

DISCREPANCIES IN THE DISTRIBUTION OF GAUSSIAN PRIMES

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ABSTRACT. Motivated by questions of Fouvry and Rudnick on the distribution of Gaussian primes, we develop a very general setting in which one can study inequities in the distribution of analogues of primes through analytic properties of infinitely many L -functions. In particular, we give a heuristic argument for the following claim : for more than half of the prime numbers that can be written as a sum of two squares, the odd square is the square of a positive integer congruent to 1 mod 4.

1. INTRODUCTION

Let p be a prime number, $p \equiv 1 \pmod{4}$, it can be written uniquely as $p = a^2 + 4b^2$ with $a, b > 0$ integers. Here we distinguish the odd and the even square to have uniqueness, and we are interested in properties of the coordinates a and $2b$. For example, an open problem in this domain is to prove that there are infinitely many prime numbers of the form $p = 1 + 4b^2$. Allowing a larger range for the coordinates, Fouvry and Iwaniec [FI97] have shown that there are infinitely many prime numbers $p = a^2 + 4b^2$ with a itself a prime number. While Friedlander and Iwaniec [FI98] established the case where one of a or $2b$ is a square. More recent results restricting the values of a and b to thinner sets include [HBL17, LSX20, Pra20, FI21].

In another direction, starting from the result of Hecke [Hec18] ensuring that the angles defined by $\arctan(\frac{2b}{a})$ are equidistributed, it is of interest to study finer statistics of this distribution. Kubilius [Kub50, Kub51] and Ankeny [Ank52] initiated the study of the distribution of these angles in shrinking sectors, and further developments followed [Kov75, Mak77, Col90, HL01, HLR20, CKL⁺20]. Recently Rudnick and Waxman [RW19] studied the function field analogue and obtained statistical results that hold for almost all arcs of shrinking length.

In this paper we discuss the following two questions:

- (1) how often is $a < 2b$ compared to $a > 2b$?
- (2) how often is $a \equiv 1 \pmod{4}$ compared to $a \equiv 3 \pmod{4}$?

The results of Hecke [Hec18] on the prime number theorem for L -functions associated to Hecke characters provide an answer to these two questions : asymptotically half of the time. Inspired by the letter of Chebyshev to Fuss [Che99], and the rich literature that followed on Chebyshev's bias (see [MS12] for a survey on the numerous contributions to the questions on "prime number races"), we investigate the discrepancy in these equidistribution results. We denote the differences of counting

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functions

$$D_1(x) = |\{p < x : p = a^2 + 4b^2, |a| > |2b|\}| - |\{p < x : p = a^2 + 4b^2, |a| < |2b|\}|,$$

$$D_2(x) = |\{p < x : p = a^2 + 4b^2, |a| \equiv 1 \pmod{4}\}| - |\{p < x : p = a^2 + 4b^2, |a| \equiv 3 \pmod{4}\}|.$$

Then, assuming the Generalized Riemann Hypothesis, the results of Hecke imply that, for $i = 1, 2$, $D_i(x) = O_\epsilon(x^{\frac{1}{2} + \epsilon})$. As observed in general for functions related to prime counting (see e.g. [Win41]), the logarithmic scale is the correct scale to study the functions D_i . We recall the definition of the upper and lower logarithmic densities of a set $\mathcal{P} \subset [1, \infty)$:

$$\bar{\delta}(\mathcal{P}) = \limsup_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y \mathbf{1}_{\mathcal{P}}(e^y) dy \quad \text{and} \quad \underline{\delta}(\mathcal{P}) = \liminf_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y \mathbf{1}_{\mathcal{P}}(e^y) dy,$$

where $\mathbf{1}_{\mathcal{P}}$ is the characteristic function of the set \mathcal{P} . If these two densities are equal, we denote $\delta(\mathcal{P})$ their common value and call it the logarithmic density of the set \mathcal{P} .

In this paper, we give a heuristic model for the distribution of the values of the functions D_1 and D_2 . We formulate two conjectures.

Conjecture 1.1. There is a bias towards negative values in the distribution of the values of the function D_1 . That is to say that for more than half of the $x \in [2, \infty)$ in logarithmic scale, more than half of the primes below x can be written as a sum of two squares with the even square larger than the odd square.

However we have $D_1(x) = \Omega_{\pm}(\frac{\sqrt{x}}{\log x})$. In particular, the function changes signs infinitely often.

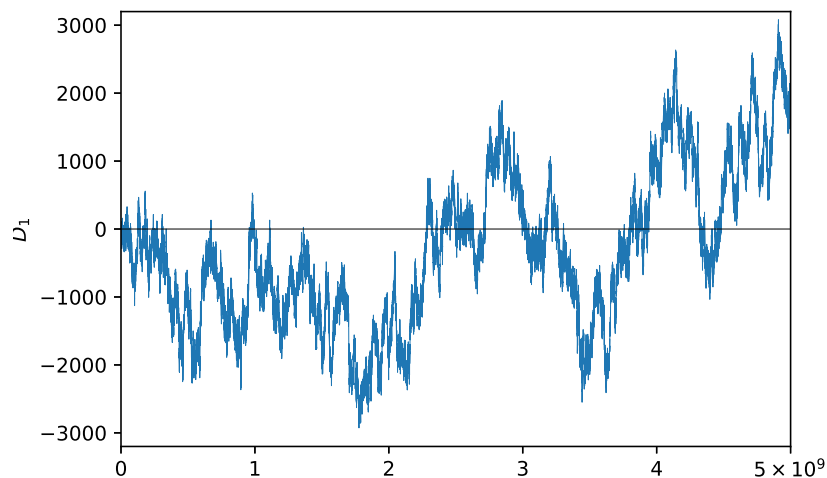
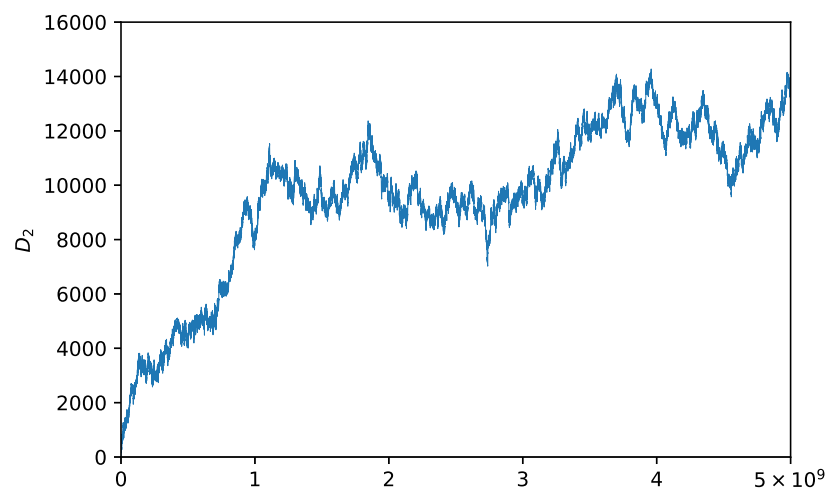
Conjecture 1.2. There is a complete bias towards positive values in the distribution of the values of the function D_2 . That is to say that, for almost all (in logarithmic scale) $x \in [2, \infty)$, more than half of the primes below x can be written as a sum of two squares $p = a^2 + 4b^2$ with $|a| \equiv 1 \pmod{4}$.

Precisely, we have $D_2(x) \geq 0$ for almost all x in logarithmic scale, and $D_2(x) = \Omega_+(\frac{\sqrt{x}}{\log x})$.

The Ω notation in the Conjectures is used in the following sense. For two functions $f, g : [1, \infty) \rightarrow \mathbf{R}$, with g positive, we write $f(x) = \Omega_+(g)$ (resp. $\Omega_-(g)$) if we have $\limsup_{x \rightarrow \infty} \frac{f(x)}{g(x)} > 0$ (resp. $\liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} < 0$).

Numerical data up to $5 \cdot 10^9$ have been computed to support both conjectures and the graphs of the prime number races are presented in Figures 1 and 2. Note in particular that no negative value of D_2 was found in the interval.

The paper is organized as follows. Section 2 is the technical heart of the paper where the setting and the precise statement of our main theoretical contribution (Theorem 2.2) are given. Here we emphasize the novel feature (compared to previous results e.g. [RS94, ANS14, Dev20]) where we have managed to obtain distribution results involving infinitely many L -functions. In Section 3, we introduce the setting to study statistics about the distribution of Gaussian primes, and we state our results towards Conjecture 1.1 and Conjecture 1.2: Theorem 3.1 and Theorem 3.2. These two results are proved in Section 4 as consequences of Theorem 2.2. Building on the ideas underlying Theorem 3.1 and Theorem 3.2, we present in Section 5 heuristics that led us to state Conjecture 1.1 and Conjecture 1.2. Finally in Section 6, we give the proofs of the theoretical contributions stated in Section 2, notably Theorem 2.2.

FIGURE 1. $D_1(x)$ for $x \in [2, 5 \cdot 10^9]$.FIGURE 2. $D_2(x)$ for $x \in [2, 5 \cdot 10^9]$.

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The computations presented here were obtained using SageMath [Sag20], with the help of Valentin Priasso to deal with larger numbers and space.

2. CHEBYSHEV'S BIAS USING INFINITELY MANY L -FUNCTIONS

2.1. Motivations and setting. In [Maz08], Mazur discussed prime number races for elliptic curves, or more generally, for the Fourier coefficients of a modular form. For example, he studied graphs of functions

$$x \mapsto |\{p \leq x : a_p(E) > 0\}| - |\{p \leq x : a_p(E) < 0\}|$$

where $a_p(E) = p+1 - |E(\mathbf{F}_p)|$, for some elliptic curve E defined over \mathbf{Q} , and observed a bias towards negative values when the algebraic rank of the elliptic curve is large. In [Sar07], Sarnak commented and explained Mazur's observations. His analysis of the prime number race involves the zeros of all the symmetric powers $L(\text{Sym}^n E, s)$ of the Hasse–Weil L -function of E/\mathbf{Q} . Sarnak notes that, in the case of an elliptic curve without complex multiplication, the corresponding distribution may have infinite variance and concludes that Mazur's race should be unbiased. However, in the case of an elliptic curve with complex multiplication, even though an infinite number of L -functions is needed to understand Mazur's bias, Sarnak states that it would be possible to observe (and compute) an actual bias. In [CFJ16], Cha, Fiorilli and Jouve develop these ideas in the context of elliptic curves over function fields. The heuristics for Conjecture 1.1 (resp. 1.2) is the implementation of these ideas in the special case of the elliptic curve $y^2 = x^3 - x$ (resp. $y^2 = x^3 + x$) with complex multiplication by the ring of Gaussian integers $\mathbf{Z}[i]$.

In the spirit of [Dev20], the main result of this article is stated in greater generality and deals with the case of prime number races using infinitely many real analytic L -functions. Before stating our theorem, let us present some definitions and notation. Our main result states the existence of a limiting logarithmic distribution for a suitable normalization of a counting function, and thus allow us to discuss the bias in the distribution of the values of this function.

