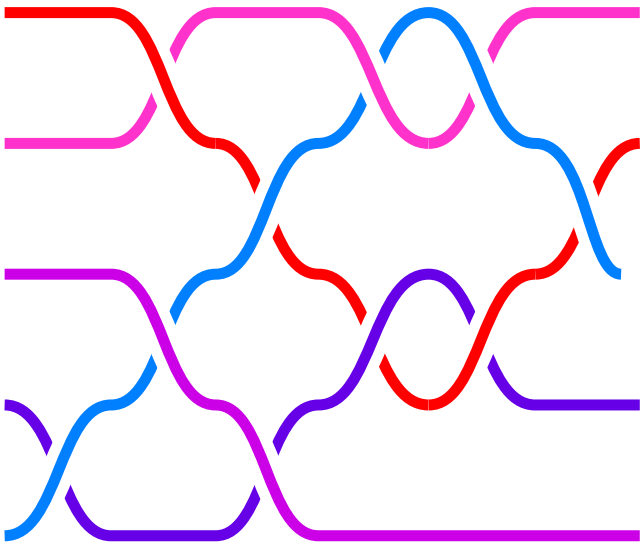

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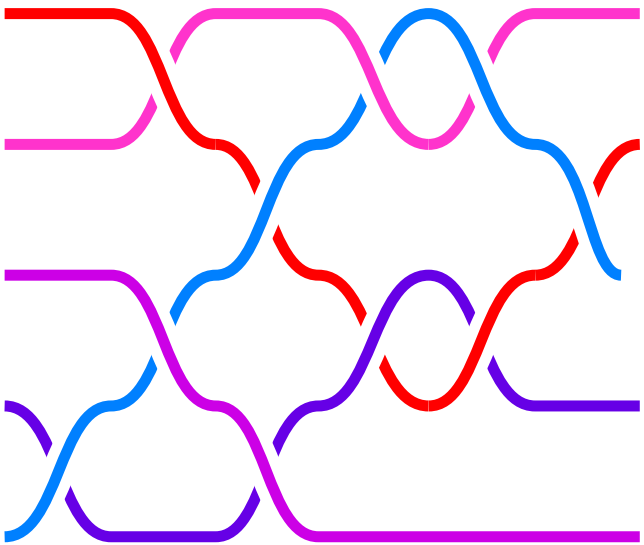

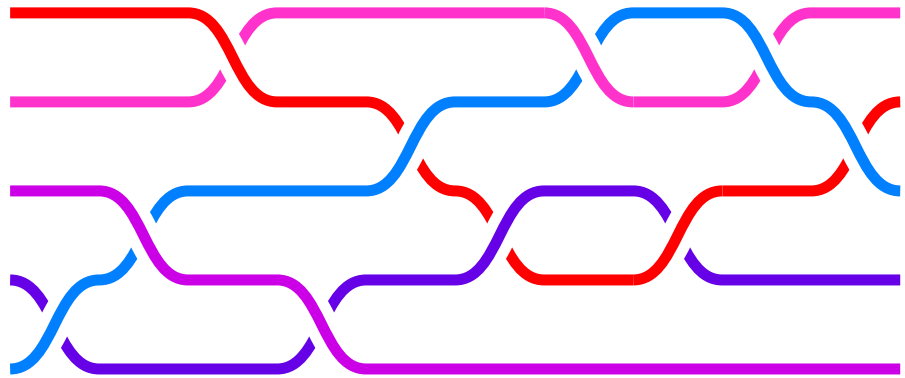

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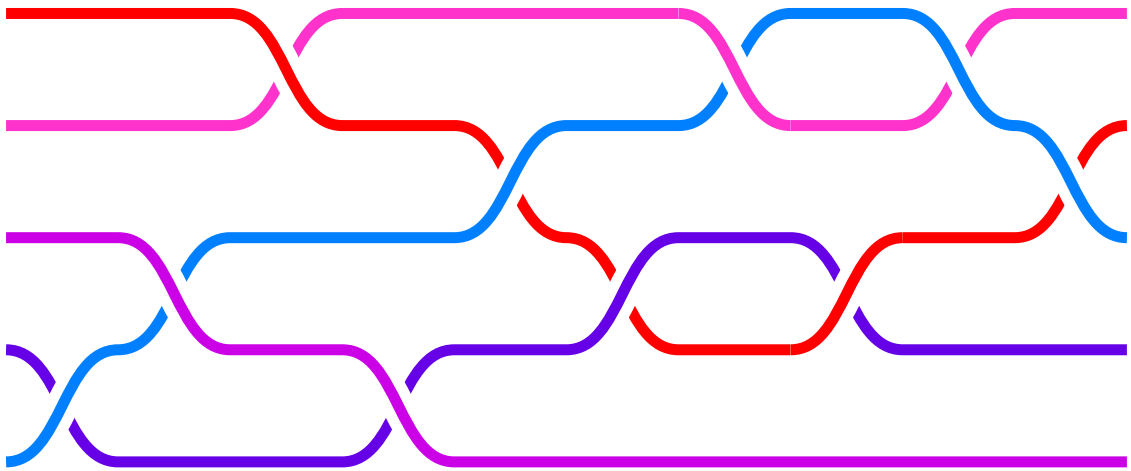

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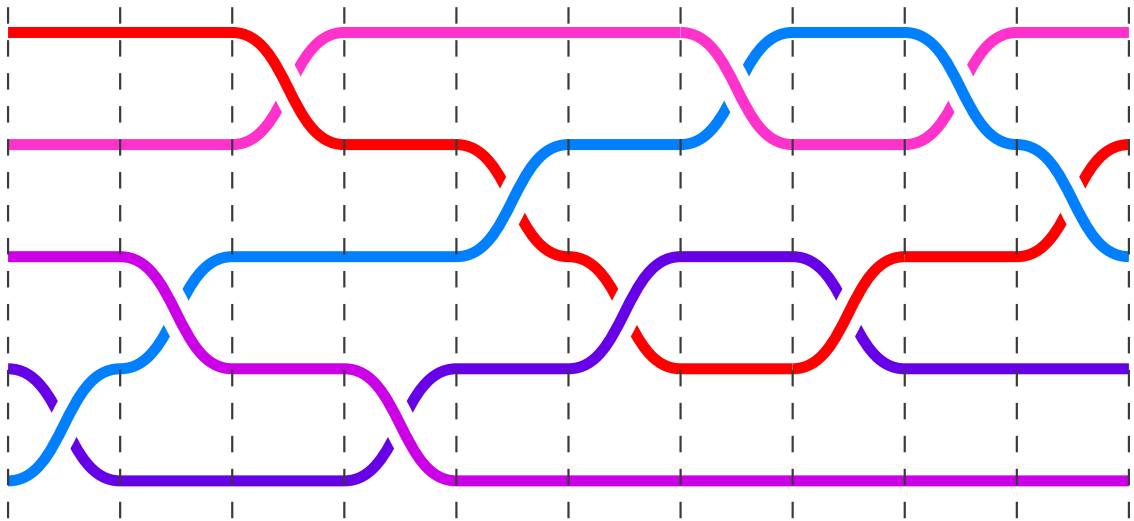

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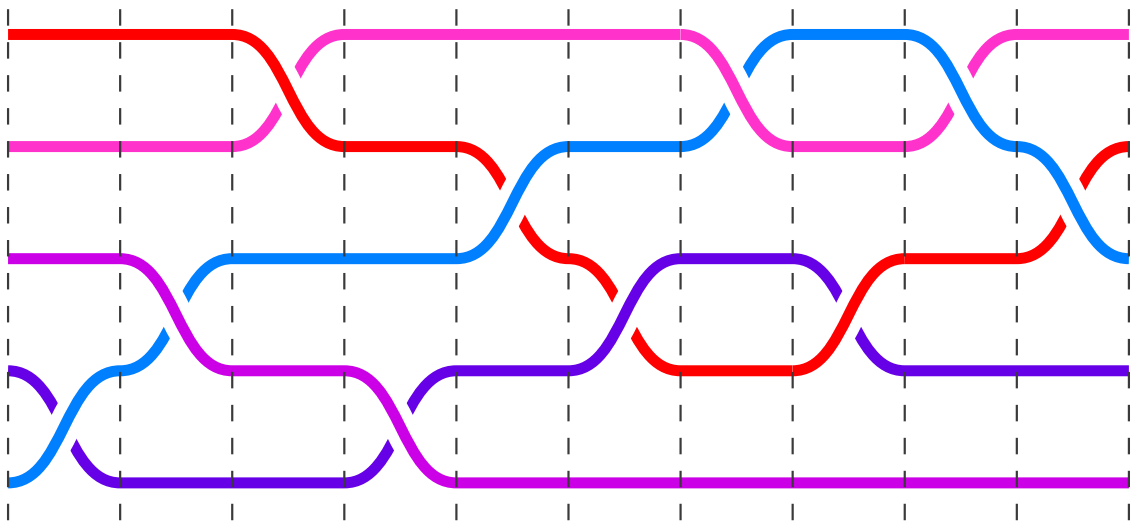

σ_1

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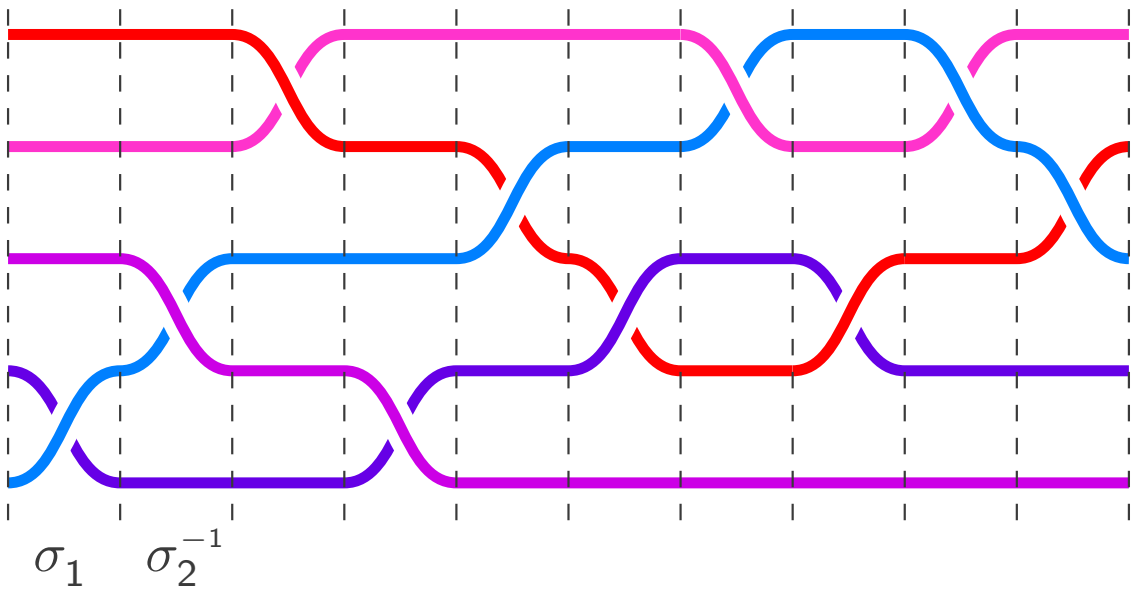

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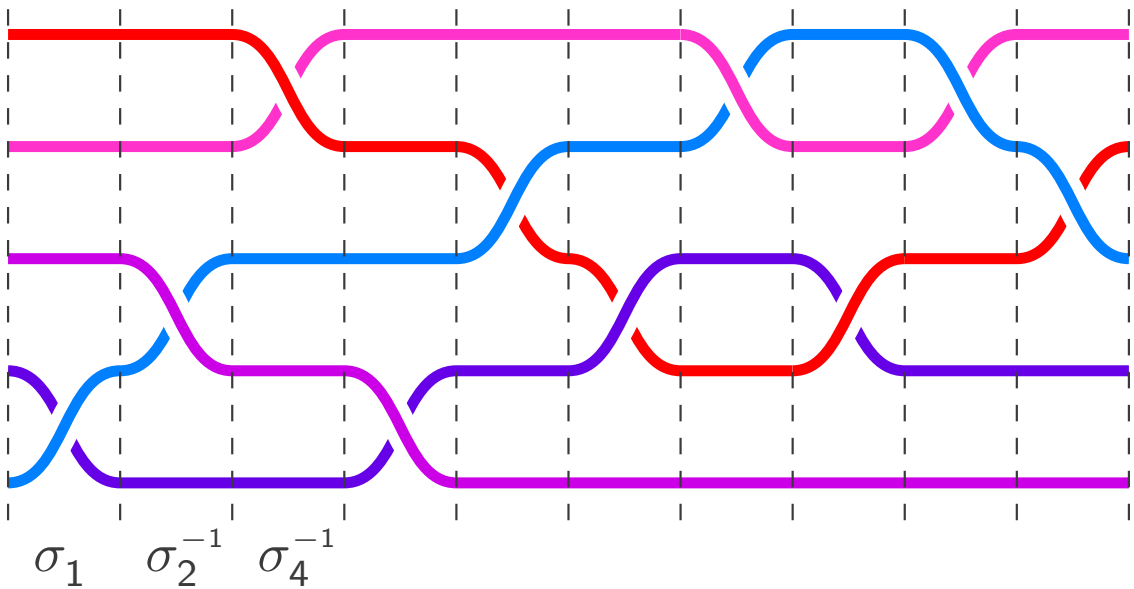

