Méthodes d'éléments finis pour la résolution d'EDP et estimateurs d'erreur a posteriori

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Outline

"Calcul Scientifique"

- Modelling physical phenomena with PDE
- Analysis of the solution of the PDE
- Solving numerically this PDE: build an approximated solution
  - ⋆ Choice of a numerical method
  - ⋆ Solve an linear system
- Analysis of the numerical solution
- Comparison of this solution with experiments results

Applying domains: aeronautic, car industry, atmosphere pollution,. . . .

Outline

Plan

Initial problem

Let $\Omega \subset \mathbb{R}^d, d = 2,3$ bounded domain with Lipschitz boundary.
Find u such that:

$$\begin{cases} -\mathrm{div}\,(a\ \nabla u) = f & \text{in } \Omega, \\ u = g_D \text{ on } \Gamma_D, & a\nabla u \cdot n = g_N \text{ on } \Gamma_N. \end{cases}$$

Variational problem

Given f a function, find $u \in V = H_0^1(\Omega)$ solution of

$$\underbrace{\int_\Omega a\nabla u \nabla v dx}_{B(u,v)} = \underbrace{\int_\Omega f v dx}_{(f,v)}, \quad \forall v \in V,$$

where B is a continuous bilinear coercive form.
$\Rightarrow$ Existence and uniqueness of the solution can be proved.

The finite element method

Appeared in the 50s.

Main ideas

Find an approximation u_h of the solution u

- Choice of an approximated space V_h:
  - Conforming approximation $V_h \subset V$,
  - Galerkin discontinuous type approximation $V_h \not\subset V$,
- Numerical approximation of the solution u by u_h in V_h,
- Estimation of the error between the exact and the approximated solution in a given norm $\|\cdot\|$ on V.

Plan

The discrete problem

- Given a set of triangulations $(\mathcal{T}_h)_h$ made of elements T: *triangles, rectangles, ... in 2D, tetrahedra, ... in 3D,*
- $h > 0$ is the mesh-size of $\mathcal{T}_h$,
- V_h is build on the triangulation $\mathcal{T}_h$,

  ⋆ Conforming approximation:
  $V_h = \left\{ v \in H_0^1(\Omega) | v_{|T} \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h \right\}$,

  ⋆ DG approximation:
  $V_h = \left\{ v \in L^2(\Omega) | v_{|T} \in \mathcal{P}_k(T), \forall T \in \mathcal{T}_h, k \in \mathbb{N} \right\}$.
- Find $u_h \in V_h$ satisfying $B_h(u_h, v_h) = (f, v_h), \forall v_h \in V_h$.

- Conforming approximation : $B_h(u_h, v_h) = \int_\Omega a\nabla u_h \nabla v_h dx$
- DG approximation :

$$\begin{aligned} B_h(u_h, v_h) &= \sum_{T\in\mathcal{T}_h} \int_T a\nabla u_h \cdot \nabla v_h + \gamma \sum_{e\in\mathcal{E}_{ID}} h_e^{-1} \int_e [\![u_h]\!]_e \cdot [\![v_h]\!]_e \\ &- \sum_{e\in\mathcal{E}_{ID}} \int_e (\{\{a\nabla_h v_h\}\} \cdot [\![u_h]\!]_e + \{\{a\nabla_h u_h\}\} \cdot [\![v_h]\!]_e). \end{aligned}$$

Error estimations

A priori error estimations

$$\|u - u_h\| \leq F(h, u),$$

F is a function depending on the regularity of the exact solution u.

A posteriori error estimations

- Introduced in 1978 by Babuška and Rheinboldt
- $\|u - u_h\| \leq \eta = \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}$ only depending on h, u_h and f.
- **Advantage:** the estimator η does not depend on the exact solution u generally unknown.

Properties of a posteriori estimators

- **Reliability:** $\|u - u_h\| \leq C\eta + osc, C > 0,$
- **Efficiency:** $\eta \leq C\|u - u_h\| + osc, C > 0,$
- **Locality:** Composed of local quantities η_T, T element of $\mathcal{T}_h$
- **Effectivity index:** $\dfrac{\|u - u_h\|}{\eta}$ must be bounded.
  Optimal when equal to 1.

Plan

Residual estimators

Let T be a triangle and denote by e any of its edge

- Verfürth, 1996
- Estimator directly deduced from the equation

$$\eta_T^2 = a_T^{-1} h_T^2 \| f_T + \mathrm{div}\,(a \nabla u_h) \|_T^2 + \sum_{e \subset \partial T} a_T^{-1/2} h_e \| [\![a \nabla u_h \cdot \mathbf{n}_e]\!]_e \|_e^2.$$

where f_T is the projection of f on the element T

Based on equilibrated fluxes

Ainsworth and Oden, 2000

Idea

Saddle point problem:

Find $(j = a\nabla u, u)$ solution in $(H_N(\mathrm{div}\,, \Omega), L^2(\Omega))$ of

$$\begin{aligned} \int_\Omega a^{-1} j\tau + \int_\Omega \mathrm{div}\,\tau u &= \int_{\Gamma_D} g_D \tau \cdot n, \ \forall \tau \in H_N(\mathrm{div}\,, \Omega), \\ \int_\Omega \mathrm{div}\, j w &= -\int_\Omega f w, \ \forall w \in L^2(\Omega). \end{aligned}$$

$\Rightarrow j_h$ approximation of j in a conforming FE space by solving local problems on the triangulation $\mathcal{T}_h$.

$$\eta_T^2 = \|a^{-1/2}(a\nabla u_h - j_h)\|^2.$$

Outline

Strategy

GOAL: Make the error decrease locating areas where the error is high and cutting the involved elements

- Take an initial triangulation $\mathcal{T}_h$
- Numerically solve the equation finding u_h
- $\|u - u_h\| \leq C\eta + osc$, so (estimator decreases $\Rightarrow$ error decreases)
- Iterative algorithm with marking procedure
  We look for elements of $\mathcal{T}_h$ where the local indicator η_T is too large:

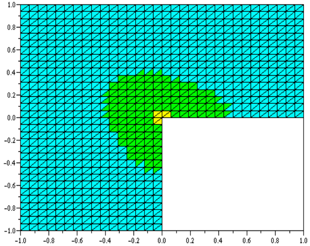
$$\text{if } \eta_T > \theta \max_{T'} \eta_{T'}, 0 < \theta \leq 1, \text{ mark it.}$$

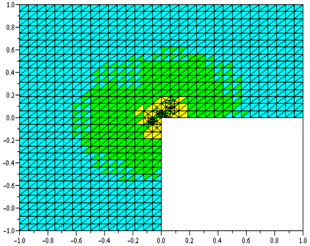
- Construction of a finer triangulation to find a better approximation u_h of u : *Each element marked is cut in 4 , in 3 or in 2.*

Example of a solution with singularity corner: the L-shape domain

- Exact solution: $u = \nabla S$ with S Laplacian singularity.
- Parameter : $a = 1$.

Meshes obtained after 6 iterations with a residual estimator.





Outline

My work

- Study of different types of a posteriori error estimators:
  ⋆ for Maxwell's system: residual, based on equilibrated fluxes,
  ⋆ for the reaction-diffusion equation: with fluxes.
- Dependance on the coefficients of the constants in the upper and lower bounds when the coefficients are piecewise constant $\Rightarrow$ robust estimators.
- Extension of estimators based on equilibrated fluxes to the discontinuous Galerkin method for the diffusion equation.