Definition 2.1. Let $F : [1, \infty) \rightarrow \mathbf{R}$ be a real function. We say that F admits a limiting logarithmic distribution if there exists a probability measure μ on the Borel sets in \mathbf{R} such that for any bounded Lipschitz continuous function g , we have

$$\lim_{Y \rightarrow \infty} \frac{1}{Y} \int_0^Y g(F(e^y)) dy = \int_{\mathbf{R}} g(t) d\mu(t).$$

If F admits a limiting logarithmic distribution μ , we say that $\mu([0, \infty))$ is the bias of F towards non-negative values.

Note that our definition of the bias differs from previous literature where it is usually defined as the logarithmic density of the set of x such that $F(x) > 0$. However, in the setting of Chebyshev's bias, it is expected (see [KR03], also [RS94] assuming the Linear Independence and [MN20, Dev20] under weaker hypotheses) that in general the distribution μ is continuous at 0 and then the two definition coincide. We thus consider it as a sufficiently good approximation.

2.2. Statement of the main result. In this paper we use the notion of *analytic L-function* defined in [Dev20, Def. 1.1]. Let us just recall that a vast majority of the L -functions used in analytic number theory (and all the L -functions referred to in this article) are proven (or at least conjectured) to be analytic L -functions. We are interested in the prime number race associated to a sequence $\mathcal{S} = \{L(f_m, \cdot) : m \geq 0\}$ of real analytic L -functions of degree d_m , and analytic conductor $\mathfrak{q}(f_m)$. To this sequence we associate real coefficients $\underline{c} = (c_m)_{m \geq 0}$ such that the series

$$\sum_{f_m \in \mathcal{S}} |c_m| d_m \log \mathfrak{q}(f_m)$$

is convergent. Moreover, we assume that the Generalized Riemann Hypothesis is true for all the L -functions $L(f_m, \cdot)$, $m \geq 0$ and their *second moment* (see [Dev20, Def. 1.1.(iii)]). That is to say that the non-trivial zeros of the functions $L(f_m, \cdot)$, $L(\text{Sym}^2 f_m, \cdot)$, and $L(\wedge^2 f_m, \cdot)$, $m \geq 0$ all have real part equal to $\frac{1}{2}$.

We note that, in this case, the difficulty compared to previous results is to deal with the fact that there can be an infinite number of zeros of the functions $L(f_m, \cdot)$, $m \geq 0$ with bounded imaginary part. Let us first fix the notation for the sets of zeros of the L -functions. For any $\rho \in \mathbf{C}$, we denote by $\text{ord}(\rho, m) = \text{ord}_{s=\rho}(L(f_m, s))$ the order of the zero at $s = \rho$ of the function $L(f_m, s)$. For $M \geq 0$, we denote

$$\text{ord}_{\mathcal{S}, \underline{c}, M}(\rho) = \sum_{m \leq M} c_m \text{ord}(\rho, m), \quad \text{ord}_{\mathcal{S}, \underline{c}}(\rho) = \sum_{m \geq 0} c_m \text{ord}(\rho, m).$$

One has (see e.g. [IK04, Prop. 5.7 (1)]),

$$\text{ord}(\rho, m) \ll \log(\mathfrak{q}(f_m)(|\rho| + 3)^{d_m}),$$

with an absolute implicit constant. Thus, the condition on the coefficients c_m ensures that the series defining $\text{ord}_{\mathcal{S}, \underline{c}}(\rho)$ is convergent for each ρ . When we assume the Generalized Riemann Hypothesis, we omit the $\frac{1}{2}$ in the notation and write for example, for $\gamma \in \mathbf{R}$, $\text{ord}_{\mathcal{S}, \underline{c}}(\gamma)$ instead of $\text{ord}_{\mathcal{S}, \underline{c}}(\frac{1}{2} + i\gamma)$. Then, for $M \geq 0$ and $T > 0$, we denote

$$\begin{aligned} \mathcal{Z}_{\mathcal{S}, \underline{c}} &= \{\gamma > 0 : \text{ord}_{\mathcal{S}, \underline{c}}(\gamma) \neq 0\}, & \mathcal{Z}_{\mathcal{S}, \underline{c}}(T) &= \mathcal{Z}_{\mathcal{S}, \underline{c}} \cap (0, T] \\ \mathcal{Z}_{\mathcal{S}, \underline{c}, M} &= \{\gamma > 0 : \text{ord}_{\mathcal{S}, \underline{c}, M}(\gamma) \neq 0\}, & \mathcal{Z}_{\mathcal{S}, \underline{c}, M}(T) &= \mathcal{Z}_{\mathcal{S}, \underline{c}, M} \cap (0, T], \end{aligned}$$

the sets of positive imaginary parts of zeros of the product of the L -functions in the family. We do not count with multiplicities in these sets.

We now have all the tools to state our main theorem.

Theorem 2.2. *Let $\mathcal{S} = \{L(f_m, \cdot) : m \geq 0\}$ be a sequence of real analytic L -functions of degree d_m and analytic conductor $\mathfrak{q}(f_m)$, and let $\underline{c} = (c_m)_{m \geq 0}$ be a sequence of real numbers such that the series*

$$\sum_{f_m \in \mathcal{S}} |c_m| d_m \log \mathfrak{q}(f_m)$$

is convergent. Assume the Riemann Hypothesis is satisfied for all $L(f_m, \cdot)$, $m \geq 0$. Then the function

$$E_{\mathcal{S}, \underline{c}}(x) := \frac{\log x}{\sqrt{x}} \left(\sum_{p \leq x} \sum_{m \geq 0} c_m \lambda_{f_m}(p) + \sum_{m \geq 0} c_m \text{ord}_{s=1}(L(f_m, s)) \text{Li}(x) \right)$$

admits a limiting logarithmic distribution $\mu_{\mathcal{S}, \underline{c}}$ with average value

$$\mathbb{E}(\mu_{\mathcal{S}, \underline{c}}) = m_{\mathcal{S}, \underline{c}} := \sum_{m \geq 0} c_m \left(\text{ord}_{s=1}(L(f_m^{(2)}, s)) - 2 \text{ord}_{s=1/2}(L(f_m, s)) \right)$$

and variance

$$\text{Var}(\mu_{\mathcal{S}, \underline{c}}) = 2 \sum_{\gamma \in \mathcal{Z}_{\mathcal{S}, \underline{c}}} \frac{|\text{ord}_{\mathcal{S}, \underline{c}}(\gamma)|^2}{\frac{1}{4} + \gamma^2}.$$

Moreover, there exists a constant $a > 0$ depending on \mathcal{S} and \underline{c} , such that we have

$$\mu_{\mathcal{S}, \underline{c}}((-\infty, -R) \cup (R, \infty)) \ll \exp(-a\sqrt{R}).$$

We prove this theorem in Section 6. A way to understand this statement, is to think that the function $y \mapsto E_{\mathcal{S}, \underline{c}}(e^y)$ takes its values with some probability law of mean value $m_{\mathcal{S}, \underline{c}}$. In general (see [Dev20, Cor. 2.4]), we expect the probability law to be symmetric with respect to its mean value, so we think of the mean value as a good indicator of the behaviour of $y \mapsto E_{\mathcal{S}, \underline{c}}(e^y)$, and the indication is more precise when the variance is relatively small.

In the case of [Sar07], E/\mathbf{Q} is an elliptic curve without complex multiplication, and we consider the L -functions $L(f_1, \cdot) = L(E, \cdot)$ and $L(f_m, \cdot) = L(\text{Sym}^m E, \cdot)$ for $m \geq 2$. The degree of each L -function is $d_m = m + 1$ and the analytic conductors satisfy $\log \mathfrak{q}(f_m) \ll m \log(m\mathfrak{q}(f_1))$ (see [Sar07, (ii)]). We observe that our condition on the convergence of $\sum_{f_m \in \mathcal{S}} c_m d_m \log \mathfrak{q}(f_m)$ corresponds to the condition stated in [CFJ16, Cor. 2.9] — precisely $c_m \ll m^{-3-\eta}$ for some $\eta > 0$ — to ensure the existence of the limiting distribution in the analogous question over function fields.

2.3. Signs changes. Most of the conditional results of [RS94, MN20, Dev20] giving more precisions on the properties of the limiting distribution $\mu_{\mathcal{S}, \underline{c}}$ can be adapted to this case. Let us present here some results concerning the support of $\mu_{\mathcal{S}, \underline{c}}$, as they can help to answer the question “does the function have infinitely many sign changes?”, and provide Omega-results. Similarly to [RS94, Th. 1.2], we have a lower bound for the tails of the distribution, in case all the coefficients have the same sign.

Proposition 2.3. *Let $\mathcal{S} = \{L(f_m, \cdot) : m \geq 0\}$ be a sequence of real analytic L -functions of degree d_m , and analytic conductor $\mathfrak{q}(f_m)$, and let $\underline{c} = (c_m)_{m \geq 0}$ be a sequence of non-negative real numbers such that the series*

$$\sum_{f \in \mathcal{S}} c_m d_m \log \mathfrak{q}(f_m)$$

is convergent, and at least one $c_m \neq 0$. Assume the Riemann Hypothesis is satisfied for all $L(f_m, \cdot)$, $m \geq 0$. Then, there exist a constant $b > 0$, depending on \mathcal{S} and \underline{c} , such that we have

$$\min(\mu_{\mathcal{S}, \underline{c}}((R, \infty)), \mu_{\mathcal{S}, \underline{c}}((-\infty, -R))) \gg \exp(-\exp bR).$$

As we may want to use coefficients of different signs, we now state a result inspired by [MN20, Th. 1.5(b and c)] and using the notion of *self-sufficient zero* that Martin and Ng introduced.

Definition 2.4. We say that an ordinate $\gamma \in \mathcal{Z}_{\mathcal{S}, \underline{c}}$ is self-sufficient if it is not in the \mathbf{Q} -span of $\mathcal{Z}_{\mathcal{S}, \underline{c}} \setminus \{\gamma\}$.

A priori, if there is no special reason for the imaginary parts of the zeros of some L -functions to be related, then we do not expect that there are any relation between them. This general idea is called the General Simplicity Hypothesis, or the Linear Independence hypothesis (LI). It is used in particular in [RS94] to give more precisions on the limiting distribution in the original case of Chebyshev's bias as well as to compute an explicit value of this bias. To state our next result, we do not need all the strength of this hypothesis. We show that if there are enough self-sufficient zeros in $\mathcal{Z}_{\mathcal{S}, \underline{c}}$ then the distribution $\mu_{\mathcal{S}, \underline{c}}$ is supported on all \mathbf{R} .