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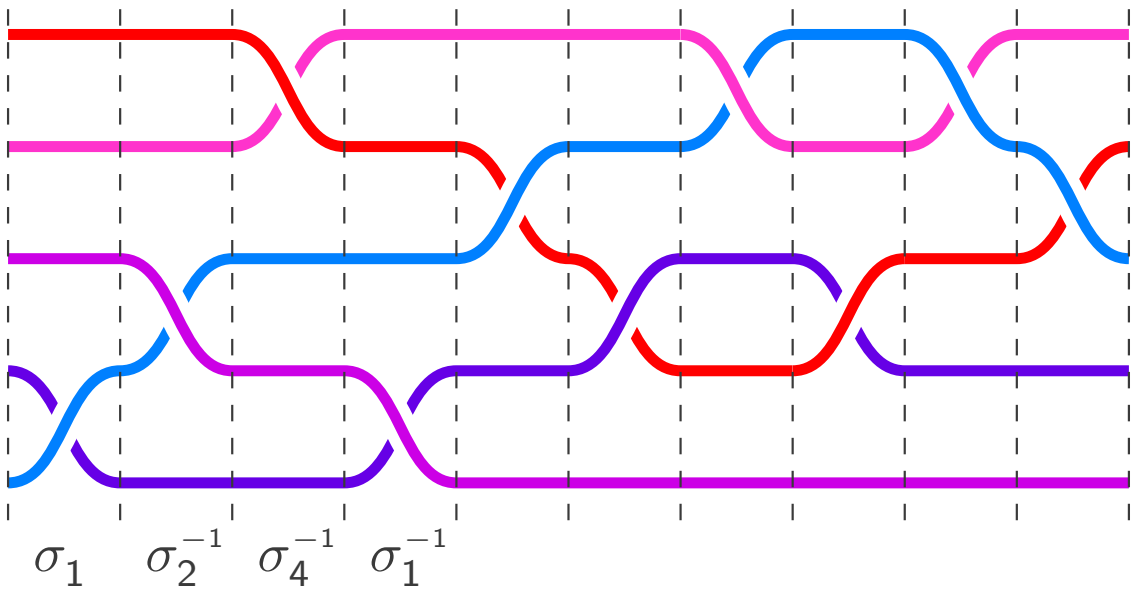

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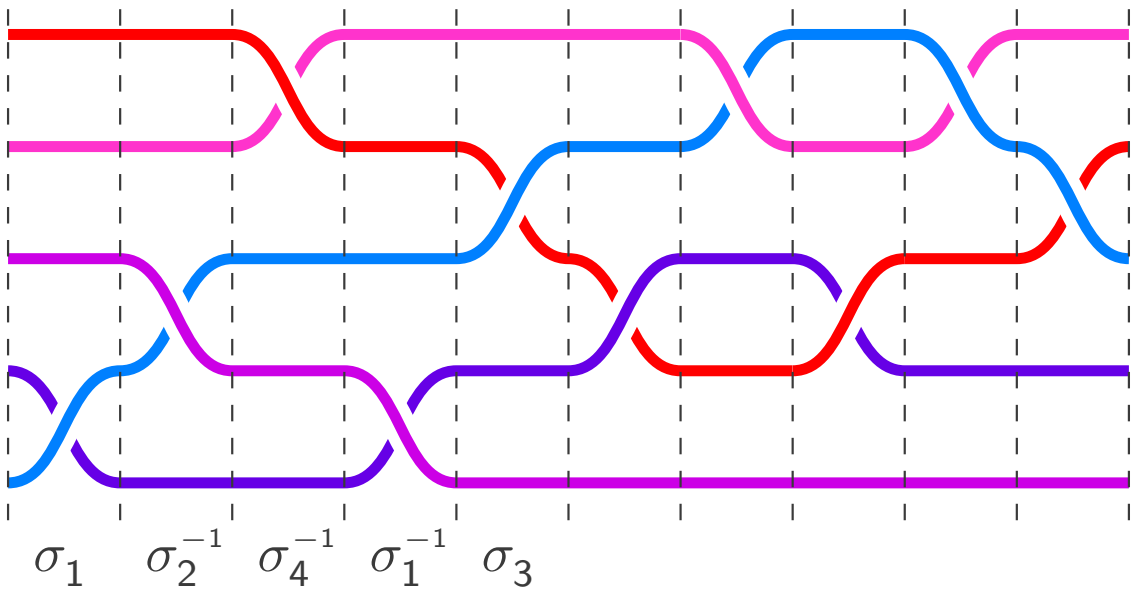

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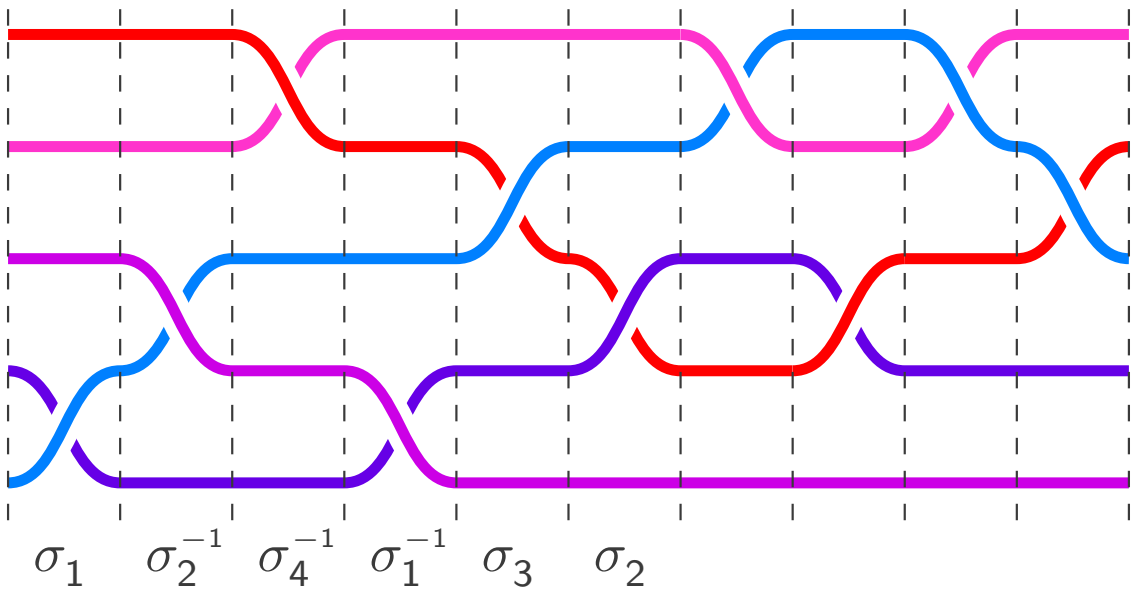

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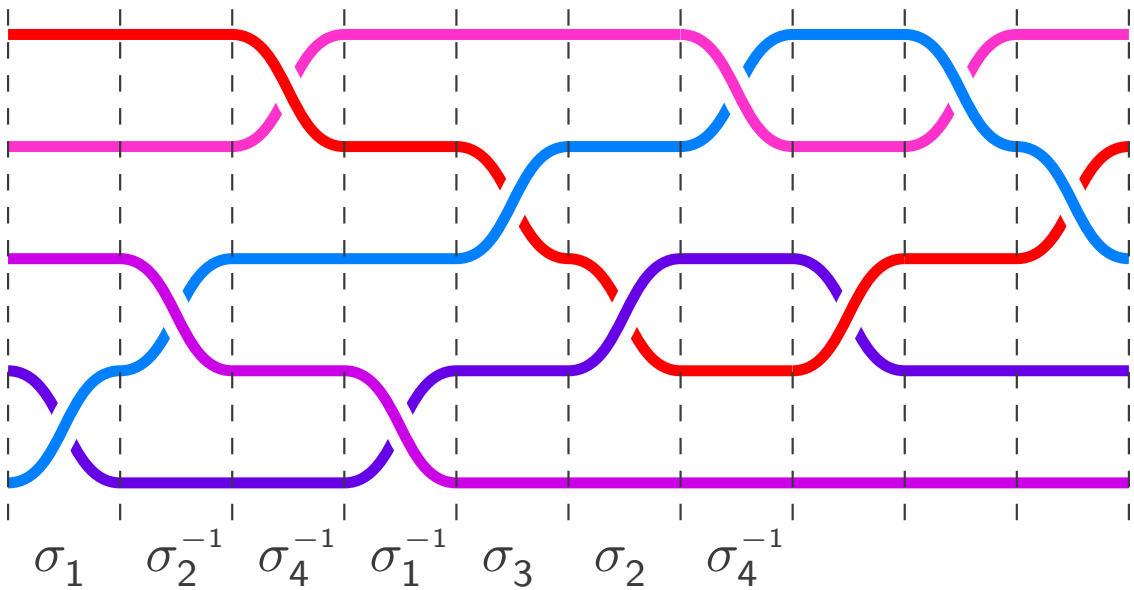

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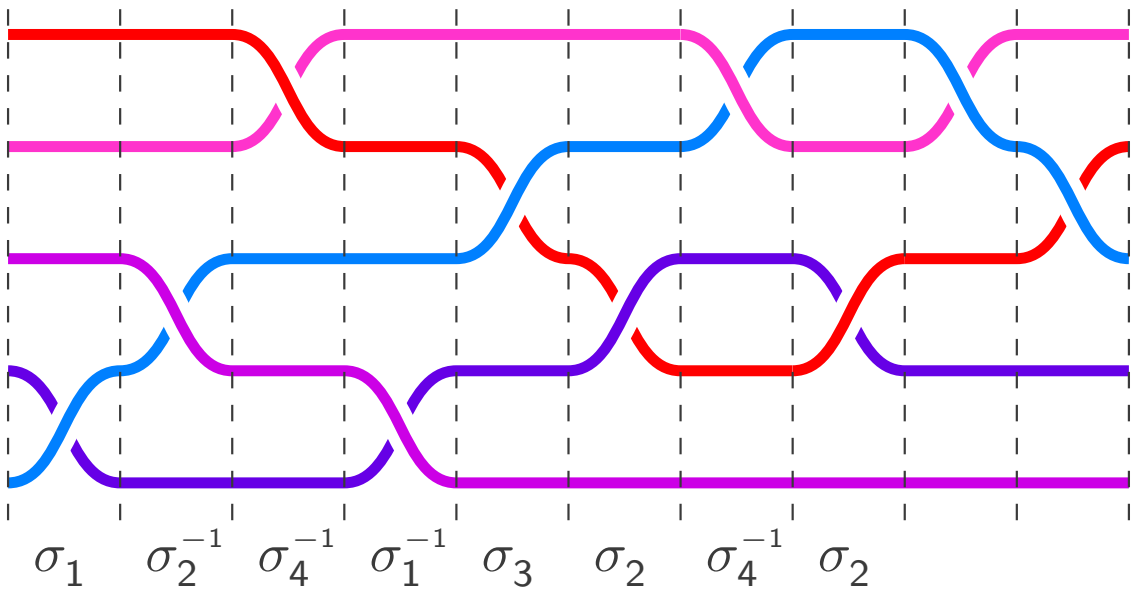
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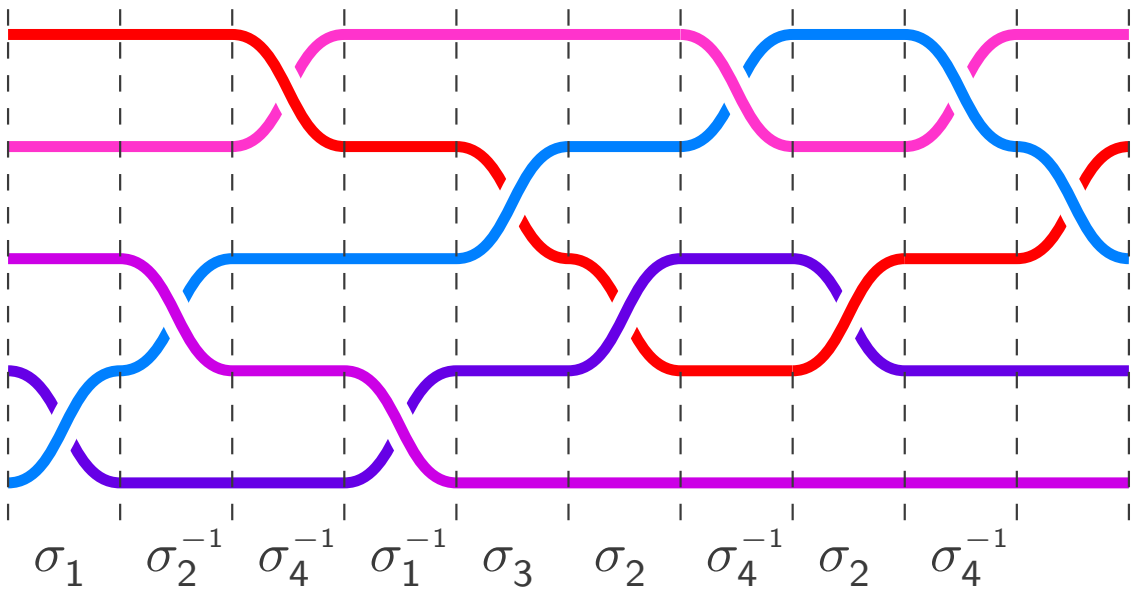
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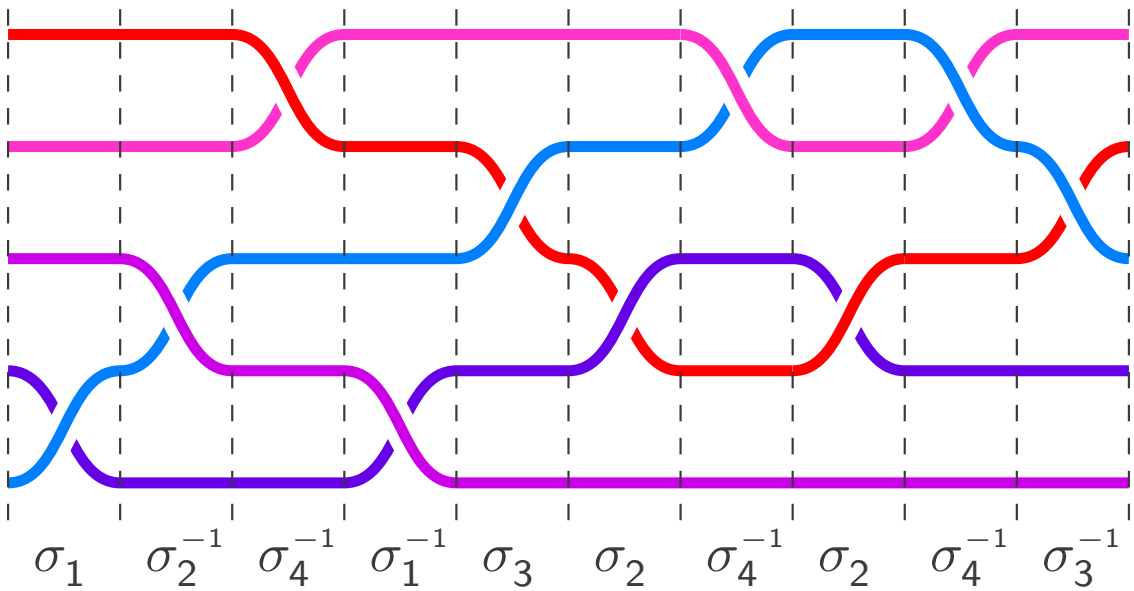
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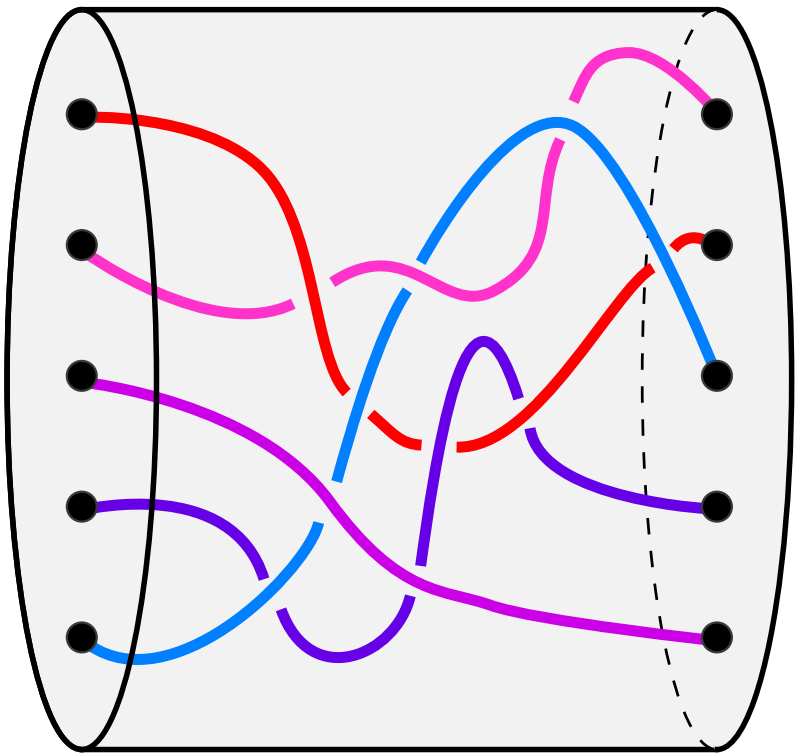