Proposition 2.5. *Let $\mathcal{S} = \{L(f_m, \cdot) : m \geq 0\}$ be a sequence of real analytic L -functions of degree d_m and analytic conductor $\mathfrak{q}(f_m)$, and let $\underline{c} = (c_m)_{m \geq 0}$ be a sequence of real numbers such that the series*

$$\sum_{f \in \mathcal{S}} |c_m| d_m \log \mathfrak{q}(f_m)$$

is convergent. Assume the Riemann Hypothesis is satisfied for all $L(f_m, \cdot)$, $m \geq 0$. Let $\mathcal{Z}_{\mathcal{S}, \underline{c}} = \{\gamma > 0 : \text{ord}_{\mathcal{S}, \underline{c}}(\gamma) \neq 0\}$ and $\mathcal{Z}_{\mathcal{S}, \underline{c}}^{\text{LI}}$ the set of self-sufficient elements in $\mathcal{Z}_{\mathcal{S}, \underline{c}}$. Assume that

$$\sum_{\gamma \in \mathcal{Z}_{\mathcal{S}, \underline{c}}^{\text{LI}}} \frac{|\text{ord}_{\mathcal{S}, \underline{c}}(\gamma)|}{\gamma}$$

diverges. Then $\text{supp}(\mu_{\mathcal{S}, \underline{c}}) = \mathbf{R}$.

In particular, under such conditions, we deduce that there cannot be a complete bias, that is to say that the function $E_{\mathcal{S}, \underline{c}}$ changes sign infinitely many times, and we obtain Omega-results.

Corollary 2.6. *Under the assumptions of Proposition 2.3 or of Proposition 2.5, we have*

$$\sum_{p \leq x} \sum_{m \geq 0} c_m \lambda_{f_m}(p) + \sum_{m \geq 0} c_m \text{ord}_{s=1}(L(f_m, s)) \text{Li}(x) = \Omega_{\pm}\left(\frac{\sqrt{x}}{\log x}\right).$$

In particular, the function $x \mapsto \sum_{p \leq x} \sum_{m \geq 0} c_m \lambda_{f_m}(p) + \sum_{m \geq 0} c_m \text{ord}_{s=1}(L(f_m, s)) \text{Li}(x)$ has infinitely many changes of signs.

Finally, we observe that in the case of a non-real¹ counter-example to the Generalized Riemann Hypothesis, oscillations results are more easily obtained thanks to Landau's Theorem.

Proposition 2.7. *Let $\mathcal{S} = \{L(f_m, \cdot) : m \geq 0\}$ be a sequence of real analytic L -functions of degree d_m and analytic conductor $\mathfrak{q}(f_m)$, and let $\underline{c} = (c_m)_{m \geq 0}$ be a sequence of real numbers such that the series*

$$\sum_{f \in \mathcal{S}} |c_m| d_m \log \mathfrak{q}(f_m)$$

is convergent. Let $\Theta = \sup\{\text{Re}(\rho) : \text{ord}_{\mathcal{S}, \underline{c}}(\rho) \neq 0\}$. Assume that $\Theta > \frac{1}{2}$ and that for each $m \geq 0$, one has $L(f_m, \Theta) \neq 0$. Then for any $\epsilon > 0$, we have

$$\sum_{p \leq x} \sum_{m \geq 0} c_m \lambda_{f_m}(p) + \sum_{m \geq 0} c_m \text{ord}_{s=1}(L(f_m, s)) \text{Li}(x) = \Omega_{\pm}(x^{\Theta - \epsilon}).$$

¹The case of a real zero of maximal real part would give only one direction of Omega-result as in [FM20, Lem. 2.1].

3. DISTRIBUTION OF THE ANGLES OF GAUSSIAN PRIMES

The representation of a prime number as a sum of two squares is explained in the ring of Gaussian integers $\mathbf{Z}[i]$ where the prime numbers $p \equiv 1 \pmod{4}$ split as $p = (a + i2b)(a - i2b)$, while the prime numbers $p \equiv 3 \pmod{4}$ are inert. The aim of Conjectures 1.1 and 1.2 is to understand fine statistics on the distribution of Gaussian primes in the plane. To do so, we study the distribution of the angles of the Gaussian primes which are defined as follows. For any $\mathfrak{p} \neq (1+i)$ prime ideal in $\mathbf{Z}[i]$, there exists a unique Gaussian integer $a+2ib \equiv 1 \pmod{(2+2i)}$ generating \mathfrak{p} (starting from $a, b \geq 0$ then either $a+2ib \equiv 1 \pmod{(2+2i)}$ or $-a-2ib \equiv 1 \pmod{(2+2i)}$). Then we say that the angle of \mathfrak{p} is the argument of this uniquely determined generator,

$$\theta_{\mathfrak{p}} = \arg(a + 2ib) \in (-\pi, \pi] \text{ where } \mathfrak{p} = (a + 2ib), \quad a + 2ib \equiv 1 \pmod{(2 + 2i)}.$$

Note that, if \mathfrak{p} is generated by $a + 2ib \equiv 1 \pmod{(2 + 2i)}$, then one also has $a - 2ib \equiv 1 \pmod{(2 + 2i)}$ and this latter Gaussian integer generates $\bar{\mathfrak{p}}$. So for a rational prime $p \equiv 1 \pmod{4}$, one can define its Gaussian angle up to its sign, $\theta_p = \pm\theta_{\mathfrak{p}} \in [0, \pi]$, where $\mathfrak{p} \mid p$. We observe also that this choice is natural : the number of \mathbf{F}_p points of the elliptic curve with affine model $y^2 = x^3 - x$ is exactly given by $p+1-2\sqrt{p}\cos(\theta_p)$. Note finally that with this definition, the natural angle associated to an inert prime $p \equiv 3 \pmod{4}$ is $\theta_p = \pi$.

Hecke proved in [Hec18] that the angles of Gaussian primes are equidistributed on the circle. In particular, for $0 \leq \alpha < \beta \leq \pi$, one has

$$\lim_{X \rightarrow \infty} \frac{|p \leq X : p \equiv 1 \pmod{4}, \alpha < \theta_p < \beta|}{|p \leq X : p \equiv 1 \pmod{4}|} = \frac{\beta - \alpha}{\pi}.$$

The limit depends only on the length of the interval. In the context of Conjecture 1.1, we are interested in the error term in this result, and in particular to show how it depends on the interval $[\alpha, \beta]$. Our main heuristic for this conjecture is the following result.

Theorem 3.1. *Assume the Generalized Riemann Hypothesis. Let ϕ be a 2π -periodic even function on \mathbf{R} , with Fourier coefficients $\hat{\phi}(m) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) e^{2i\pi mt} dt$ for $m \in \mathbf{Z}$, satisfying $\hat{\phi}(m) \ll |m|^{-1-\epsilon}$ for some $\epsilon > 0$. Then the function*

$$E_{\phi}(x) := \frac{\log x}{\sqrt{x}} \left(\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \phi(\theta_p) - \text{Li}(x) \int_0^{\pi} \phi(t) \frac{dt}{2\pi} \right)$$

admits a limiting logarithmic distribution μ_{ϕ} as $x \rightarrow \infty$.

Assume also that the vanishing at $s = \frac{1}{2}$ of the Hecke L-functions involved is exactly given by the sign of its functional equation. Then the average value of μ_{ϕ} is equal to

$$\frac{-\hat{\phi}(0)}{2} - \frac{\phi(0) + \phi(\pi)}{4} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{\cos(t)}{\cos(2t)} dt,$$

where the integral is understood as the Cauchy principal value.

Theorem 3.1 is just an ϵ away from Conjecture 1.1. Indeed, we would like to choose ϕ to be a step function but the m -th Fourier coefficient of such a function is of size $|m|^{-1}$.

However, as the function ϕ can be chosen very generally, it actually gives fine information on the distribution of the angles θ_p . Figure 3 shows the difference

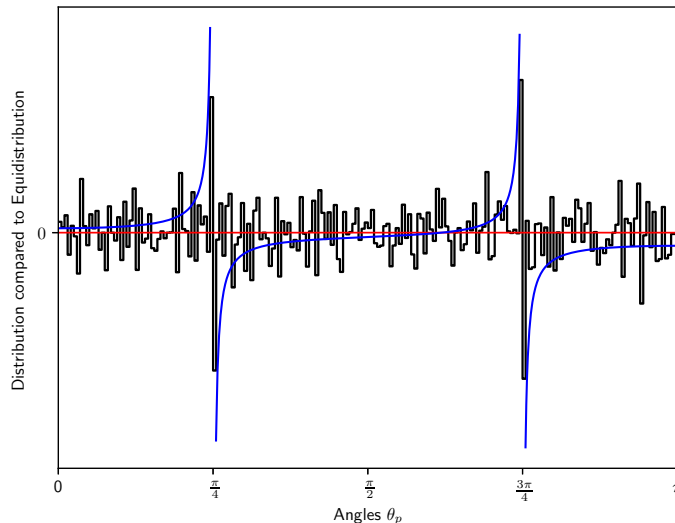


FIGURE 3. Relative distribution of the angles θ_p for $p \leq 10^8$: we count the number of angles θ_p in 200 subintervals of $[0, \pi]$ and withdraw the mean value; in red equidistribution; in blue the “secondary term” $\frac{\cos x}{\cos 2x} - \frac{1}{2}$.

between a histogram of the distribution of the angles θ_p and equidistribution. We observe some irregularities in the otherwise relatively well equidistributed behaviour happening at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ corresponding to the poles of $\frac{\cos(t)}{\cos(2t)}$. This irregularity might only be due to our “unfolding” of the angle θ_p (observe that the angle $2\theta_p$ or even $4\theta_p$ is more usually studied in the literature [Kub50, Col90, RW19] and does not exhibit such a bias). We note that our choice of $a + 2ib \equiv 1 \pmod{2 + 2i}$ imply that the only possibilities for $|a|$ and $|2b|$ to be very close to each other are $a = \pm 2b + 1$ while $a = \pm 2b - 1$ are excluded. This could explain the behaviour of the distribution around $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

In the case ϕ is π -periodic with mean value 0, we find that the mean value of μ_ϕ is equal to $\frac{-\phi(\pi)}{2}$, this indicates a bias in the distribution of the angles θ_p due to the fact that our sum defining E_ϕ does not include the inert primes $p \equiv 3 \pmod{4}$.

Note finally that this is a case where the function field analogue differs from the original question. Indeed, [PG20, Th. 1.8] shows that there is a bias in the distribution of the analogue of angles of Gaussian primes in function fields that is in the direction of sectors parametrized by non-squares. Such a phenomenon does not seem to appear here.