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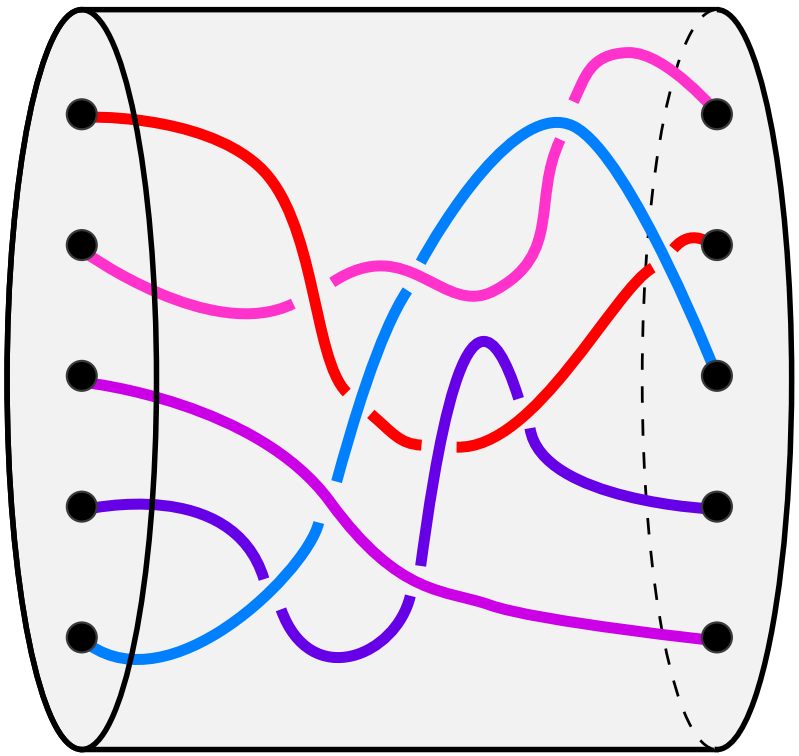

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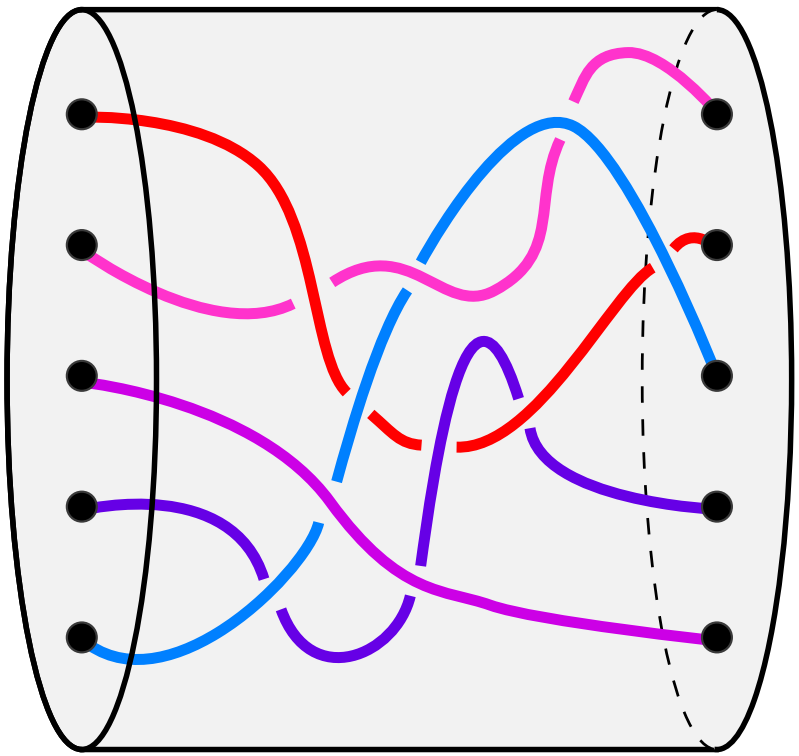

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- Theorem (E. Artin 1925) : The braid group B_n is presented by

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i\sigma_j & \equiv & \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i & \equiv & \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle \tag{1}$$

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- Theorem (L. Sabalka 2004) :

$$\mathcal{G}(B_3, \Sigma_3) = \frac{t^4 + 3t^3 + t + 1}{(t^2 + 2t - 1)(t^2 + t - 1)}.$$

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- Open question : Do $\mathcal{G}(B_n, \Sigma_n)$ and $\mathcal{S}(B_n, \Sigma_n)$ are rationals for $n \geqslant 4$?

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where $P_n(t)$ is given by

$$P_n(t) = \sum_{i=1}^{n} (-1)^{i+1} t^{\frac{i(i-1)}{2}} P_{n-i}(t)$$

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- And for all spherical Artin–Tits semigroups by R. Flores and J. González-Meneses in 2018.

Dual generators

- Definition : For $1 \leqslant p \leqslant q$ we put

$$a_{p,q} = \sigma_p \dots \sigma_{q-2}\ \sigma_{q-1}\ \sigma_{q-2}^{-1} \dots \sigma_p^{-1}.$$

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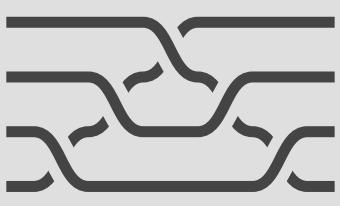
$a_{1,4} =$ 

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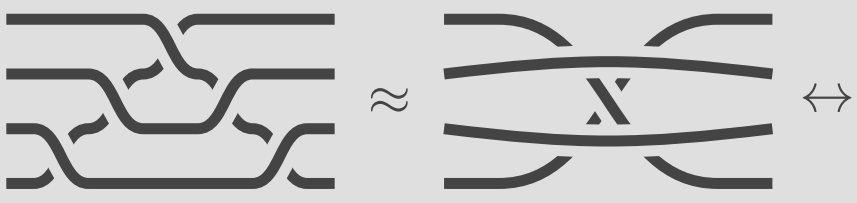
$a_{1,4} =$ 

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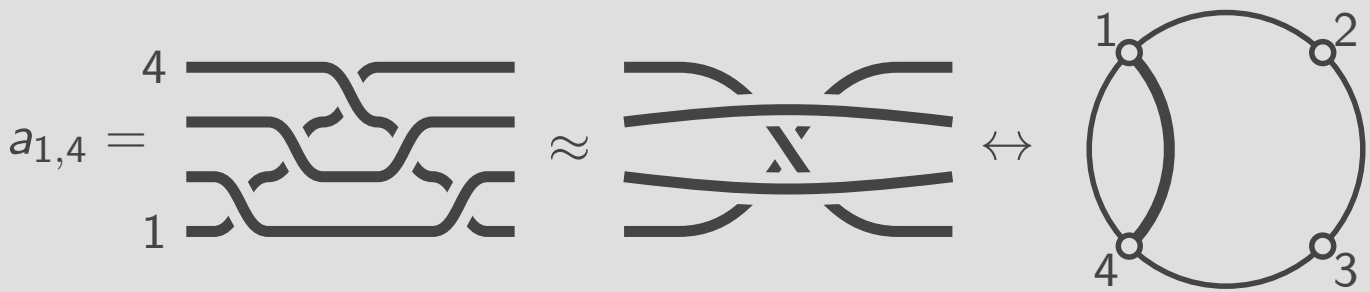

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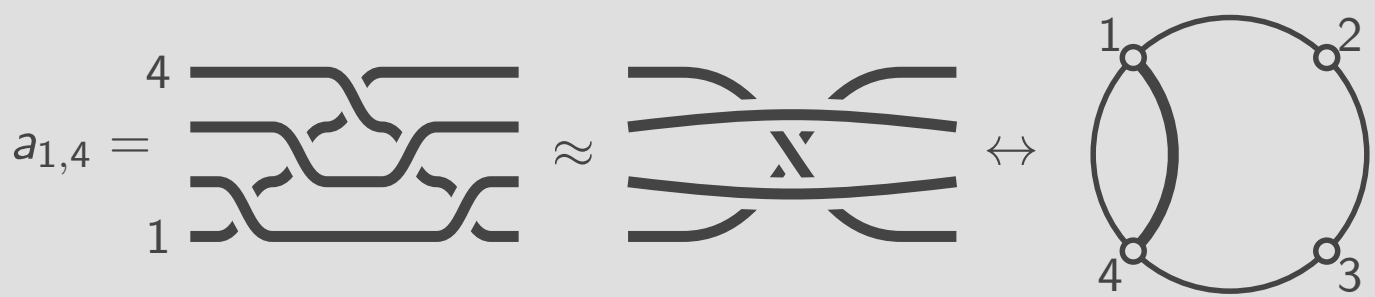
$a_{1,4} =$ 

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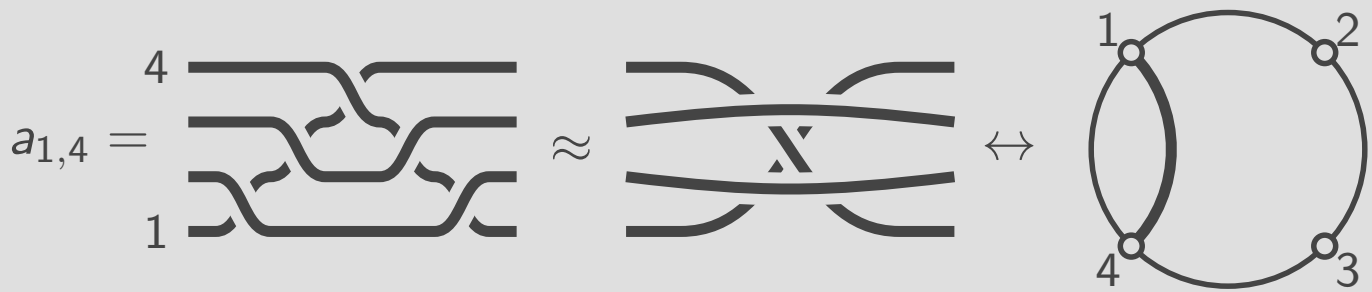

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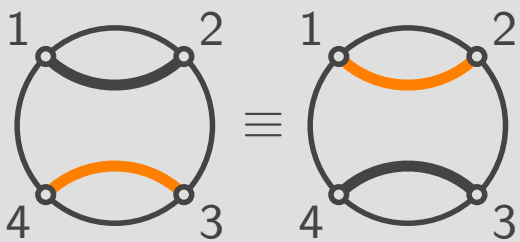

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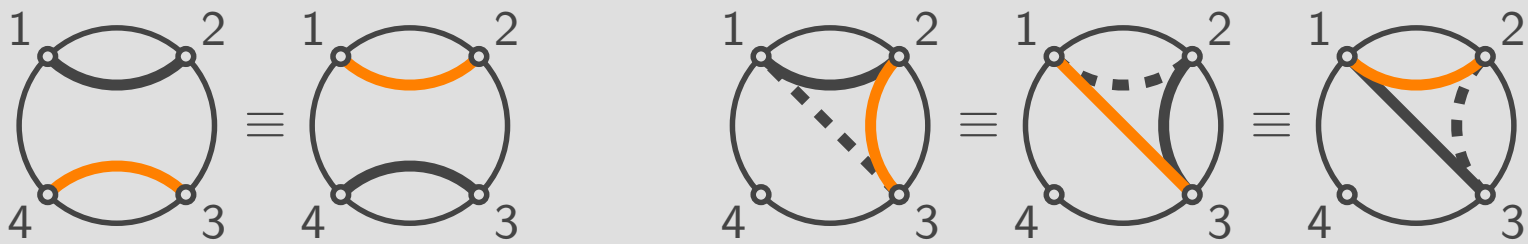

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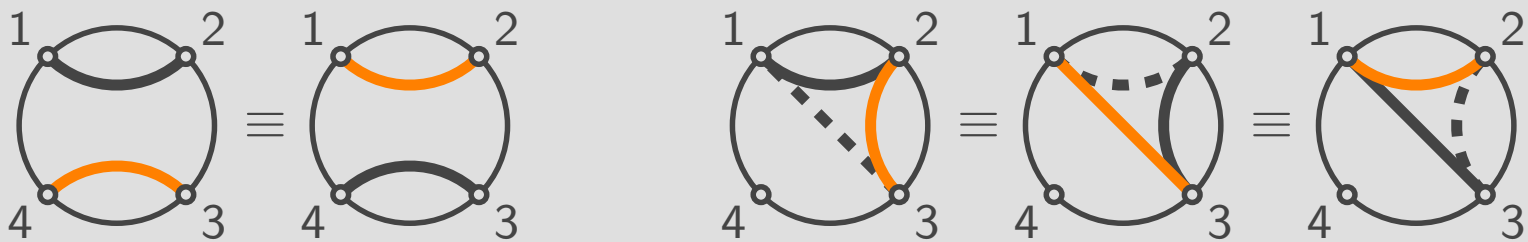

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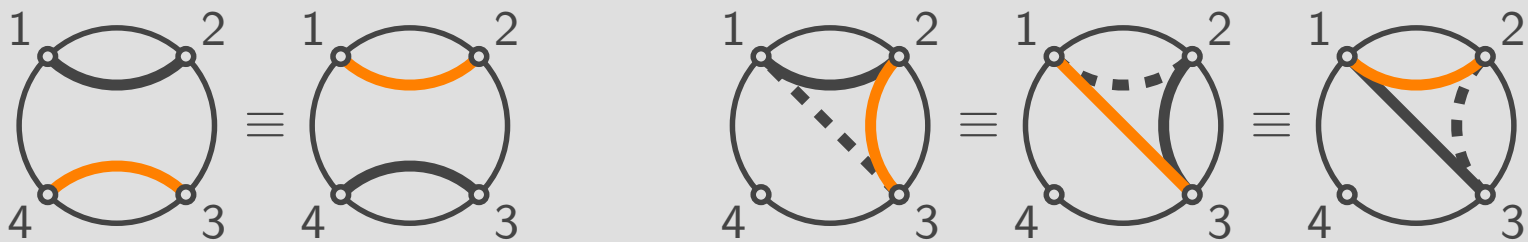

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- Fact : We have $\Sigma_2^* = \Sigma_2^+ = \{\sigma_1^{\pm 1}\}$ and

$$\mathcal{S}(B_2, \{\sigma_1^{\pm 1}\}) = \mathcal{G}(B_2, \{\sigma_1^{\pm 1}\}) = \mathcal{G}(\mathbb{Z}, \{\pm 1\}) = \frac{1+t}{1-t}.$$

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  - No good algorithms.

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Assume $\ell \geqslant 2$.

- Lemma : Let u be a geodesic S_n-word of length $\ell - 1$ and $\alpha \in S_n$.
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A first algorithm

- Algorithm $(W_{\ell-2} \vdash B_n(S_n; \ell-2),\ W_{\ell-1} \vdash B_n(S_n; \ell-1))$:

  for $u \in W_{\ell-1}$ **do**
    for $\alpha \in S_n$ **do**
      $v \leftarrow u\,\alpha$
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        $W_\ell \leftarrow W_\ell \sqcup \{v\}$ *A new braid $\overline{v}$ of $B_n(S_n; \ell)$ is found.*
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- Question : How to test if a S_n-word u appears in a subset W of S_n^* ?

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  - Better but we can do more in this direction.

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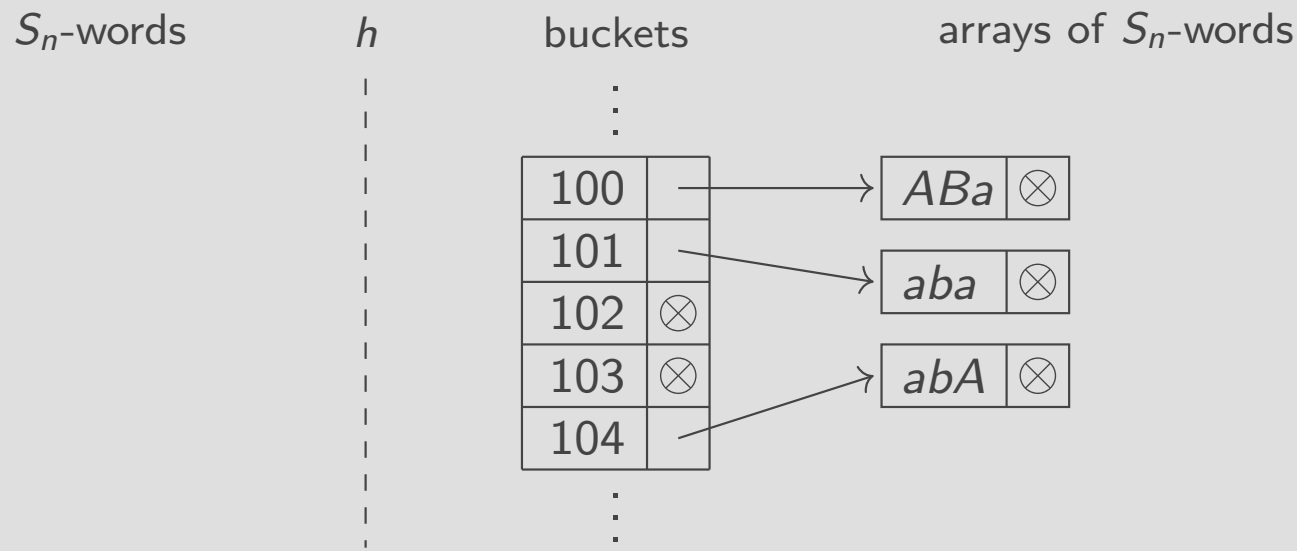

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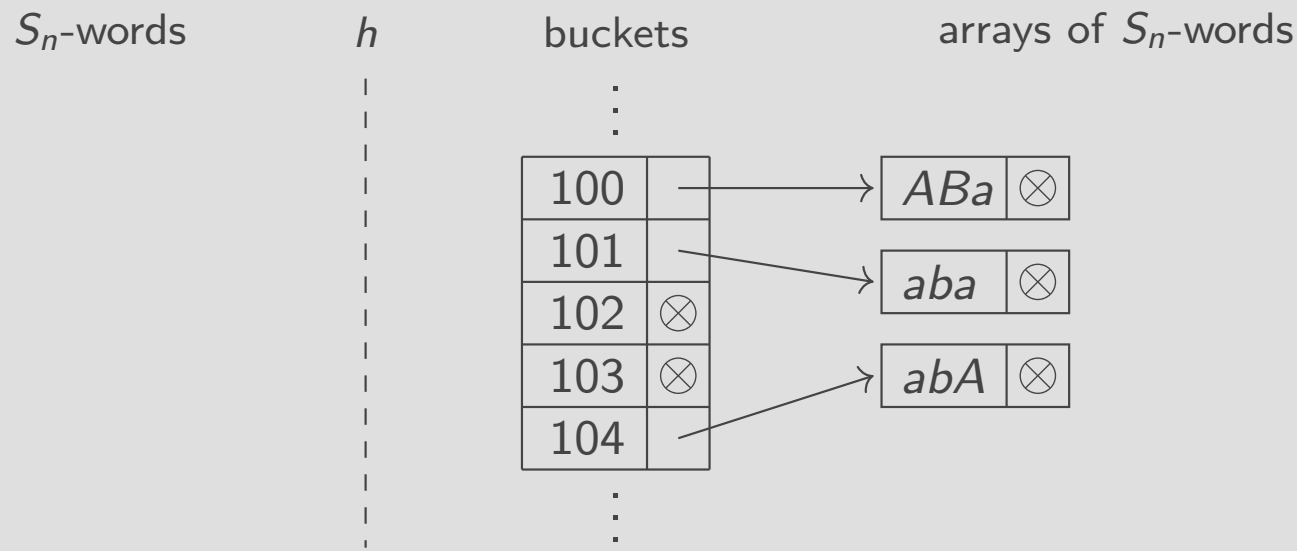

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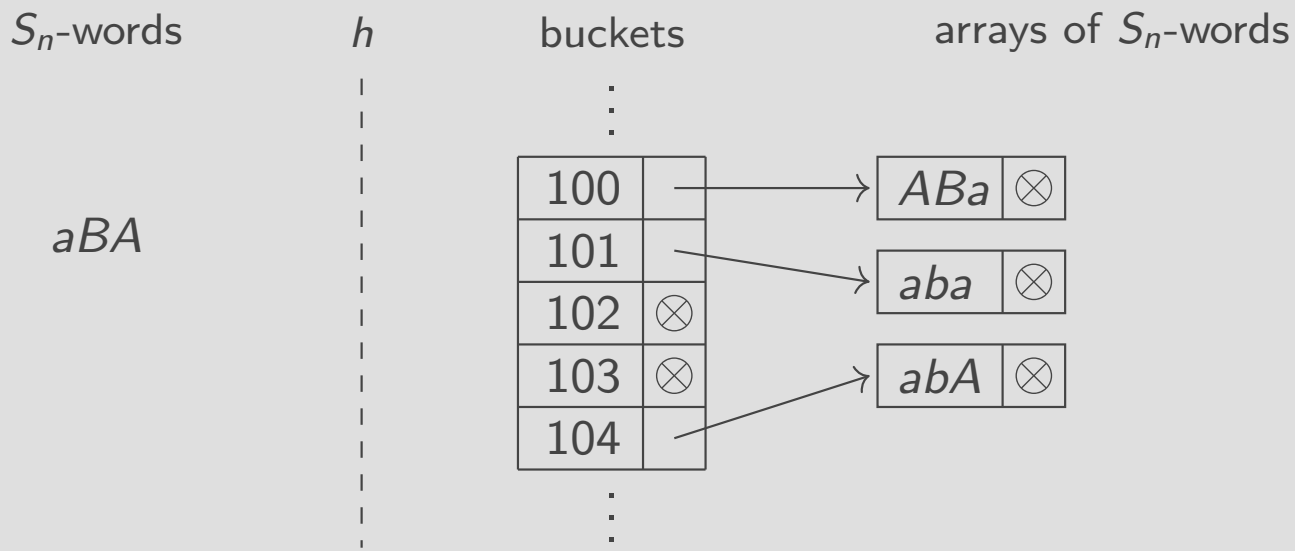

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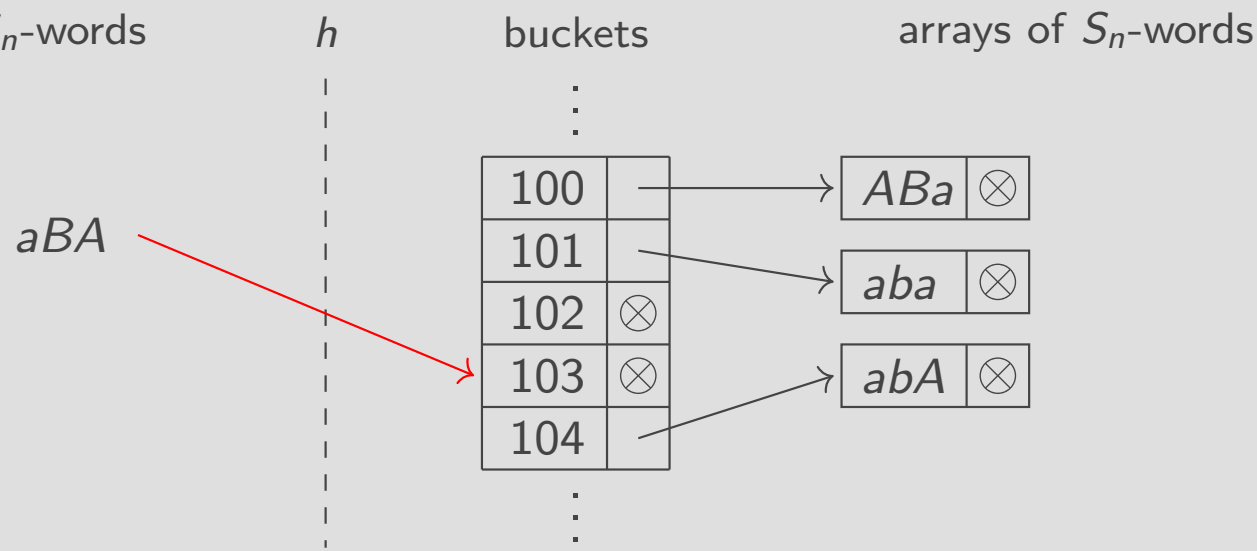

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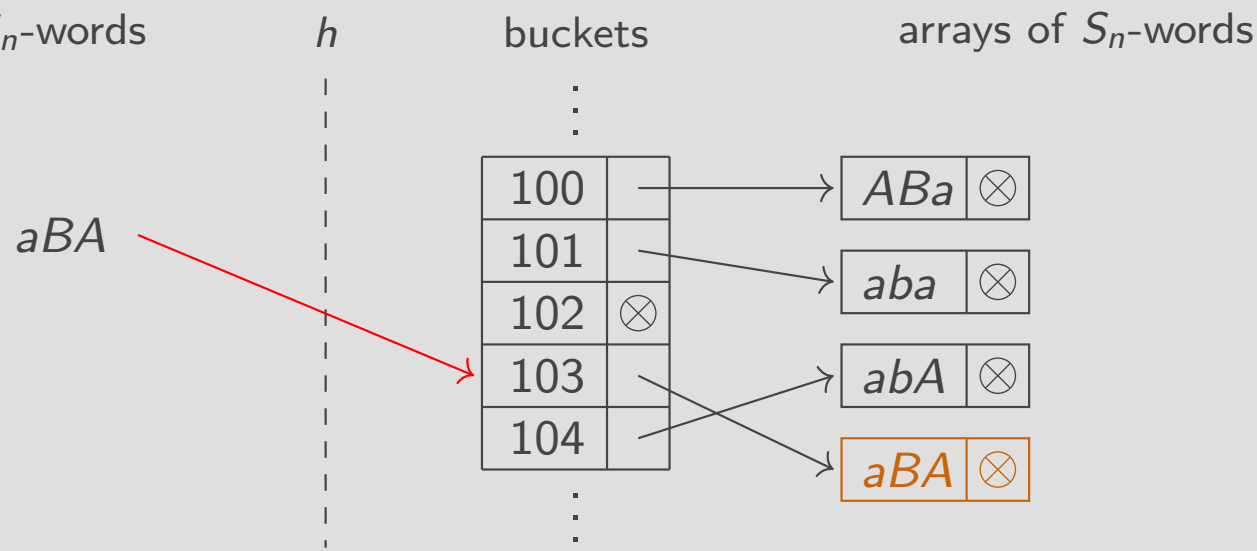