In the context of Conjecture 1.2, let us first note that for $p = a^2 + 4b^2$ with $a + 2ib \equiv 1 \pmod{2 + 2i}$ then $p \equiv 1 \pmod{8} \Leftrightarrow a \equiv 1 \pmod{4}$ and $p \equiv 5 \pmod{8} \Leftrightarrow a \equiv$

$-1 \pmod 4$, so we have

$$\begin{aligned} \sum_{p=a^2+4b^2 \leq x} \mathbf{1}_{|a| \equiv 1 \pmod 4}(p) - \mathbf{1}_{|a| \equiv -1 \pmod 4}(p) &= \sum_{\substack{p \leq x \\ p \equiv 1 \pmod 8}} \phi(\theta_p) - \sum_{\substack{p \leq x \\ p \equiv 5 \pmod 8}} \phi(\theta_p) \\ &= \sum_{\substack{p \leq x \\ p \equiv 1 \pmod 4}} \phi(\tilde{\theta}_p), \end{aligned}$$

where $\phi = \mathbf{1}_{(0, \frac{\pi}{2})} - \mathbf{1}_{(\frac{\pi}{2}, \pi)}$ (so that for $a \equiv 1 \pmod 4$ one counts $+1$ when a is positive, and -1 when it is negative; and the other way round for $a \equiv -1 \pmod 4$)

and we define $\tilde{\theta}_p = \begin{cases} \theta_p & \text{if } p \equiv 1 \pmod 8 \\ \pi - \theta_p & \text{if } p \equiv 5 \pmod 8 \end{cases}$. Similarly to Theorem 3.1, we have an analogous result in the case ϕ is just slightly smoother than a step function.

Theorem 3.2. *Assume the Generalized Riemann Hypothesis. Let ϕ be a 2π -periodic even function on \mathbf{R} with Fourier coefficients $\hat{\phi}(m) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) e^{2i\pi mt} dt$ for $m \in \mathbf{Z}$, satisfying $\hat{\phi}(m) \ll |m|^{-1-\epsilon}$ for some $\epsilon > 0$. Then the function*

$$F_{\phi}(x) := \frac{\log x}{\sqrt{x}} \left(\sum_{\substack{p \leq x \\ p \equiv 1 \pmod 8}} \phi(\theta_p) - \sum_{\substack{p \leq x \\ p \equiv 5 \pmod 8}} \phi(\theta_p) \right)$$

admits a limiting logarithmic distribution ν_{ϕ} as $x \rightarrow \infty$.

Assume also that the vanishing at $s = \frac{1}{2}$ of the Hecke L -functions involved is exactly given by the sign of its functional equation. Then the average value of ν_{ϕ} is equal to

$$\frac{-\hat{\phi}(0)}{2} - \frac{\phi(0) + \phi(\pi)}{4} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{1}{2 \cos(t)} dt,$$

where the integral is understood as the Cauchy principal value.

Taking ϕ such that $\phi(\pi - \theta) = -\phi(\theta)$ in Theorem 3.2 gives fine information on the distribution of the angles $\tilde{\theta}_p$. Figure 4 shows the difference between a histogram of the distribution of the angles θ_p and equidistribution. We observe some irregularity in the otherwise relatively well equidistributed behaviour happening at $\frac{\pi}{2}$ corresponding to the pole of $\frac{1}{\cos(t)}$. Note that along the prime numbers $p = 1 + 4b^2$ we have $\tilde{\theta}_p \rightarrow \frac{\pi}{2}^-$. The jump in the distribution goes well with the idea that there are infinitely many such prime numbers.

Theorem 3.1 and 3.2 both follow from the decomposition of the function ϕ in Fourier series and an application of Theorem 2.2 to a sequence of L -functions associated to the powers of a given Hecke character, with coefficients the Fourier coefficients of ϕ . We give the details on the precise Hecke characters in Section 4. We also comment there on the hypothesis that allows us to give a formula for the mean value of the limiting distribution.

4. L -FUNCTIONS OF HECKE CHARACTERS ON $\mathbf{Z}[i]$

Let us first review some properties of Hecke characters that will be useful in this paper. Our references for this section are mostly [Roh11], [Neu99, Chap. VII.6] and [IK04, Chap. 3.8, Chap. 5.10]. Let ξ be a unitary primitive Hecke character

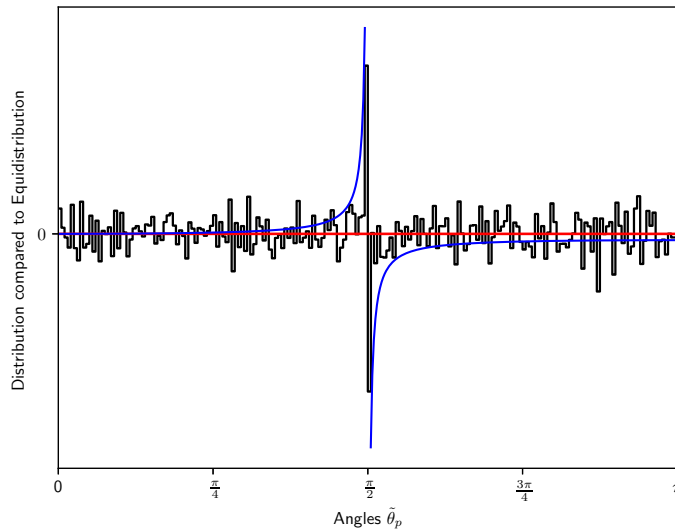


FIGURE 4. Relative distribution of the angles $\tilde{\theta}_p$ for $p \leq 10^8$: we count the number of angles $\tilde{\theta}_p$ in 200 subintervals of $[0, \pi]$ and withdraw the mean value; in red equidistribution; in blue the “secondary term” $\frac{1}{\cos x} - \frac{1}{2}$.

on $\mathbf{Z}[i]$ of conductor \mathfrak{f} and frequency ℓ , we define the Hecke L -function associated to ξ as the following Dirichlet series on the half-plane $\text{Re}(s) > 1$:

$$(1) \quad L(s, \xi) = \sum_{\mathfrak{n}} \xi(\mathfrak{n}) N(\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} (1 - \xi(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}$$

where \mathfrak{n} runs over all non-zero ideals of $\mathbf{Z}[i]$. Hecke proved that $L(s, \xi)$ extends meromorphically to the whole plane \mathbf{C} with at most a simple pole at $s = 1$ when ξ is the trivial character. Moreover, the completed L -function

$$\Lambda(s, \xi) := (4N\mathfrak{f})^{s/2} \pi^{-s - (|\ell|+1)/2} \Gamma\left(\frac{s + \frac{|\ell|}{2}}{2}\right) \Gamma\left(\frac{s + 1 + \frac{|\ell|}{2}}{2}\right) L(s, \xi),$$

admits a functional equation $\Lambda(\xi, s) = W(\xi) \Lambda(\bar{\xi}, 1 - s)$, where $W(\xi)$ is a complex number of modulus 1 (the sign of the functional equation) that can be given explicitly via a Gauss sum [IK04, (3.85)]. In the case $\xi(\bar{\mathfrak{p}}) = \bar{\xi}(\mathfrak{p})$, one has $\Lambda(\xi, s) = \Lambda(\bar{\xi}, s)$ and $W(\xi) = \pm 1$. Then, as it is believed to be true in 100% of the cases (see for example [Gre85] and [Wax21] for partial results in a family of L -functions associated to Hecke characters close to the ones we are considering in this paper), it is common to assume that the multiplicity of the zero at $s = \frac{1}{2}$ of $L(s, \xi)$ is determined by the sign of the functional equation. Precisely, in Theorem 3.1 and 3.2, to give a formula for the mean value of the limiting distribution, we assume that

$$\text{ord}_{s=1/2}(L(s, \xi)) = \frac{1 - W(\xi)}{2}$$

for each Hecke character ξ involved. With these notation and assumptions fixed, let us now prove Theorem 3.1.

Proof of Theorem 3.1 as a consequence of Theorem 2.2. Let us first write ϕ as a sum of its Fourier series. Since it is even of period 2π , one has $\phi(\theta) = \sum_{m \geq 0} c_m(\phi) \cos(m\theta)$ where the Fourier coefficients are

$$c_0(\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) dt$$

$$c_m(\phi) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(t) \cos(mt) dt, \text{ for } m \geq 1.$$

In equation (3), the definition of the angle of Gaussian primes is given so that the function $\mathfrak{p} \rightarrow e^{i\theta_{\mathfrak{p}}}$ comes from a Hecke character. Precisely, let $\mathfrak{m} = (2+2i)$, there are exactly 4 invertible elements in $\mathbf{Z}[i]/\mathfrak{m}$ corresponding to the 4 units of $\mathbf{Z}[i] : \pm 1, \pm i$. Thus, every ideal $\mathfrak{a} \subset \mathbf{Z}[i]$ co-prime to \mathfrak{m} has a unique generator $\alpha \equiv 1 \pmod{\mathfrak{m}}$. Let ξ be the Hecke character on the multiplicative groups of fractional ideals of $\mathbf{Z}[i]$ modulo \mathfrak{m} defined by

$$\xi((\alpha)) = \begin{cases} \frac{\alpha}{|\alpha|} & \text{if } \alpha \equiv 1 \pmod{\mathfrak{m}} \\ 0 & \text{if } (\alpha, \mathfrak{m}) \neq 1. \end{cases}$$

So that for any prime ideal \mathfrak{p} co-prime to 2, one has $\xi(\mathfrak{p}) = e^{i\theta_{\mathfrak{p}}}$. Then ξ is a unitary Hecke character of frequency 1, and conductor \mathfrak{m} , its finite part is the Dirichlet character $\chi : u \mapsto u^{-1}$ for $u \in \{\pm 1, \pm i\}$ representing the invertible congruence classes modulo \mathfrak{m} . Coming back to our decomposition, for any unramified splitting rational prime $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, one has, for any $m \in \mathbf{N}$

$$2 \cos(m\theta_p) = \xi^m(\mathfrak{p}) + \xi^m(\bar{\mathfrak{p}}).$$

Moreover, for any integer $m \geq 1$, the character ξ^m has frequency m and finite part $\chi(u) = u^{-m}$ for $u \in \{\pm 1, \pm i\}$ (again seen as elements in $\mathbf{Z}[i]/\mathfrak{m}$), it is primitive of conductor respectively $(2+2i)$ if m is odd, (2) if $m \equiv 2 \pmod{4}$ and non-primitive of conductor (1) if $m \equiv 0 \pmod{4}$. Let us denote ξ_m the primitive character associated to ξ^m ,

$$\xi^m = \begin{cases} \xi_m & \text{if } m \not\equiv 0 \pmod{4}, \\ \xi_m \chi_0 & \text{if } m \equiv 0 \pmod{4} \end{cases}$$

where χ_0 is the principal character modulo $(1+i)$. In particular,

$$\sum_{N\mathfrak{p} \leq x} \xi^m(\mathfrak{p}) = \sum_{N\mathfrak{p} \leq x} \xi_m(\mathfrak{p}) + O(1),$$

so we only loose a small error term in considering ξ_m instead of ξ^m .