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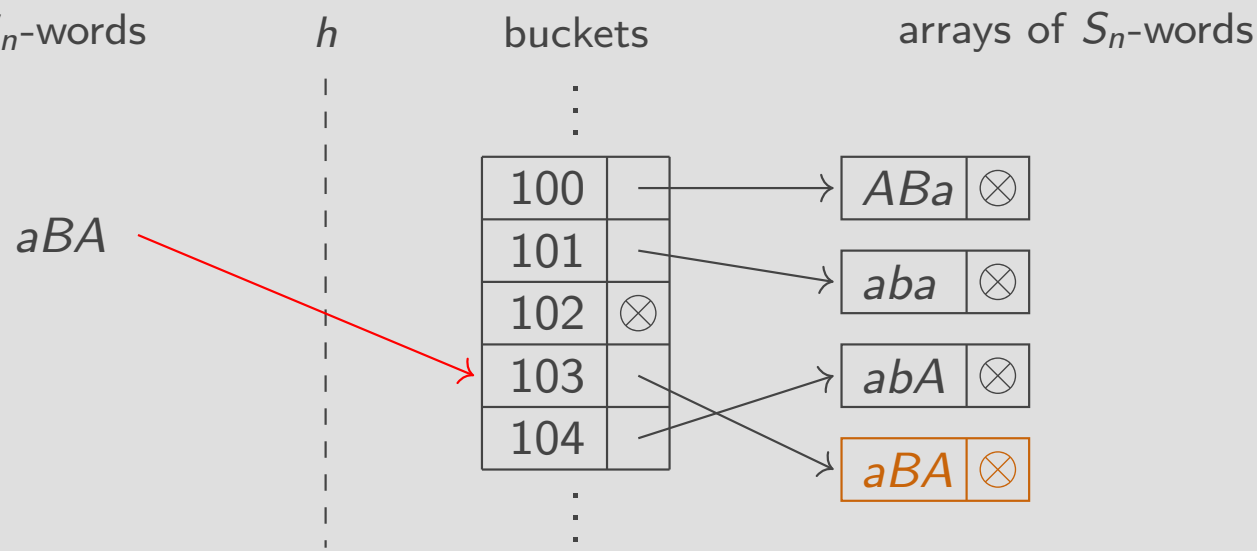

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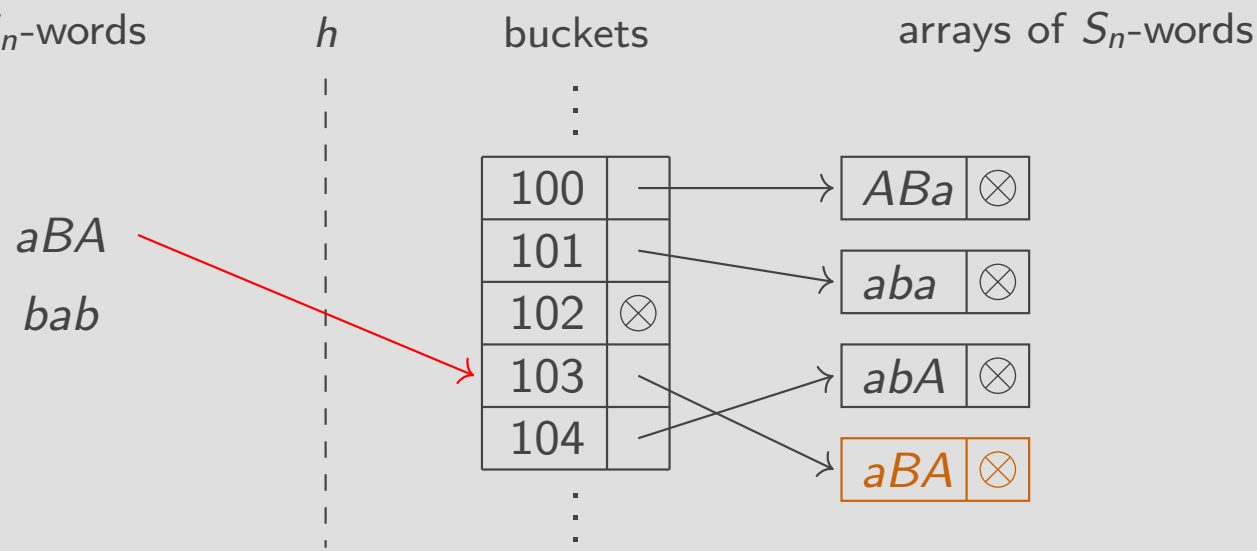

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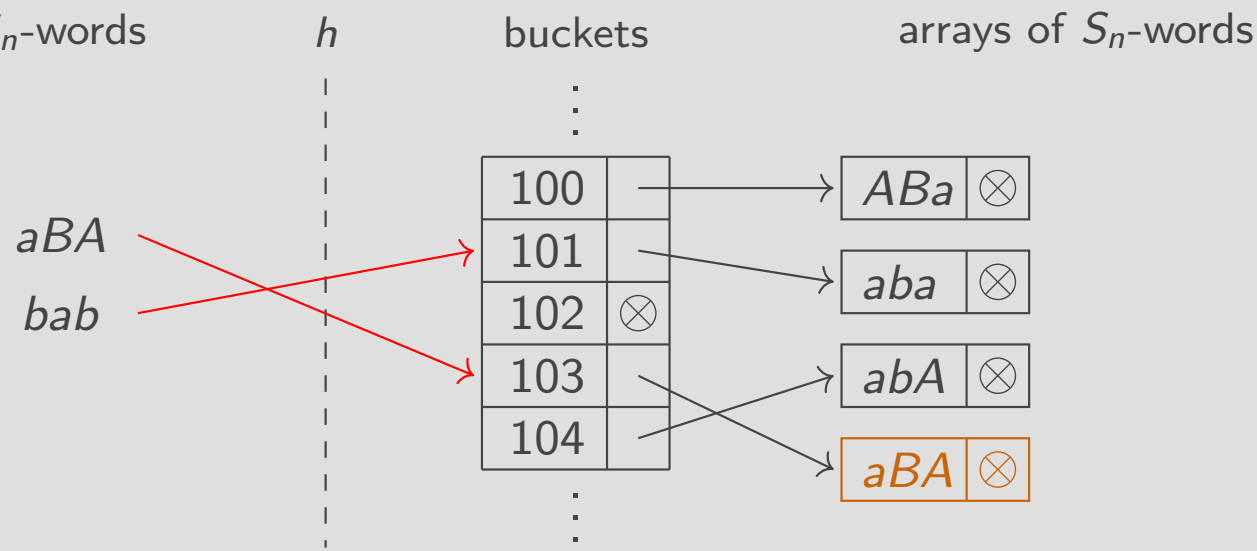

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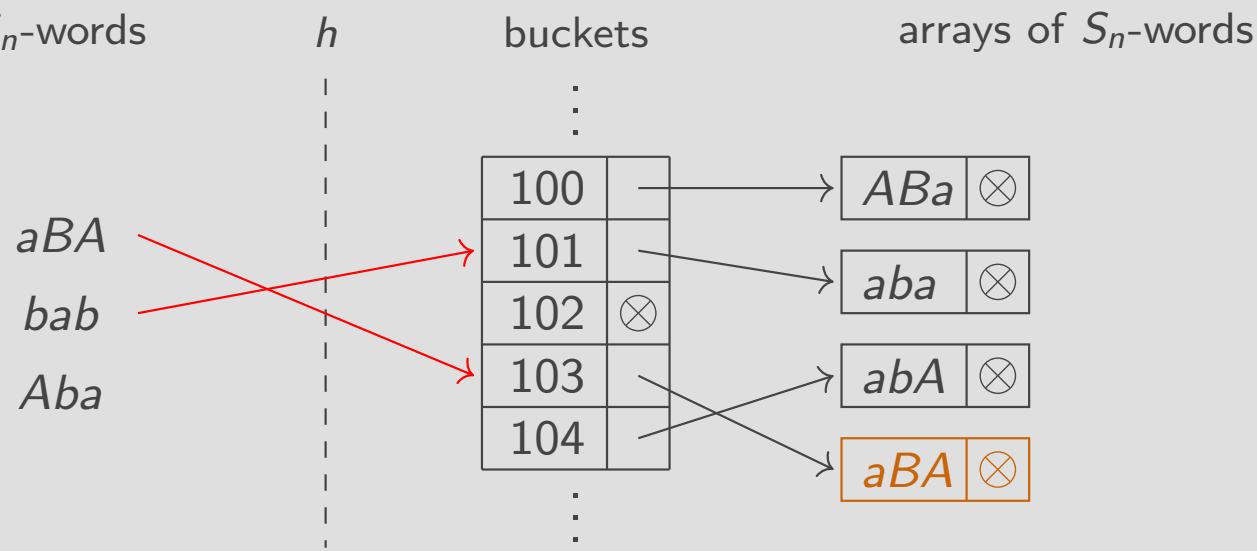

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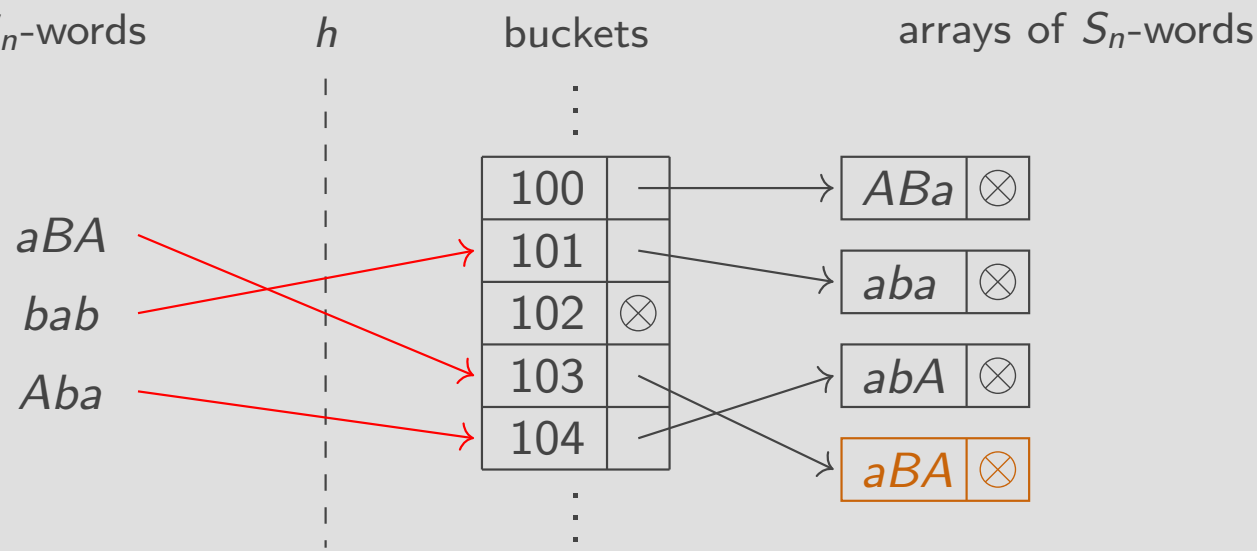

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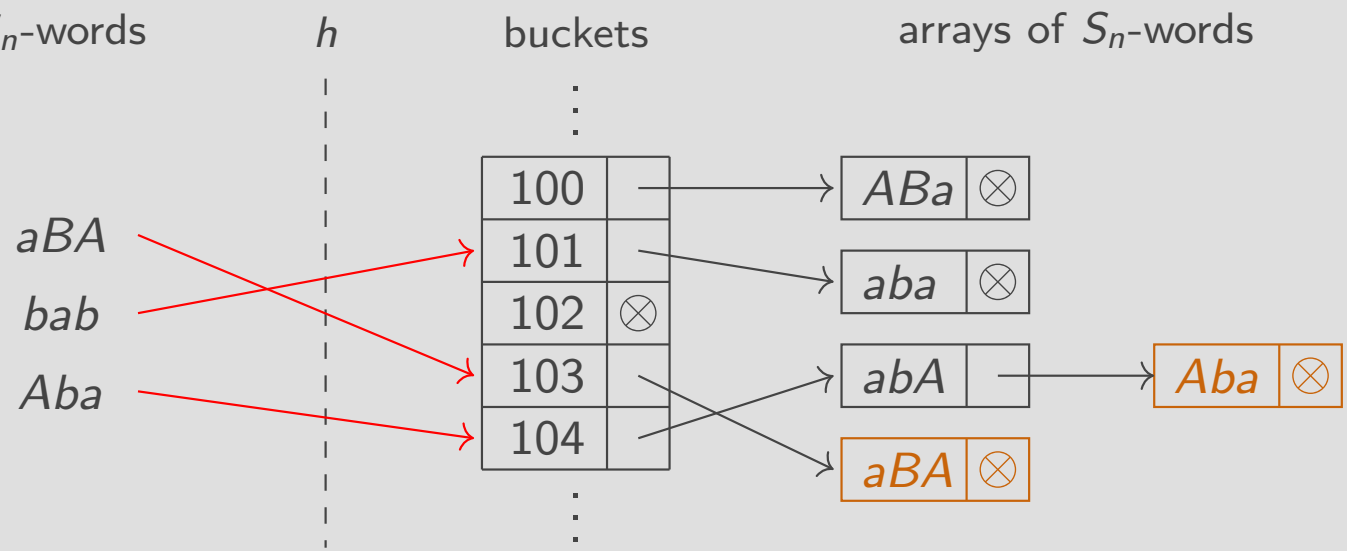

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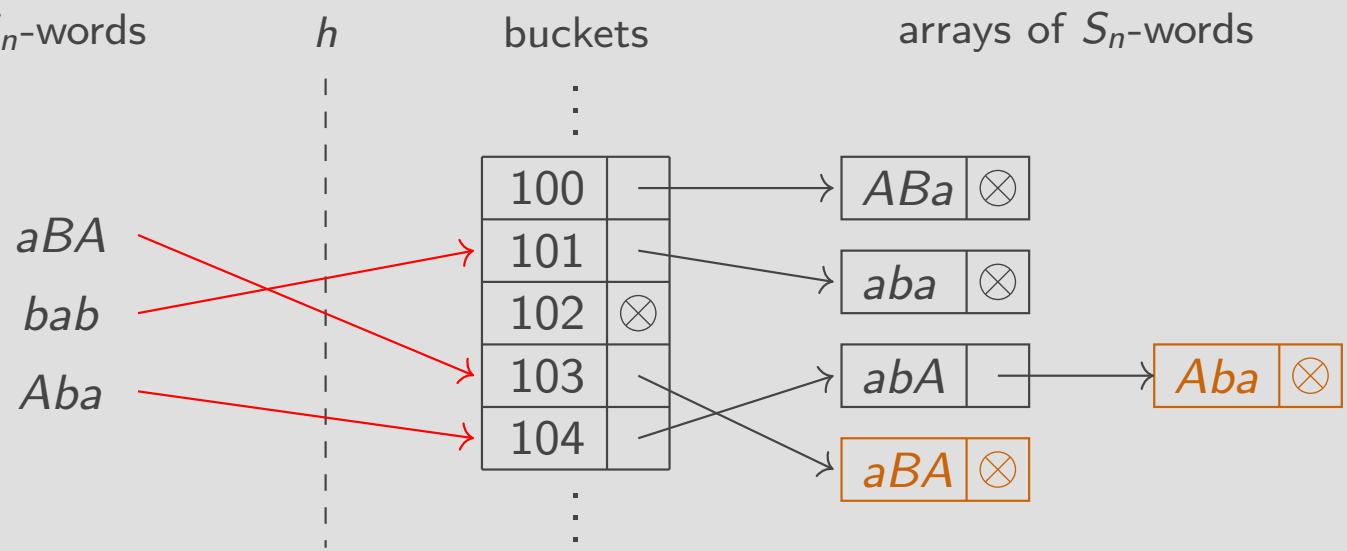

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Dynnikov's coordinates

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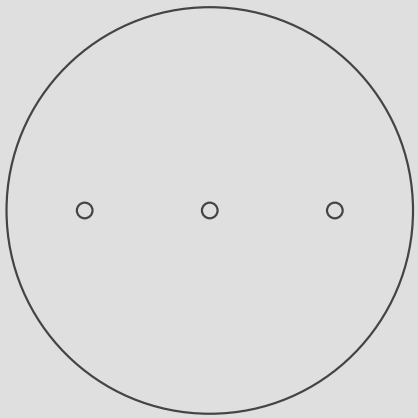

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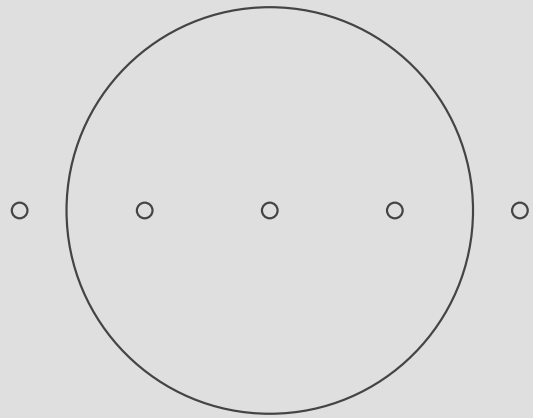

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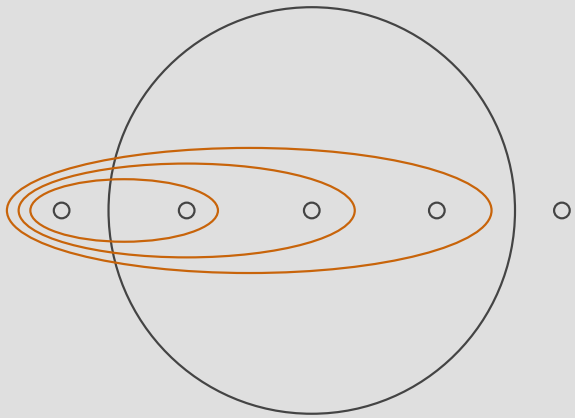

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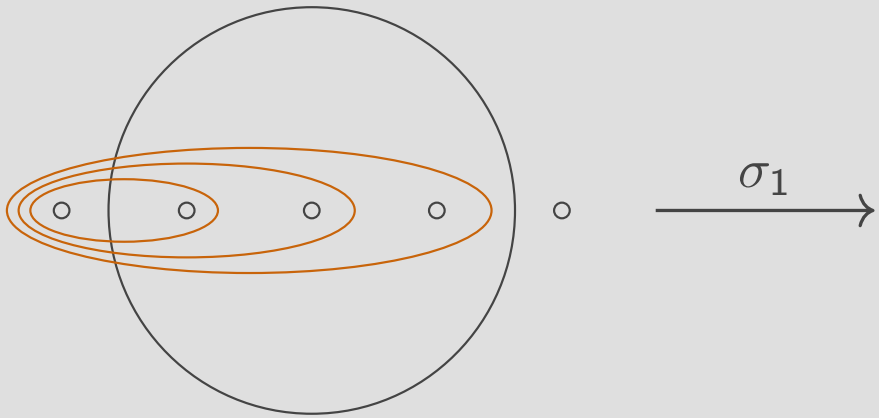

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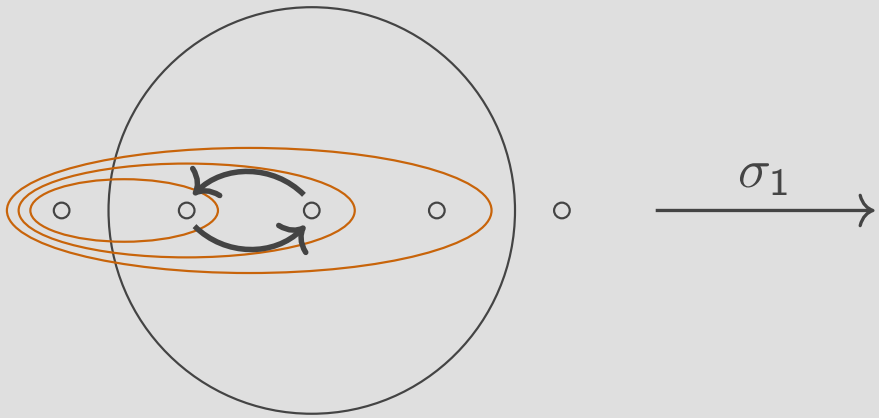

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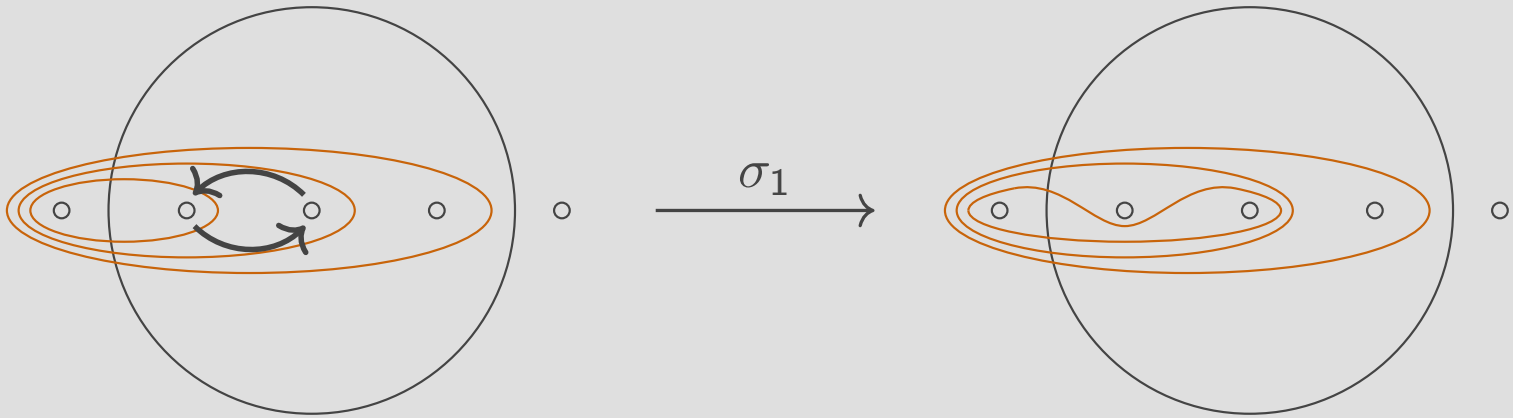

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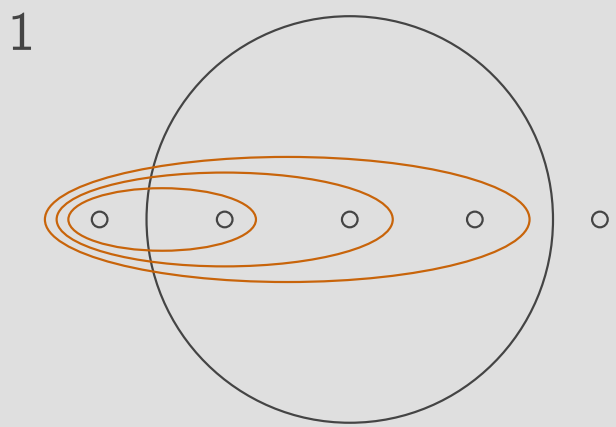

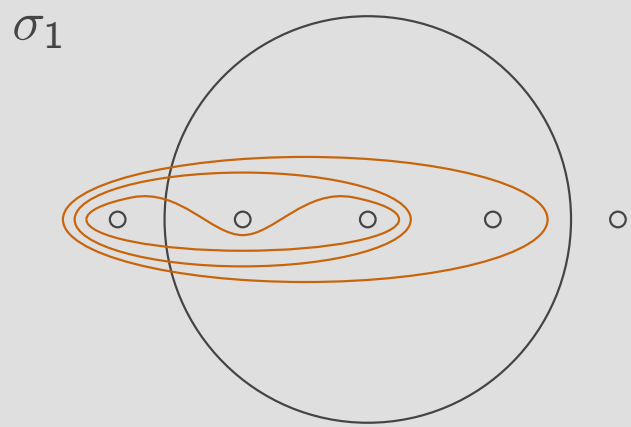

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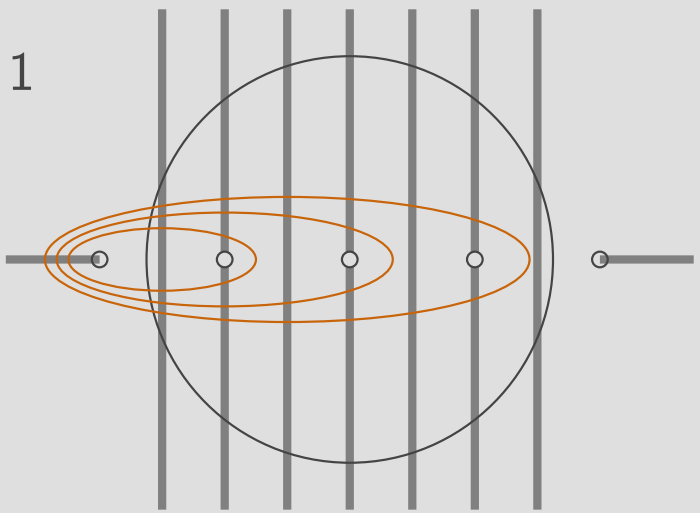

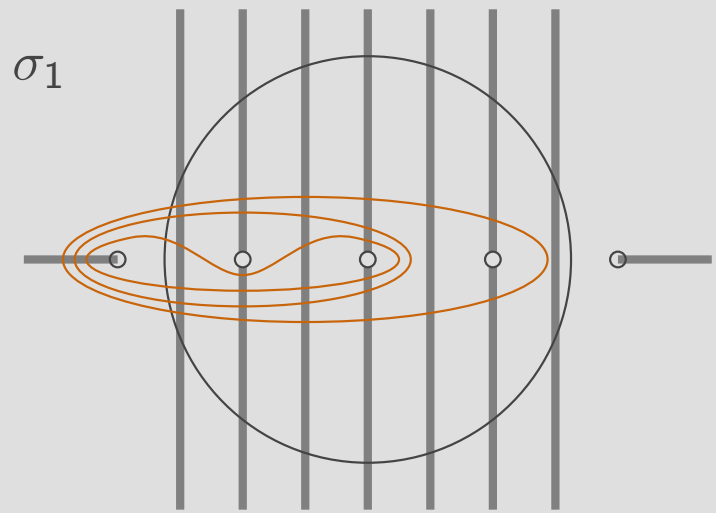

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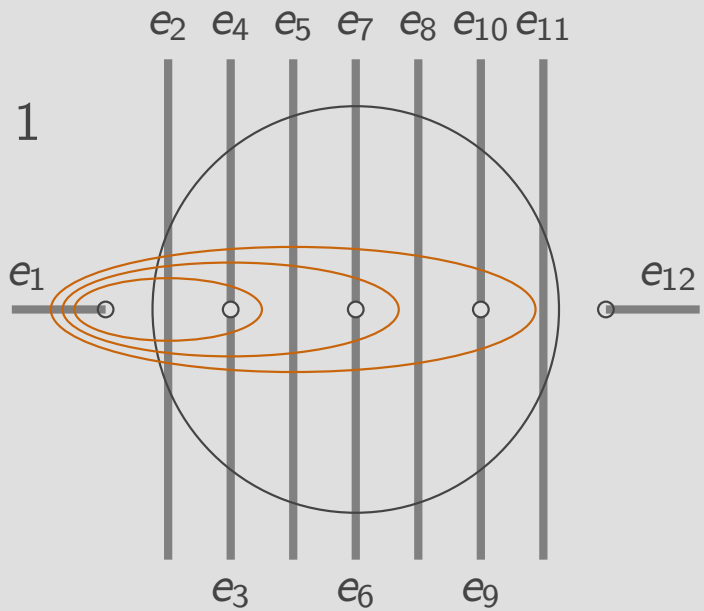