Note that $L(s, \xi_0) = \zeta_{\mathbf{Q}(i)}(s)$ and for $m \geq 1$,

$$L(s, \xi_m) = (1 - \xi_m(1+i)2^{-s})^{-1} \prod_{p \equiv 1 \pmod{4}} (1 - 2 \cos(m\theta_p) p^{-s} + p^{-2s})^{-1}$$

$$\times \prod_{p \equiv 3 \pmod{4}} (1 - (-1)^m p^{-2s})^{-1}$$

are analytic L -functions in the sense of [Dev20, Def. 1.1] (see e.g. [IK04, 5.10]). Hence, we apply Theorem 2.2 with $\mathcal{S} = \mathcal{S}_1 = \{L(s, \xi_m) : m \geq 0\}$, and $\underline{c} = \{c_m(\phi) : m \geq 0\}$. Each of these L -functions has degree $d_m = 2$, and for $m \geq 1$ we have

$$q(\xi^m) \leq 4 \times N(2 + 2i)\left(\frac{m}{2} + 3\right)\left(\frac{m}{2} + 4\right) \ll m^2,$$

the hypothesis on the Fourier coefficients of ϕ is exactly here to ensure that the series $\sum_{m \geq 1} |c_m(\phi)| \log m^2$ is convergent.

Thus, we apply Theorem 2.2. Under the Riemann Hypothesis for $L(s, \xi_m)$, $m \geq 0$, the function

$$\begin{aligned} E_\phi(x) &= \frac{\log x}{2\sqrt{x}} \left(\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \sum_{m \geq 0} c_m(\phi) 2 \cos(m\theta_p) + \sum_{m \geq 0} c_m(\phi) \operatorname{ord}_{s=1}(L(s, \xi_m)) \operatorname{Li}(x) \right) \\ &= \frac{\log x}{\sqrt{x}} \left(\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{4}}} \phi(\theta_p) - \frac{c_0(\phi) \operatorname{Li}(x)}{2} \right) \end{aligned}$$

admits a limiting distribution μ_ϕ (where we used the fact that for all $m \geq 1$, the function $L(\cdot, \xi_m)$ does not have a pole nor a zero at $s = 1$). Moreover, Theorem 2.2 yields the following expression for the mean value of μ_ϕ :

$$\mathbb{E}(\mu_\phi) = \frac{1}{2} \sum_{m \geq 0} c_m(\phi) \left(\operatorname{ord}_{s=1}(L(s, \xi_m^{(2)})) - 2 \operatorname{ord}_{s=1/2}(L(s, \xi_m)) \right).$$

In the case $m = 0$, we have $L(s, \xi_0) = \zeta_{\mathbf{Q}(i)}(s) = \zeta(s)L(s, \chi_4)$. Up to the factor at 2, we have $L(s, \chi_4^{(2)}) = \zeta(s)$, thus, we get that $\operatorname{ord}_{s=1}(L(s, \xi_0^{(2)})) = -2$. For $m \geq 1$, we have that

$$\begin{aligned} L(s, \xi^{m(2)}) &= \prod_{p \equiv 1 \pmod{4}} (1 - e^{i2m\theta_p} p^{-s})^{-1} (1 - e^{-i2m\theta_p} p^{-s})^{-1} \prod_{p \equiv 3 \pmod{4}} (1 - (-1)^m p^{-s})^{-2} \\ &= \begin{cases} L(s, \xi^{2m}) \frac{\zeta(s)}{L(s, \chi_4)} (1 - 2^{-s}) & \text{for } m \text{ even,} \\ L(s, \xi^{2m}) \frac{L(s, \chi_4)}{\zeta(s)} (1 - 2^{-s})^{-1} & \text{for } m \text{ odd.} \end{cases} \end{aligned}$$

For each $m \geq 1$, $\operatorname{ord}_{s=1}(L(s, \xi^{2m})) = 0$, so, the function $L(s, \xi_m^{(2)})$ has a pole of order 1 at $s = 1$ when m is even and a zero of order 1 at $s = 1$ when m is odd. We have

$$\sum_{m \geq 0} c_m(\phi) \operatorname{ord}_{s=1}(L(s, \xi_m^{(2)})) = -c_0(\phi) - \sum_{m \geq 0} (-1)^m c_m(\phi) = -\hat{\phi}(0) - \phi(\pi).$$

Now, we need to determine the values of $\operatorname{ord}_{s=1/2}(L(s, \xi_m))$. The sign of the functional equation for $L(s, \xi_m)$ depends on the congruence class of m modulo 8. Precisely, the formula [IK04, 3.85] for the sign of the functional equation gives

$$W(\xi_m) = i^{-m} N(\mathfrak{f}_m)^{-\frac{1}{2}} \xi_{m, \infty}(\gamma_m) \sum_{x \in \mathbf{Z}[i]/\mathfrak{f}_m} \xi_{m, \text{fin}}(x) e^{2i\pi \operatorname{tr}\left(\frac{x}{\gamma_m}\right)}$$

where \mathfrak{f}_m is the conductor of ξ_m — that is $(2 + 2i)$ if m is odd, (2) if $m \equiv 2 \pmod{4}$ and (1) if $m \equiv 0 \pmod{4}$ —, $\xi_{m, \infty}$ and $\xi_{m, \text{fin}}$ are respectively the infinite and finite

part of ξ_m , and $\gamma_m \in \mathbf{Z}[i]$ is such that $(\gamma_m) = 2\mathfrak{f}_m$. In particular for odd m ,

$$\begin{aligned} W(\xi_m) &= i^{-m} 8^{-\frac{1}{2}} e^{i\pi \frac{m}{4}} \sum_{x \in \{\pm 1, \pm i\}} x^{-m} e^{2i\pi \operatorname{tr}(\frac{x}{4+4i})} \\ &= \frac{e^{-i\pi \frac{m}{4}}}{2\sqrt{2}} (2i - 2i^{m+1}) = \begin{cases} e^{i\pi \frac{1-m}{4}} & \text{if } m \equiv 1 \pmod{4}, \\ e^{i\pi \frac{3-m}{4}} & \text{if } m \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

For $m \equiv 2 \pmod{4}$, we have

$$W(\xi_m) = i^{-m} 4^{-\frac{1}{2}} \sum_{x \in \{1, i\}} x^{-m} e^{2i\pi \operatorname{tr}(\frac{x}{4})} = 1,$$

and for $m \equiv 0 \pmod{4}$,

$$W(\xi_m) = i^{-m} 1^{-\frac{1}{2}} e^{2i\pi \operatorname{tr}(\frac{1}{2})} = 1.$$

So that

$$W(\xi_m) = \begin{cases} 1 & \text{if } m \equiv 0, 1, 2, 3, 4, 6 \pmod{8}, \\ -1 & \text{if } m \equiv 5, 7 \pmod{8}. \end{cases}$$

Assuming that the sign of the functional equation determines the order of vanishing at $\frac{1}{2}$ for each L -function, it yields

$$\sum_{m \geq 0} c_m(\phi) \operatorname{ord}_{s=1/2}(L(s, \xi_m)) = \sum_{\substack{m \geq 0 \\ m \equiv 5, 7 \pmod{8}}} c_m(\phi).$$

We conclude the proof by writing this sum as an integral. Using Lemma 4.1 below, we have for smooth ϕ supported outside $\frac{\pi}{4}\mathbf{Z}$,

$$\begin{aligned} \sum_{\substack{m \geq 0 \\ m \equiv 5, 7 \pmod{8}}} (\hat{\phi}(m) + \hat{\phi}(-m)) &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) 2 \frac{\cos(t)}{\sin(4t)} (\sin((8N+10)t) - \sin(2t)) dt \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{\cos(t)}{\cos(2t)} dt. \end{aligned}$$

Moreover, the sum at $t \in \frac{\pi}{4}\mathbf{Z}$ gives the value $\frac{\phi(0) - \phi(\pi)}{4}$ which concludes the proof. \square

Lemma 4.1. *Let $q, a \in \mathbf{N}$, $q > 0$ and $t \notin \frac{2\pi}{q}\mathbf{Z}$. One has*

$$\sum_{m=0}^N (e^{i(qm+a)t} + e^{i(-qm-a)t}) = \frac{\sin((\frac{q}{2}(2N+1) + a)t) - \sin((a - \frac{q}{2})t)}{\sin(\frac{q}{2}t)}$$

Proof. For $t \notin \frac{2\pi}{q}\mathbf{Z}$ one has

$$\begin{aligned} \sum_{m=0}^N (e^{i(qm+a)t} + e^{i(-qm-a)t}) &= e^{iat} \frac{1 - e^{iq(N+1)t}}{1 - e^{iqt}} + e^{-iat} \frac{1 - e^{-iq(N+1)t}}{1 - e^{-iqt}} \\ &= 2 \cos((\frac{q}{2}N + a)t) \frac{\sin(\frac{q}{2}(N+1)t)}{\sin(\frac{q}{2}t)} \\ &= \frac{\sin((\frac{q}{2}(2N+1) + a)t) - \sin((a - \frac{q}{2})t)}{\sin(\frac{q}{2}t)}. \end{aligned}$$

□

The proof of Theorem 3.2 follows the same lines, we only give the details on the differences.

Proof of Theorem 3.2 as a consequence of Theorem 2.2. To separate the congruence classes of a , we will use characters of larger modulus. For each $m \geq 0$ let ψ_m be the Hecke character on the multiplicative groups of fractional ideals of $\mathbf{Z}[i]$ modulo (4) defined by

$$\psi_m((\alpha)) = \begin{cases} \left(\frac{\alpha}{|\alpha|}\right)^m & \text{if } \alpha \equiv 1 \pmod{4} \\ -\left(\frac{\alpha}{|\alpha|}\right)^m & \text{if } \alpha \equiv 3 + 2i \pmod{4} \\ 0 & \text{if } (\alpha, (4)) \neq 1. \end{cases}$$

Then ψ_m is a primitive unitary Hecke character of frequency m , and conductor (4), its finite part is the Dirichlet character

$$\psi_{m,\text{fin}} : u \mapsto \begin{cases} u^{-m} & \text{for } u \in \{\pm 1, \pm i\} \\ -\left(\frac{u}{3+2i}\right)^{-m} & \text{for } u \in \{3+2i, -3-2i, -2+3i, 2-3i\} \end{cases}$$

representing the 8 invertible congruence classes modulo (4). We have, for any unramified splitting rational prime $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, and for any $m \in \mathbf{N}$

$$\psi_m(\mathfrak{p}) + \psi_m(\bar{\mathfrak{p}}) = \begin{cases} 2 \cos(m\theta_p) & \text{if } p \equiv 1 \pmod{8} \\ -2 \cos(m\theta_p) & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

So that, for ϕ even and 2π -periodic, one has

$$\phi(\theta_p) \mathbf{1}_{p \equiv 1 \pmod{8}} - \phi(\theta_p) \mathbf{1}_{p \equiv 5 \pmod{8}} = \sum_{m \geq 0} c_m(\phi) (\psi_m(\mathfrak{p}) + \psi_m(\bar{\mathfrak{p}})).$$

We apply Theorem 2.2 with $\mathcal{S} = \mathcal{S}_2 = \{L(s, \psi_m) : m \geq 0\}$, each of these L -function has degree $d_m = 2$, and conductor $\mathfrak{q}(\psi_m) \ll (m+1)^2$ for $m \geq 0$, finally we take $c_m = c_m(\phi)$ for $m \geq 0$. This gives the existence of the limiting logarithmic distribution ν_ϕ under the Riemann Hypothesis for the $L(s, \psi_m)$, $m \geq 0$. Moreover, its mean value is given by

$$\mathbb{E}(\nu_\phi) = \frac{1}{2} \sum_{m \geq 0} c_m(\phi) \left(\text{ord}_{s=1}(L(s, \psi_m^{(2)})) - 2 \text{ord}_{s=1/2}(L(s, \psi_m)) \right).$$

Note that $L(s, \psi_0) = L(s, \chi)L(s, \chi')$, where χ and χ' are the primitive characters modulo 8 in particular, up to the factor at 2, $L(s, \psi_0^{(2)}) = \zeta^2(s)$ has a pole of order 2 at $s = 1$. For $m \geq 1$, we have

$$L(s, \psi_m) = \prod_{p \equiv 3 \pmod{4}} (1 - (-1)^m p^{-2s})^{-1} \prod_{p \equiv 1 \pmod{8}} (1 - 2 \cos(m\theta_p) p^{-s} + p^{-2s})^{-1} \\ \prod_{p \equiv 5 \pmod{8}} (1 + 2 \cos(m\theta_p) p^{-s} + p^{-2s})^{-1}.$$

Thus, we have

$$L(s, \psi_m^{(2)}) = L(s, \xi^{m(2)}) = \begin{cases} L(s, \xi^{2m}) \frac{\zeta(s)}{L(s, \chi_4)} (1 - 2^{-s}) & \text{for } m \text{ even,} \\ L(s, \xi^{2m}) \frac{L(s, \chi_4)}{\zeta(s)} (1 - 2^{-s})^{-1} & \text{for } m \text{ odd.} \end{cases}$$

The function $L(s, \psi_m^{(2)})$ has a pole of order 1 at $s = 1$ when m is even and a zero of order 1 at $s = 1$ when m is odd. We deduce

$$\sum_{m \geq 0} c_m(\phi) \operatorname{ord}_{s=1}(L(s, \psi_m^{(2)})) = -c_0(\phi) - \sum_{m \geq 0} (-1)^m c_m(\phi) = -\hat{\phi}(0) - \phi(\pi).$$

Now, we need to determine the values of $\operatorname{ord}_{s=1/2}(L(s, \psi_m))$. As before, we assume that it is enough to compute the sign of the functional equation. We have

$$\begin{aligned} W(\psi_m) &= i^{-m} N(4)^{-\frac{1}{2}} \psi_{m, \infty}(8) \sum_{x \in \mathbf{Z}[i]/(4)} \psi_{m, \text{fin}}(x) e^{2i\pi \operatorname{tr}(\frac{x}{8})} \\ &= \frac{i^{-m}}{4} \sum_{x \in \{\pm 1, \pm i\}} x^{-m} (e^{2i\pi \operatorname{tr}(\frac{x}{8})} - e^{2i\pi \operatorname{tr}(\frac{x(3+2i)}{8})}) \\ &= \frac{1}{2} ((1 - (-1)^m) i^{1-m} + (1 + (-1)^m)) \\ &= \begin{cases} 1 & \text{if } m \equiv 0 \pmod{2} \text{ or } m \equiv 1 \pmod{4}, \\ -1 & \text{if } m \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

This gives that the mean value of ν_ϕ is

$$\frac{-\hat{\phi}(0) - \phi(\pi)}{2} - \sum_{m \equiv 3 \pmod{4}} \hat{\phi}(m) + \hat{\phi}(-m).$$

We apply Lemma 4.1 to obtain for smooth ϕ supported outside $\frac{\pi}{2}\mathbf{Z}$,

$$\begin{aligned} \sum_{\substack{m \geq 0 \\ m \equiv 3 \pmod{4}}} (\hat{\phi}(m) + \hat{\phi}(-m)) &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{\sin((4N+5)t) - \sin(t)}{\sin(2t)} dt \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) \frac{1}{2 \cos(t)} dt. \end{aligned}$$

Then considering the sum at $t \in \frac{\pi}{2}\mathbf{Z}$, gives the value $\frac{\phi(0) - \phi(\pi)}{4}$ which concludes the proof of Theorem 3.2. \square

To conclude this section, let us come back to our hypothesis that the sign of the functional equation of an L -function determines its analytic rank. In the case of the L -functions used in this section, that is the $L(\cdot, \xi_m)$ and $L(\cdot, \psi_m)$ for $m \geq 1$, we first observe that they correspond to the L -functions of certain cusp forms of level a power of 2 (the norm of the conductor of the character multiplied by 4) and of weight $m + 1$. Those L -functions are referenced in the L -functions and Modular Forms Database [LMF21], at least for $m \leq 24$, where the analytic rank is calculated and satisfies our assumption.

5. HEURISTIC FOR THE CONJECTURES

Let us now develop our heuristic argument for Conjecture 1.1 and 1.2. The main idea is to replace ϕ in Theorems 3.1 and 3.2 by a difference of indicator function. Precisely, we take $\phi_1 = \mathbf{1}_{[0, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \pi]} - \mathbf{1}_{(\frac{\pi}{4}, \frac{3\pi}{4})}$ in Theorem 3.1 and $\phi_2 = \mathbf{1}_{[0, \frac{\pi}{2}]} - \mathbf{1}_{(\frac{\pi}{2}, \pi]}$ in Theorem 3.2, where the functions ϕ_i are defined on $[0, \pi]$ and we

extend their definition to \mathbf{R} so that they are even and 2π -periodic. Then, we have

$$c_m(\phi_1) = \begin{cases} \frac{8}{m\pi} & \text{if } m \equiv 2 \pmod{8} \\ -\frac{8}{m\pi} & \text{if } m \equiv 6 \pmod{8} \\ 0 & \text{otherwise} \end{cases} \text{ and } c_m(\phi_2) = \begin{cases} \frac{4}{m\pi} & \text{if } m \equiv 1 \pmod{4} \\ -\frac{4}{m\pi} & \text{if } m \equiv 3 \pmod{4} \\ 0 & \text{otherwise.} \end{cases}$$

As observed earlier, these Fourier coefficients do not satisfy the hypothesis of decay needed to apply Theorem 3.1 or 3.2. Let us however continue the heuristic argument by ignoring this.

Then we would obtain logarithmic limiting distributions μ_{ϕ_1} and ν_{ϕ_2} with mean values equal to

$$\mathbb{E}(\mu_{\phi_1}) = -\frac{\phi_1(0) + \phi_1(\pi)}{4} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_1(t) \frac{\cos(t)}{\cos(2t)} dt = \frac{-1}{2}$$

and $\mathbb{E}(\nu_{\phi_2}) = -\frac{\phi_2(0) + \phi_2(\pi)}{4} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_2(t) \frac{1}{2\cos(t)} dt = \infty.$

The variance of μ_{ϕ_1} is given by

$$\text{Var}(\mu_{\phi_1}) = 2 \sum_{\gamma \in \mathcal{Z}_{\mathcal{S}_1, \underline{c}}} \frac{|\sum_{m \geq 0} c_m(\phi_1) \text{ord}(\gamma, m)|^2}{\frac{1}{4} + \gamma^2} \in \mathbf{R} \cup \{\infty\},$$

where \mathcal{S}_1 is the set of Hecke L -functions used in the proof of Theorem 3.1, and $\underline{c} = \{c_m(\phi_1) : m \geq 0\}$. We obtain the same formula for $\text{Var}(\nu_{\phi_2})$ with ϕ_1 replaced by ϕ_2 and \mathcal{S}_1 replaced by \mathcal{S}_2 , the set of Hecke L -functions used in the proof of Theorem 3.2.

Let $\phi = \phi_1$ or ϕ_2 and $\mathcal{S} = \mathcal{S}_1$ or \mathcal{S}_2 . Let us assume that there exists $B > 0$ such that for all $\gamma \in \mathcal{Z}_{\mathcal{S}, \underline{c}}$, we have $|\sum_{m \geq 0} c_m(\phi) \text{ord}(\gamma, m)| < B \max\{|c_m(\phi) \text{ord}(\gamma, m)| : m \geq 0\}$. That is, we assume that for each $\gamma > 0$ there are not too many L -functions in \mathcal{S} that vanish at $\frac{1}{2} + i\gamma$. This hypothesis is reminiscent of the bounded multiplicity hypothesis used by Fiorilli in [Fio14a] and is supported by the general idea that zeros of L -functions should be independent while being weaker than the Linear Independence hypothesis. Then we have

$$\begin{aligned} \text{Var}(\mu_{\phi_1}) &< B^2 \sum_{m \geq 0} \sum_{\gamma \in \mathcal{Z}_m} \frac{|c_m(\phi) \text{ord}(\gamma, m)|^2}{\frac{1}{4} + \gamma^2} \\ &= B^2 \sum_{m \geq 0} |c_m(\phi)|^2 \sum_{\gamma \in \mathcal{Z}_m} \frac{|\text{ord}(\gamma, m)|^2}{\frac{1}{4} + \gamma^2}. \end{aligned}$$

Recall that

$$\sum_{\gamma \in \mathcal{Z}_m} \frac{|\text{ord}(\gamma, m)|^2}{\frac{1}{4} + \gamma^2} \ll (\log \mathfrak{q}_m)^3 \ll (\log m)^3,$$

and that $c_m(\phi) \ll \frac{1}{m}$. It yields $\text{Var}(\mu_{\phi_1}) < \infty$ and similarly, $\text{Var}(\nu_{\phi_2}) < \infty$.

This concludes our heuristic for Conjecture 1.1, we found a limiting logarithmic distribution with negative mean value and bounded variance. This indicates a bias towards negative values in the distribution of the values of the function D_1 . Moreover, if we assume that the set $\mathcal{Z}_{\mathcal{S}_1, \underline{c}}$ has many self-sufficient elements, as in Proposition 2.5 — assumption that is again supported by the fact that the zeros of L -functions should be independent — then we deduce the Omega-result with the aid of Corollary 2.6.

In the case of Conjecture 1.2, we obtained a limiting logarithmic distribution that has infinite mean value and bounded variance, this indicates a very strong bias in the direction of positive values. Let us approach this heuristic by another way and let us write $\phi_{2,N}(\theta) = \sum_{m \leq N} c_m(\phi_2) \cos(m\theta)$. By Chebyshev's inequality, as in [Fio14b, Lem. 2.10] (see also [Dev20, Cor. 5.8]) we have

$$\nu_{\phi_{2,N}}([0, \infty)) \geq 1 - \frac{\text{Var}(\mu_{\phi_{2,N}})}{\mathbb{E}(\nu_{\phi_{2,N}})} = 1 - O((\log N)^{-1}).$$

This heuristically indicates that, in the limit when $N \rightarrow \infty$ there is a complete bias, namely we expect $\nu_{\phi_2}([0, \infty)) = 1$, or in other terms, the function D_2 is almost always (in logarithmic scale) positive.