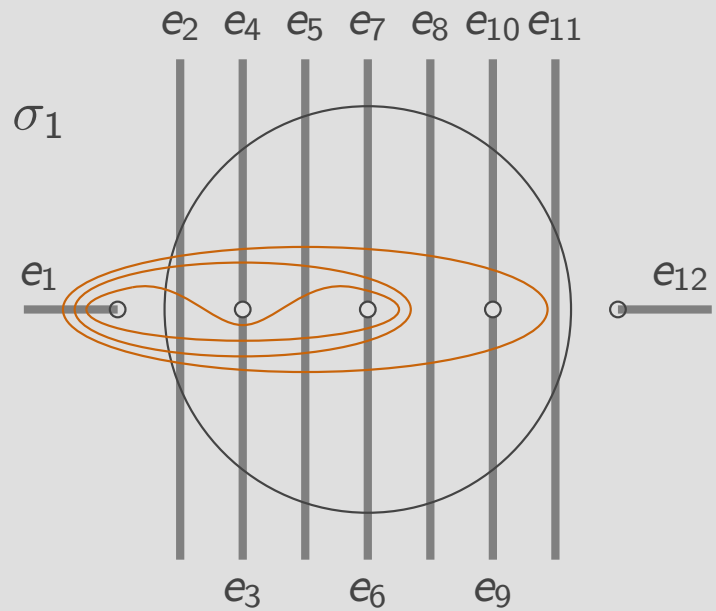

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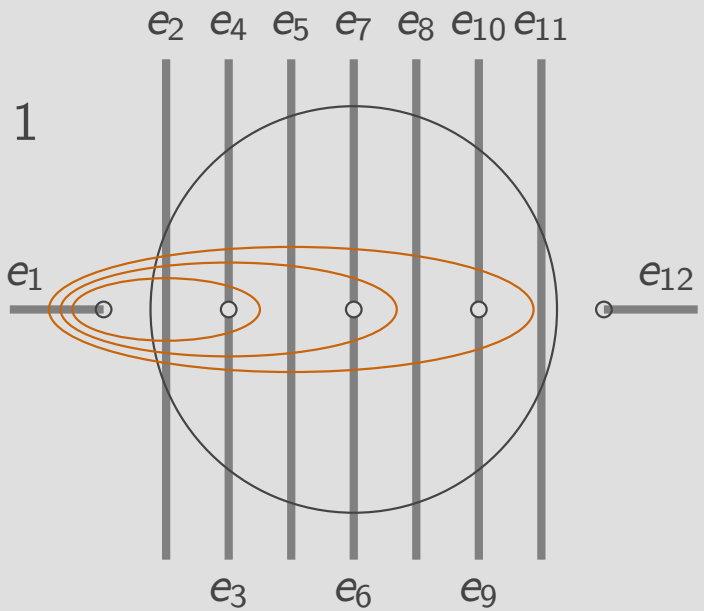

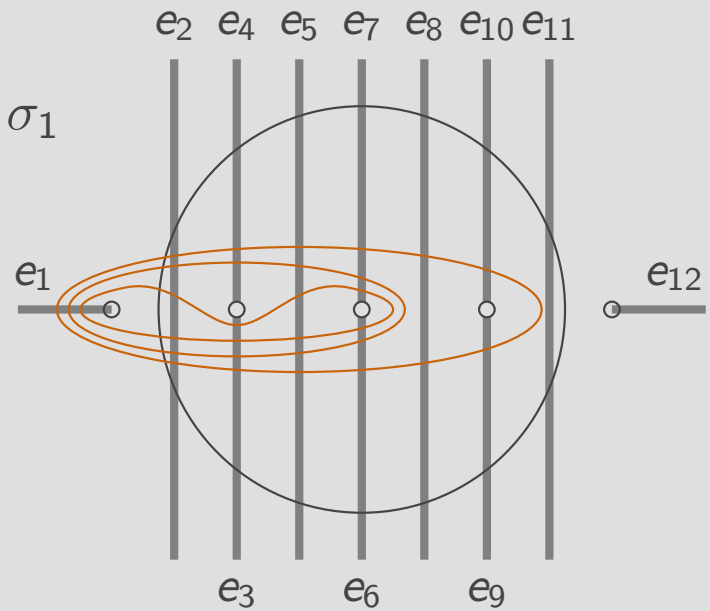

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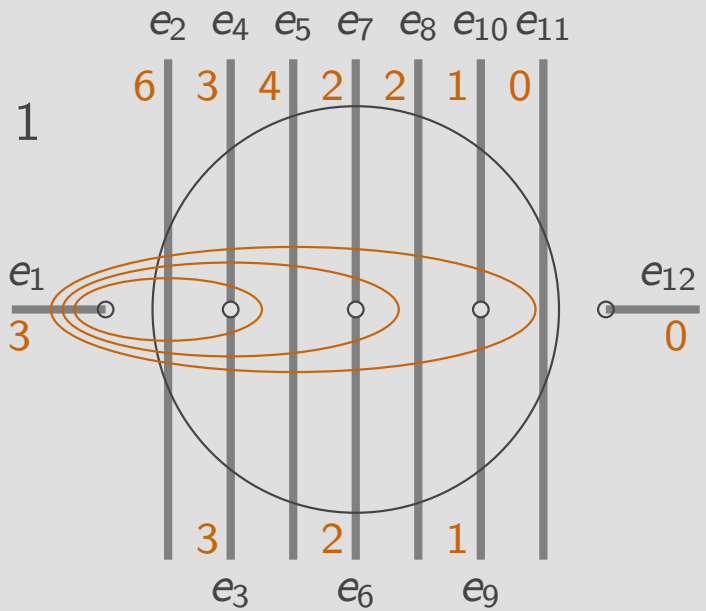

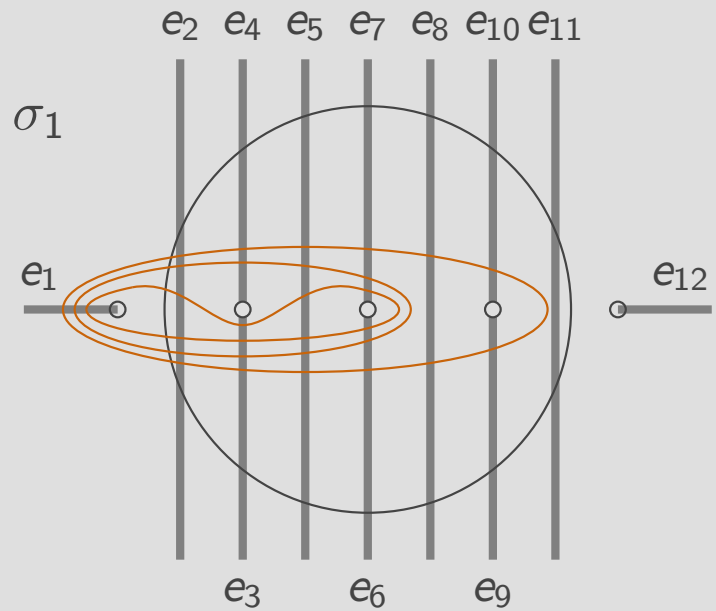

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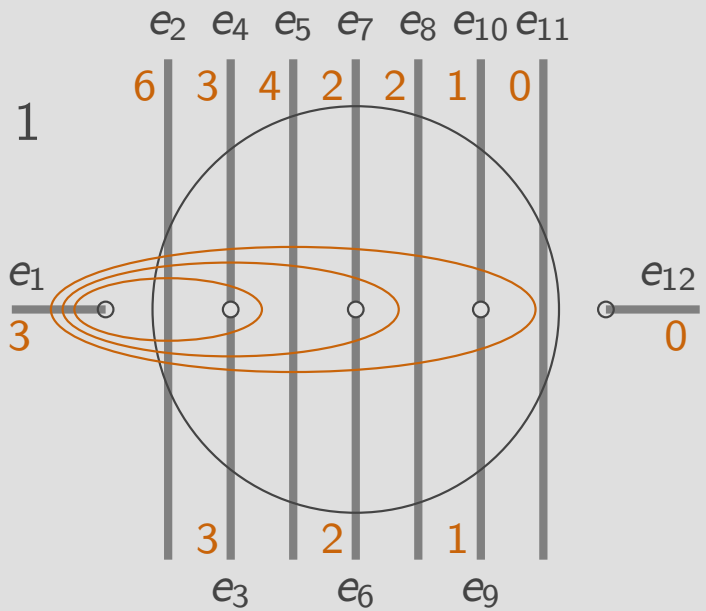

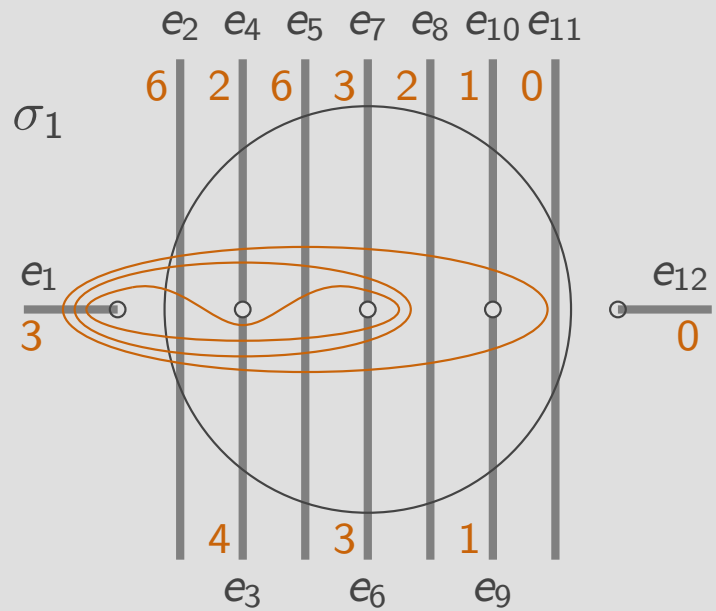

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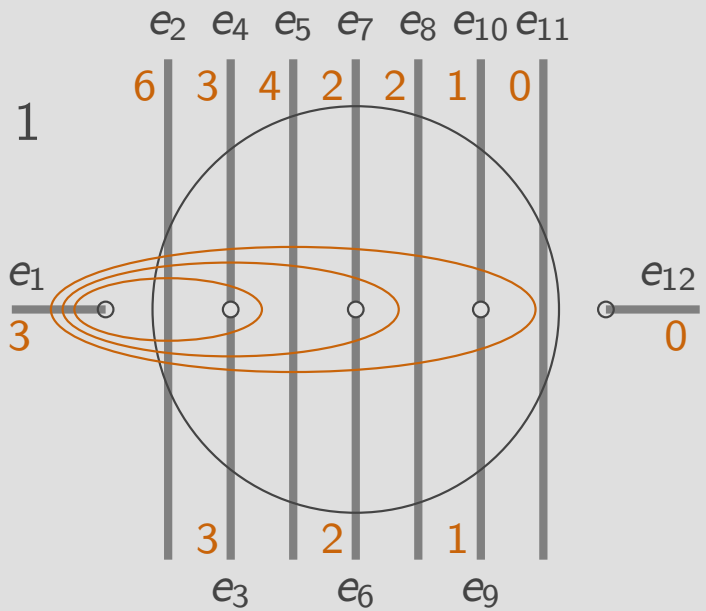

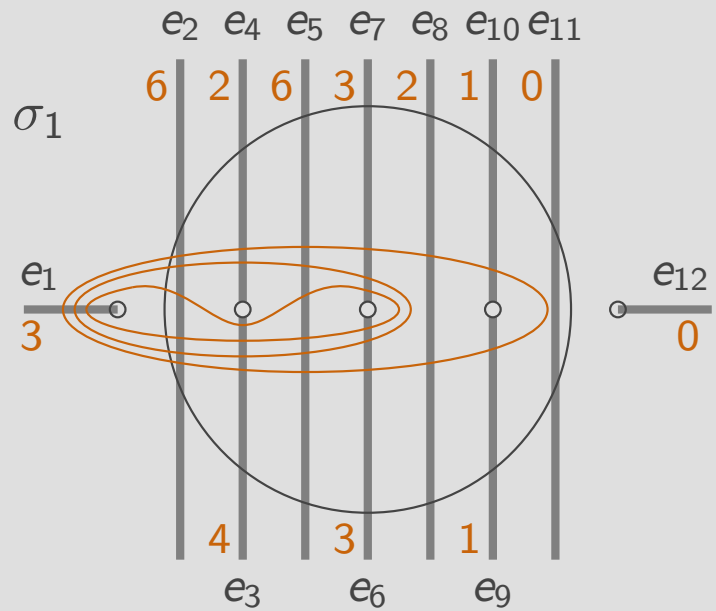

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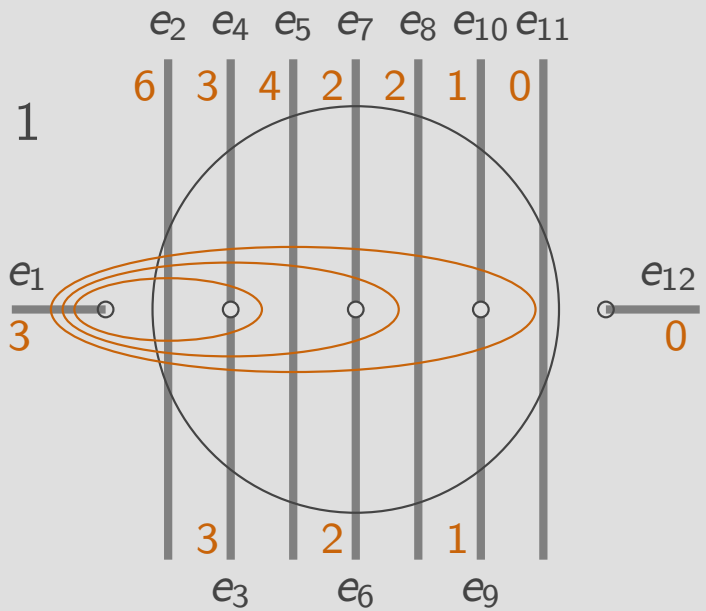