6. PROOF OF THEOREM 2.2

To prove Theorem 2.2, we follow the proof of [ANS14, Th. 1.2] or [Dev20, Th. 2.1], but keeping explicit the dependency on the L -function. We show that Theorem 2.2 is the consequence of the following result.

Proposition 6.1. *Under the hypotheses of Theorem 2.2, for each $M > 0$ and $T > 2$, let*

$$G_{\mathcal{S}, \underline{c}, M, T}(x) = m_{\mathcal{S}, \underline{c}} - \sum_{\gamma \in \mathcal{Z}_{\mathcal{S}, \underline{c}, M}(T)} 2 \operatorname{Re} \left(\operatorname{ord}_{\mathcal{S}, \underline{c}}(\gamma) \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} \right).$$

The function $G_{\mathcal{S}, \underline{c}, M, T}$ admits a limiting logarithmic distribution $\mu_{\mathcal{S}, \underline{c}, M, T}$. Moreover, there exists a function $M(T)$ satisfying $M(T) \rightarrow \infty$ as $T \rightarrow \infty$ such as for any bounded Lipschitz continuous function g , one has

$$\lim_{T \rightarrow \infty} \int_{\mathbf{R}} g(t) d\mu_{\mathcal{S}, \underline{c}, M(T), T}(t) = \int_{\mathbf{R}} g(t) d\mu_{\mathcal{S}, \underline{c}}(t).$$

For every M, T fixed, the set $\mathcal{Z}_{\mathcal{S}, \underline{c}, M}(T)$ is finite. Thus, by hypothesis, the function $G_{\mathcal{S}, \underline{c}, M, T}$ is well-defined and it admits a limiting logarithmic distribution as a consequence of Kronecker–Weyl equidistribution Theorem (see e.g. [Dev20, Th. 4.2], [Hum10, Lem. 4.3], or [MN20, Lem. B.3]). The convergence of the measures needs more work on the estimation of the error terms. Let us first recall the following precise form of [ANS14, Prop. 4.2], [Dev20, (4.5)].

Proposition 6.2. *Let $L(f, s)$ be an analytic L -function of degree d , we denote by $L(f^{(2)}, s)$ its second moment L -function (as in [Dev20, Def. 1.1.(iii)]). Assume the Riemann Hypothesis holds for $L(f, s)$ and $L(f^{(2)}, s)$. Let $T > 0$, and*

$$G_{f, T}(x) = m_f - \sum_{\gamma \in \mathcal{Z}_f(T)} 2 \operatorname{Re} \left(\operatorname{ord}(\gamma, L(f, s)) \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} \right).$$

We have the following estimate for all $x > 0$:

$$\begin{aligned} E_f(x) &:= \frac{\log x}{\sqrt{x}} \left(\sum_{p \leq x} \lambda_f(p) + \operatorname{ord}_{s=1}(L(f, s)) \operatorname{Li}(x) \right) \\ &= G_{f, T}(x) - \epsilon_f(x, T) + O \left(d \frac{\log \mathfrak{q}(f)}{\log x} \right) \end{aligned}$$

where the function $\epsilon_f(x, T)$ satisfies

$$(2) \quad \int_2^Y |\epsilon_f(e^y, T)|^2 dy \ll Y \frac{(\log(\mathfrak{q}(f)T^d))^2}{T} + \frac{(\log(\mathfrak{q}(f)T^d))^2 \log T}{T}$$

with an absolute implicit constant.

Proof. The proof is contained in [Dev20], where the dependency in the L -function is not always written explicitly. In particular from [Dev20, Prop. 4.4] we have

$$\psi(f, x) + \text{ord}_{s=1}(L(f, s))x = - \sum_{\substack{L(f, \rho)=0 \\ |\text{Im}(\rho)| \leq T}} \frac{x^\rho}{\rho} - x^{1/2} \epsilon_f(x, T) + O(\log(\mathfrak{q}(f)x^d) \log x)$$

with an absolute implicit constant. Then taking care of the sum over squares of primes, we use the Ramanujan–Petersson Conjecture and the Prime Number Theorem to obtain:

$$\theta(f, x) := \sum_{p \leq x} \lambda_f(p) \log p = \psi(f, x) - \sum_{p^2 \leq x} \left(\sum_{j=1}^d \alpha_j(p)^2 \right) \log p + O(dx^{1/3}).$$

To evaluate the second term, we use the Riemann Hypothesis for the function $L(f^{(2)}, s) = L(\text{Sym}^2 f, s)L(\wedge^2 f, s)^{-1}$. One has $\frac{L'(f^{(2)}, s)}{L(f^{(2)}, s)} = \frac{L'(\text{Sym}^2 f, s)}{L(\text{Sym}^2 f, s)} - \frac{L'(\wedge^2 f, s)}{L(\wedge^2 f, s)}$, thus

$$\begin{aligned} \sum_{p^2 \leq x} \left(\sum_{j=1}^d \alpha_j(p)^2 \right) \log p &= - \text{ord}_{s=1}(L(\text{Sym}^2 f, s))x^{\frac{1}{2}} + O(x^{\frac{1}{4}} \log x \log(x^{d(d+1)/2} \mathfrak{q}(\text{Sym}^2 f))) \\ &\quad + \text{ord}_{s=1}(L(\wedge^2 f, s))x^{\frac{1}{2}} + O(x^{\frac{1}{4}} \log x \log(x^{d(d-1)/2} \mathfrak{q}(\wedge^2 f))) + O(dx^{1/4}) \\ &= - \text{ord}_{s=1}(L(f^{(2)}, s))x^{\frac{1}{2}} + O(dx^{\frac{1}{4}} \log x \log(x^d \mathfrak{q}(f))). \end{aligned}$$

Finally, using Stieltjes integral, we write $E_f(x) = \frac{\log x}{x^{1/2}} \int_2^x \frac{d(\theta(f, t) + \text{ord}_{s=1}(L(f, s))t)}{\log t}$. After integration by parts this yields

$$\begin{aligned} E_f(x) &= \frac{1}{\sqrt{x}} (\psi(f, x) + x \text{ord}_{s=1}(L(f, s))) + \text{ord}_{s=1}(L(f^{(2)}, s)) \\ &\quad + O\left(\frac{\log x}{\sqrt{x}} \int_2^x \frac{\psi(f, t) + t \text{ord}_{s=1}(L(f, s)) + \sqrt{t} \text{ord}_{s=1}(L(f^{(2)}, s))}{t(\log t)^2} dt\right) \\ &\quad + O(dx^{-1/6} \log x \log(\mathfrak{q}(f)x^d)). \end{aligned}$$

Using the explicit formula

$$\begin{aligned} \psi(f, x) + \text{ord}_{s=1}(L(f, s))x &= - \sum_{\substack{L(f, \rho)=0 \\ |\text{Im}(\rho)| \leq X}} \frac{x^\rho}{\rho} \\ &\quad + O\left(d \log x + \frac{x}{X} (d(\log x)^2 + \log(\mathfrak{q}(f)X^d)) + \log(\mathfrak{q}(f)X^d) \log X\right), \end{aligned}$$

and another integration by parts to evaluate the second term we have

$$\begin{aligned} & \int_2^x \frac{\psi(f, t) + t \operatorname{ord}_{s=1}(L(f, s)) + \sqrt{t} \operatorname{ord}_{s=1}(L(f^{(2)}, s))}{t(\log t)^2} dt \\ & \ll \operatorname{ord}_{s=1}(L(f^{(2)}, s)) \frac{x^{1/2}}{(\log x)^2} + \sum_{\substack{L(f, \rho)=0 \\ |\operatorname{Im}(\rho)| \leq x}} \frac{|x^\rho|}{|\rho^2|(\log x)^2} + \log(\mathfrak{q}(f)x^d) \log x \end{aligned}$$

where after the integration we take $X = x$. The sum over the zeros is convergent and this concludes the proof of Proposition 6.2. \square

Then we sum over the L -functions, we obtain the following result.

Proposition 6.3. *Under the hypotheses of Theorem 2.2, there exists a function $M(T) = M_{\mathcal{S}, \underline{c}}(T)$, with $M(T) \rightarrow \infty$ as $T \rightarrow \infty$, such that we have the following estimate for all $x > 0$ and $T > 0$.*

$$E_{\mathcal{S}, \underline{c}}(x) = G_{\mathcal{S}, \underline{c}, M(T), T}(x) - \epsilon_{\mathcal{S}, \underline{c}}(x, T) + O_{\mathcal{S}, \underline{c}}\left(\frac{1}{\log x} + \frac{1}{T}\right),$$

where the function $\epsilon_{\mathcal{S}, \underline{c}}(x, T)$ satisfies

$$(3) \quad \int_2^Y |\epsilon_{\mathcal{S}, \underline{c}}(e^y, T)|^2 dy \ll_{\mathcal{S}, \underline{c}} Y \frac{(\log T)^2}{T} + \frac{(\log T)^3}{T}.$$

Proof of Proposition 6.3. By definition, one has

$$E_{\mathcal{S}, \underline{c}}(x) = \sum_{m \geq 0} c_m E_{f_m}(x).$$

Thus, using Proposition 6.2, one has

$$E_{\mathcal{S}, \underline{c}}(x) = \sum_{m=0}^{\infty} c_m \left(G_{f_m, T}(x) - \epsilon_{f_m}(x, T) + O\left(d_m \frac{\log \mathfrak{q}(f_m)}{\log x}\right) \right).$$

For each x and T , the three series are convergent, we separate

$$\begin{aligned} E_{\mathcal{S}, \underline{c}}(x) &= G_{\mathcal{S}, \underline{c}, M, T}(x) - \sum_{m > M} \left(c_m \sum_{\substack{\gamma \in \mathcal{Z}_{f_m}(T) \\ \gamma \notin \mathcal{Z}_{\mathcal{S}, \underline{c}, M}(T)}} 2 \operatorname{Re} \left(\operatorname{ord}(\gamma, m) \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} \right) \right) \\ &\quad - \sum_{m=0}^{\infty} c_m \epsilon_{f_m}(x, T) + O\left(\sum_{m=0}^{\infty} |c_m| d_m \frac{\log \mathfrak{q}(f_m)}{\log x} \right). \end{aligned}$$

For each $m > M$, one has

$$\sum_{\gamma \in \mathcal{Z}_{f_m}(T) \setminus \mathcal{Z}_{\mathcal{S}, \underline{c}, M}(T)} 2 \operatorname{Re} \left(\operatorname{ord}(\gamma, m) \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} \right) \ll \log T \log(\mathfrak{q}(f_m)T).$$

Thus

$$\sum_{m > M} c_m \sum_{\substack{\gamma \in \mathcal{Z}_{f_m}(T) \\ \gamma \notin \mathcal{Z}_{\mathcal{S}, \underline{c}, M}(T)}} 2 \operatorname{Re} \left(\operatorname{ord}(\gamma, m) \frac{x^{i\gamma}}{\frac{1}{2} + i\gamma} \right) \ll (\log T)^2 \sum_{m > M} |c_m| \log(\mathfrak{q}(f_m)).$$

For each T the series is convergent, so there exist $M = M_{\mathcal{S}, \underline{c}}(T)$ such that

$$\sum_{m>M} |c_m| \log(\mathfrak{q}(f_m)) \leq \frac{1}{(\log T)^2 T}.$$

Let

$$\epsilon_{\mathcal{S}, \underline{c}}(x, T) = \sum_{m=0}^{\infty} c_m \epsilon_{f_m}(x, T).$$

One has

$$\begin{aligned} \int_2^Y |\epsilon_{\mathcal{S}, \underline{c}}(e^y, T)|^2 dy &\leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_m| \cdot |c_n| \int_2^Y |\epsilon_{f_m}(e^y, T)| \cdot |\epsilon_{f_n}(e^y, T)| dy \\ &\ll \left\{ \sum_{m=0}^{\infty} |c_m| \left(Y \frac{(\log(\mathfrak{q}(f_m)T^{d_m}))^2}{T} + \frac{(\log(\mathfrak{q}(f_m)T^{d_m}))^2 \log T}{T} \right)^{1/2} \right\}^2 \\ (4) \quad &\ll_{\mathcal{S}, \underline{c}} Y \frac{(\log T)^2}{T} + \frac{(\log T)^3}{T}. \end{aligned}$$

Finally, since the series

$$\sum_{m \geq 0} |c_m| d_m \log \mathfrak{q}(f_m)$$

is convergent, the proof is complete. \square

We can now come back to the proof of Proposition 6.1.

Proof of Proposition 6.1. By Proposition 6.3, $E_{\mathcal{S}, \underline{c}}$ is a B^2 -almost periodic function well approximated by the $G_{\mathcal{S}, \underline{c}, M(T), T}$'s. Thus Proposition 6.1 follows from² [ANS14, Th. 2.9]. \square

Then Theorem 2.2 follows.

Proof of Theorem 2.2. The existence of the limiting logarithmic distribution $\mu_{\mathcal{S}, \underline{c}}$ is stated in Proposition 6.1. In the process of the proof, we used the fact that the function $E_{\mathcal{S}, \underline{c}}$ is a B^2 -almost periodic function, by [Bes55, Chap. II, §6, 4°] it admits a mean value which is

$$\mathbb{E}(\mu_{\mathcal{S}, \underline{c}}) = \lim_{T \rightarrow \infty} \mathbb{E}(\mu_{\mathcal{S}, \underline{c}, M(T), T}) = m_{\mathcal{S}, \underline{c}}.$$

Then it follows from [Bes55, Chap. II, §9, 1°] that it admits a second moment which is given via Parseval's identity. The formula for the variance follows as in [ANS14, Th. 1.14] and [Fio14b, Lem. 2.5, 2.6].

For the decay of the tails of the distribution, the proof is similar to the proof of [RS94, Th. 1.2] (see also [Dev20, Lem. 4.8]) noting that the measure $\mu_{\mathcal{S}, \underline{c}, M(T), T}$ is supported inside an interval of the form $[-A(\log T)^2, A(\log T)^2]$, for a positive constant A depending on \mathcal{S} and \underline{c} . \square

Let us now prove the results on the support of $\mu_{\mathcal{S}, \underline{c}}$ and on sign changes that depends on supplementary conditions.

²Note that there is a misprint in the proof of [ANS14, Th. 2.9], (2.10) should read $\frac{1}{Y} \int_0^Y |\vec{\phi}(y) - \vec{P}_M(y)| dy < \epsilon$ for Y large enough, the constant A_ϵ may have to be enlarged to include smaller Y 's, see also [Bai20, Th. 1.17], correcting this in more details.

Proof of Proposition 2.3. The proof is similar to the proof of [RS94, Th. 1.2], Following the notation of [RS94, Sec. 2.2], let $\epsilon > 0$, $t \geq \log 2 + \frac{1}{2}\epsilon$, and

$$F_\epsilon(t) = \frac{1}{\epsilon} \int_{t-\frac{\epsilon}{2}}^{t+\frac{\epsilon}{2}} E_{\mathcal{S},\underline{c}}(e^y) dy.$$

Using Proposition 6.3 and the bound

$$|\epsilon_{\mathcal{S},\underline{c}}(x, T)| \leq \sum_{m=0}^{\infty} |c_m \epsilon_{f_m}(x, T)| \ll \frac{\log x}{\sqrt{x}} + \frac{\sqrt{x}}{T} \left((\log x)^2 + \log T \right),$$

letting $T \rightarrow \infty$, we have

$$F_\epsilon(t) = \frac{4}{\epsilon} \sum_{\gamma \in \mathcal{Z}_{\mathcal{S},\underline{c}}} \text{ord}_{\mathcal{S},\underline{c}}(\gamma) \frac{\sin(t\gamma) \sin(\frac{\epsilon}{2}\gamma)}{\gamma^2} + O(1).$$

The sum $\sum_{\gamma \in \mathcal{Z}_{\mathcal{S},\underline{c}}} \frac{|\text{ord}_{\mathcal{S},\underline{c}}(\gamma)|}{\gamma^2}$ converges, so there exists $T = T(\epsilon)$ such that the function

$$\tilde{F}_\epsilon(t) = \frac{4}{\epsilon} \sum_{\gamma \in \mathcal{Z}_{\mathcal{S},\underline{c},M(T)}(T)} \text{ord}_{\mathcal{S},\underline{c}}(\gamma) \frac{\sin(t\gamma) \sin(\frac{\epsilon}{2}\gamma)}{\gamma^2}$$

satisfies

$$F_\epsilon(t) = \tilde{F}_\epsilon(t) + O(1).$$

It is then enough to show that $\tilde{F}_\epsilon(t)$ is large on a large set. As this is a finite sum, the proof follows from the same argument as in [RS94, Sec. 2.2], under the condition that $\text{ord}_{\mathcal{S},\underline{c}}(\gamma) \geq 0$ for all γ . \square

Proof of Proposition 2.5. The proof is similar to the proof of [MN20, Th. 1.5(c)]. Using [MN20, Lem. 3.8 and Prop. 3.10], we write $\mu_{\mathcal{S},\underline{c}} = \mu^{\text{LI}} * \mu^N$ where $\hat{\mu}^{\text{LI}}(\xi) =$

$$\prod_{\gamma \in \mathcal{Z}_{\mathcal{S},\underline{c}}^{\text{LI}}} J_0 \left(\left| \frac{2 \text{ord}_{\mathcal{S},\underline{c}}(\gamma) \xi}{\frac{1}{2} + i\gamma} \right| \right) \text{ and } \mu^N \text{ has positive mass in a small interval centred}$$

at 0. In particular the law of μ^{LI} is the same as the law of $\sum_{\gamma \in \mathcal{Z}_{\mathcal{S},\underline{c}}^{\text{LI}}} \left| \frac{2 \text{ord}_{\mathcal{S},\underline{c}}(\gamma)}{\frac{1}{2} + i\gamma} \right| X_\gamma$ where the X_γ are independent random variables each of which is uniformly distributed on the unit circle. Applying [MN20, Lem. 6.2] with the assumption $\sum_{\gamma \in \mathcal{Z}_{\mathcal{S},\underline{c}}^{\text{LI}}} \left| \frac{2 \text{ord}_{\mathcal{S},\underline{c}}(\gamma)}{\frac{1}{2} + i\gamma} \right| = \infty$, we conclude that $\text{supp}(\mu^{\text{LI}}) = \mathbf{R}$ and Proposition 2.5 follows. \square

Proof of Proposition 2.7. The idea of the proof is similar to the proof of [MV07, Th. 15.2] and is based on a theorem of Landau (see [MV07, Th. 15.1], also [KP86]). Fix $\epsilon > 0$, we consider the real functions

$$f_\pm : x \mapsto \sum_{n \leq x} \sum_{m \geq 0} c_m \frac{\Lambda_{f_m}(n)}{\log n} + \sum_{m \geq 0} c_m \text{ord}_{s=1}(L(f_m, s)) \text{Li}(x) \pm x^{\Theta - \epsilon},$$

where Θ is defined in the statement of Proposition 2.7 and for each $m \geq 0$, Λ_{f_m} is the von Mangoldt function associated to f_m . Precisely, one has

$$\Lambda_{f_m} = \begin{cases} \sum_{j=1}^{d_m} \alpha_{j,m}(p)^k \log p & \text{if } n = p^k \\ 0 & \text{if } n \text{ is not a prime power,} \end{cases}$$

where the $\alpha_{j,m}$ are the local roots of $L(f_m, \cdot)$. In particular, using the Ramanujan–Petersson Conjecture, the Prime Number Theorem and the fact that the series $\sum_{m \geq 0} |c_m| d_m$ converges, we see that the functions f_{\pm} are well-defined and we have that

$$(5) \quad f_{\pm}(x) = \frac{\sqrt{x}}{\log x} E_{S, \underline{c}}(x) \pm x^{\Theta - \epsilon} + O_{S, \underline{c}}(x^{\frac{1}{2}}) = O_{S, \underline{c}, \epsilon'}(x^{\Theta + \epsilon'}),$$

with $\epsilon' > 0$ arbitrarily small. For $\operatorname{Re}(s) > \Theta$, write

$$\begin{aligned} F_{\pm}(s) &= \int_1^{\infty} f_{\pm}(x) x^{-s-1} ds \\ &= \frac{1}{s} \sum_{m \geq 0} c_m \left(\log(L(f_m, s)) - \operatorname{ord}_{s=1}(L(f_m, s)) \log(s-1) \right) + \frac{r_{S, \underline{c}}(s)}{s} \pm \frac{1}{s - \Theta + \epsilon} \end{aligned}$$

where we used absolute convergence to exchange the order of summation between the integral and the sum over $m \geq 0$, and where the function $r_{S, \underline{c}}$ is entire. The second expression gives an analytic continuation of F_{\pm} to a larger set avoiding lines at the left of points $\beta + i\gamma$ with $\operatorname{ord}_{S, \underline{c}}(\beta + i\gamma) \neq 0$ where the functions have logarithmic singularities. In particular, by hypothesis, the functions F_{\pm} are regular at $s = \Theta$, but are not regular in any half-plane $\operatorname{Re}(s) > \Theta - \epsilon'$ with $\epsilon' > 0$. Landau's Theorem then implies that the functions f_{\pm} have infinitely many changes of signs. We deduce that there are infinitely many $x > 0$ such that $f_{-}(x) > 0$, using (5), we obtain

$$\sum_{p \leq x} \sum_{m \geq 0} c_m \lambda_{f_m}(p) + \sum_{m \geq 0} c_m \operatorname{ord}_{s=1}(L(f_m, s)) \operatorname{Li}(x) = \Omega_{+}(x^{\Theta - \epsilon}),$$

and similarly f_{+} takes negative values infinitely many times, which then yields the Ω_{-} -result and concludes the proof. \square

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