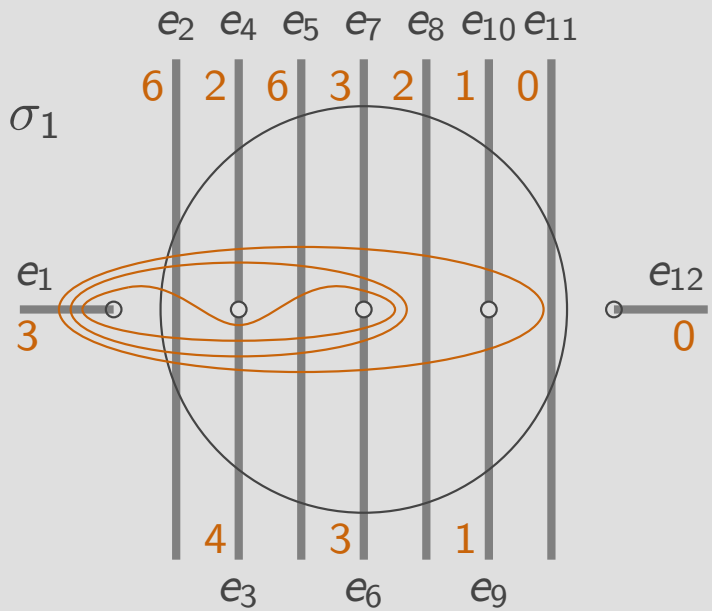

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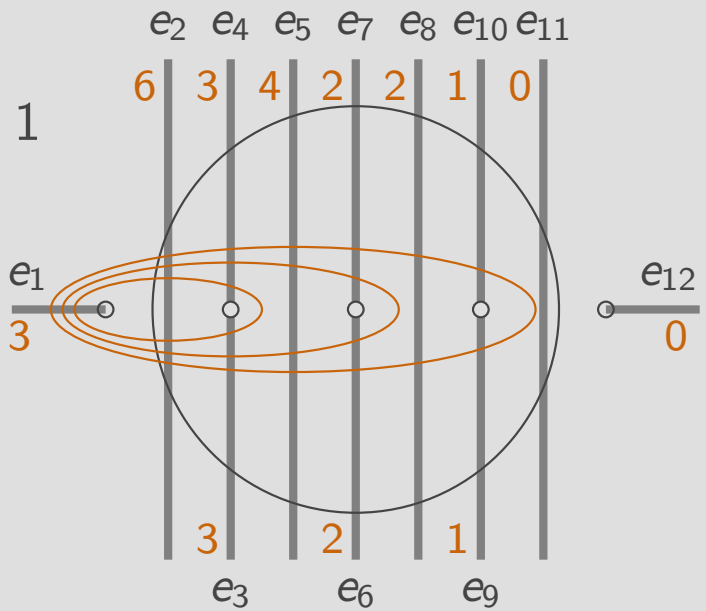

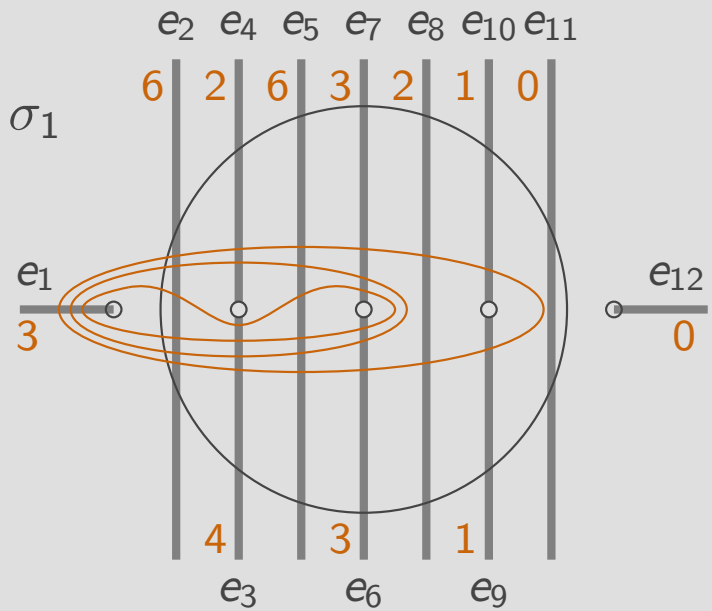

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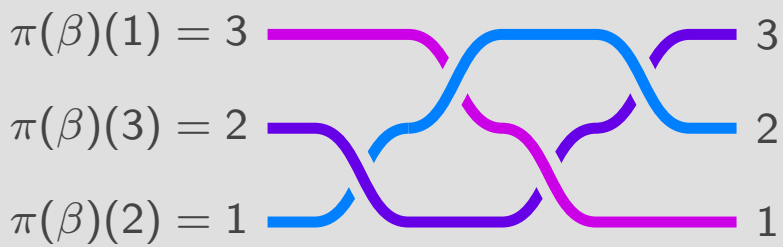

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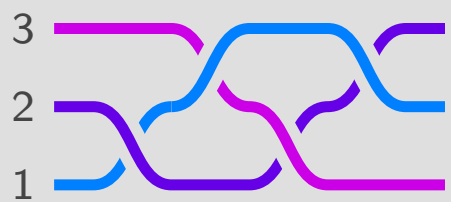

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  The linking number $\ell_{i,j}(\beta)$ is the algebraic number of crossings involving the strands i and j in β.

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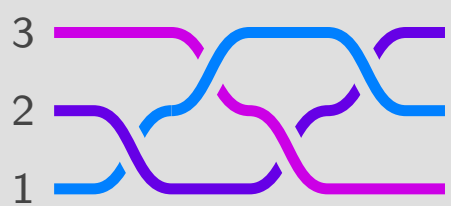

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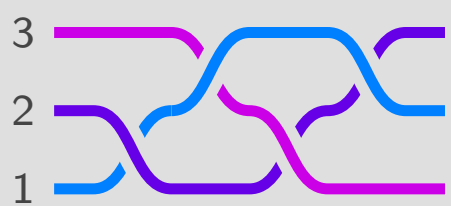

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- Corollary : For $\alpha \in S_n$, $\ell_{i,j}(\beta \cdot \alpha)$ depends only of $\ell_{*,*}(\beta)$ and $\pi(\beta)$.

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$$\tau(\beta) = (\pi(\beta), \ell_{1,2}(\beta), \ldots, \ell_{n-1,n}(\beta)) \in \mathfrak{S}_n \times \mathbb{Z}^{\frac{n(n+1)}{2}}$$

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Parallelization

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- Algorithm (Storing a representative set of $B_n(S_n, \ell, t)$) :

$W_{\ell,t} \leftarrow \emptyset$
$W_{\ell-2,t} \leftarrow \mathtt{Load}(\ell - 2, t)$
for $\alpha \in S_n$ **do**
  $t' \leftarrow t \odot \alpha^{-1}$
  $W_{\ell-1,t'} \leftarrow \mathtt{Load}(\ell - 1, t')$
  for $u \in W_{\ell-1,t'}$ **do**
    $v \leftarrow u\,\alpha$
    if $v \not\triangleleft W_{\ell-2,t}$ and $v \not\triangleleft W_{\ell,t}$ **then**
      $W_{\ell,t} \leftarrow W_{\ell,t} \sqcup \{v\}$
    end if
  end for
end for
$\mathtt{Save}(W_{\ell,t}, \ell, t)$

Stable bijection

- Definition : A bijection μ of S_n^* is S_n-stable whenever :
  - μ preserves the word length,
  - for all u, v in S_n^* we have $\mu(u) \equiv \mu(v) \Leftrightarrow u \equiv v$
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  - Can be effictively used from an algorithmic point of view.

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- Validation of L. Sabalka formulas :

$$\mathcal{S}(B_3, \Sigma_3) = \frac{(t+1)(2t^3 - t^2 + t - 1)}{(t-1)(2t-1)(t^2+t-1)},$$

$$\mathcal{G}(B_3, \Sigma_3) = \frac{t^4 + 3t^3 + t + 1}{(t^2+2t-1)(t^2+t-1)}.$$

Three strand – Dual case

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ℓ	$s(B_3, \Sigma_3^*; \ell)$	$g(B_3, \Sigma_3^*; \ell)$
0	1	1
1	6	6
2	20	30
3	54	126
4	134	498
5	318	1 926
6	734	7 410
7	1 662	28 566
8	3 710	110 658
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- Conjecture :

$$\mathcal{S}(B_3, \Sigma_3^*) = \frac{(t+1)(2t^2-1)}{(t-1)(2t-1)^2}, \quad \mathcal{G}(B_3, \Sigma_3^*) = \frac{12t^3 - 2t^2 + 3t - 1}{(2t-1)(3t-1)(4t-1)}.$$

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  - With growth rates of 2 and 4 respectively.

Four strands - Artin case

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0	1	1	13	9 007 466	281 799 158
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4	338	646	17	745 421 660	78 004 083 510
5	1 110	2 870	18	2 242 595 598	316 591 341 866
6	3 542	12 558	19	6 741 618 346	1 283 041 428 650
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8	34 362	229 338	21	60 800 088 680	20 994 893 965 398
9	105 546	963 570	22	182 422 321 452	84 795 261 908 498
10	322 400	4 016 674	23	547 032 036 564	342 173 680 884 002
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ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$	ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$
0	1	1	13	9 007 466	281 799 158
1	6	6	14	27 218 486	1 153 638 466
2	26	30	15	82 133 734	4 710 108 514
3	98	142	16	247 557 852	19 186 676 438
4	338	646	17	745 421 660	78 004 083 510
5	1 110	2 870	18	2 242 595 598	316 591 341 866
6	3 542	12 558	19	6 741 618 346	1 283 041 428 650
7	11 098	54 026	20	20 252 254 058	5 193 053 664 554
8	34 362	229 338	21	60 800 088 680	20 994 893 965 398
9	105 546	963 570	22	182 422 321 452	84 795 261 908 498
10	322 400	4 016 674	23	547 032 036 564	342 173 680 884 002
11	980 904	16 641 454	24	1 639 548 505 920	1 379 691 672 165 334
12	2 975 728	68 614 150	25	4 911 638 066 620	5 559 241 797 216 166

- No good conjectures.

Four strands - Artin case

ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$	ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$
0	1	1	13	9 007 466	281 799 158
1	6	6	14	27 218 486	1 153 638 466
2	26	30	15	82 133 734	4 710 108 514
3	98	142	16	247 557 852	19 186 676 438
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9	105 546	963 570	22	182 422 321 452	84 795 261 908 498
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11	980 904	16 641 454	24	1 639 548 505 920	1 379 691 672 165 334
12	2 975 728	68 614 150	25	4 911 638 066 620	5 559 241 797 216 166

- No good conjectures.
- The storage of all braids of B_4 with geodesic Σ_4-length $\leqslant 25$

Four strands - Artin case

ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$	ℓ	$s(B_4, \Sigma_4; \ell)$	$g(B_4, \Sigma_4; \ell)$
0	1	1	13	9 007 466	281 799 158
1	6	6	14	27 218 486	1 153 638 466
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5	1 110	2 870	18	2 242 595 598	316 591 341 866
6	3 542	12 558	19	6 741 618 346	1 283 041 428 650
7	11 098	54 026	20	20 252 254 058	5 193 053 664 554
8	34 362	229 338	21	60 800 088 680	20 994 893 965 398
9	105 546	963 570	22	182 422 321 452	84 795 261 908 498
10	322 400	4 016 674	23	547 032 036 564	342 173 680 884 002
11	980 904	16 641 454	24	1 639 548 505 920	1 379 691 672 165 334
12	2 975 728	68 614 150	25	4 911 638 066 620	5 559 241 797 216 166

- No good conjectures.
- The storage of all braids of B_4 with geodesic Σ_4-length $\leqslant 25$ requires 26 To of disk space.

Four strands – dual case

Four strands – dual case

ℓ	$s(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$	ℓ	$S(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$
0	1	1	9	7 348 366	708 368 540
1	12	12	10	35 773 324	6 128 211 364
2	84	132	11	173 885 572	52 826 999 612
3	478	1 340	12	844 277 874	454 136 092 148
4	2 500	12 788	13	4 095 929 948	3 895 624 824 092
5	12 612	117 452	14	19 858 981 932	33 359 143 410 468
6	62 570	1 053 604	15	96 242 356 958	285 259 736 104 444
7	303 356	9 311 420	16	466 262 144 180	2 436 488 694 821 748
8	1 506 212	81 488 628	17	2 258 320 991 652	20 790 986 096 580 060

Four strands – dual case

ℓ	$s(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$	ℓ	$S(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$
0	1	1	9	7 348 366	708 368 540
1	12	12	10	35 773 324	6 128 211 364
2	84	132	11	173 885 572	52 826 999 612
3	478	1 340	12	844 277 874	454 136 092 148
4	2 500	12 788	13	4 095 929 948	3 895 624 824 092
5	12 612	117 452	14	19 858 981 932	33 359 143 410 468
6	62 570	1 053 604	15	96 242 356 958	285 259 736 104 444
7	303 356	9 311 420	16	466 262 144 180	2 436 488 694 821 748
8	1 506 212	81 488 628	17	2 258 320 991 652	20 790 986 096 580 060

- Conjecture :

$$\mathcal{S}(B_4, \Sigma_4^*) = -\frac{(t+1)(10t^6 - 10t^5 - 3t^4 + 11t^3 - 4t^2 - 3t + 1)}{(t-1)(5t^2 - 5t + 1)(10t^4 - 20t^3 + 19^2 - 8t + 1)}$$

Four strands – dual case

ℓ	$s(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$	ℓ	$S(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$
0	1	1	9	7 348 366	708 368 540
1	12	12	10	35 773 324	6 128 211 364
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4	2 500	12 788	13	4 095 929 948	3 895 624 824 092
5	12 612	117 452	14	19 858 981 932	33 359 143 410 468
6	62 570	1 053 604	15	96 242 356 958	285 259 736 104 444
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- Conjecture :

$$\mathcal{S}(B_4, \Sigma_4^*) = -\frac{(t+1)(10t^6 - 10t^5 - 3t^4 + 11t^3 - 4t^2 - 3t + 1)}{(t-1)(5t^2 - 5t + 1)(10t^4 - 20t^3 + 19^2 - 8t + 1)}$$

  - The growth rate of $\mathcal{S}(B_4, \Sigma_4^*)$ is $\simeq 4.8$.

Four strands – dual case

ℓ	$s(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$	ℓ	$S(B_4, \Sigma_4^*; \ell)$	$g(B_4, \Sigma_4^*; \ell)$
0	1	1	9	7 348 366	708 368 540
1	12	12	10	35 773 324	6 128 211 364
2	84	132	11	173 885 572	52 826 999 612
3	478	1 340	12	844 277 874	454 136 092 148
4	2 500	12 788	13	4 095 929 948	3 895 624 824 092
5	12 612	117 452	14	19 858 981 932	33 359 143 410 468
6	62 570	1 053 604	15	96 242 356 958	285 259 736 104 444
7	303 356	9 311 420	16	466 262 144 180	2 436 488 694 821 748
8	1 506 212	81 488 628	17	2 258 320 991 652	20 790 986 096 580 060

- Conjecture :

$$\mathcal{S}(B_4, \Sigma_4^*) = -\frac{(t+1)(10t^6 - 10t^5 - 3t^4 + 11t^3 - 4t^2 - 3t + 1)}{(t-1)(5t^2 - 5t + 1)(10t^4 - 20t^3 + 19^2 - 8t + 1)}$$

  - The growth rate of $\mathcal{S}(B_4, \Sigma_4^*)$ is $\simeq 4.8$.
  - No good conjecture for $\mathcal{G}(B_4, \Sigma_4^*)$.

Thank you !