ERRATUM : "LIMITING PROPERTIES OF THE DISTRIBUTION OF PRIMES IN AN ARBITRARILY LARGE NUMBER OF RESIDUE CLASSES"

LUCILE DEVIN

Abstract. As pointed out by Alexandre Bailleul, the paper mentioned in the title contains a mistake in Theorem 2.2. The hypothesis on the linear relation of the almost periods is not sufficient. In this note we fix the problem and its minor consequences on other results in the same paper.

1. Corrected statement of Theorem 2.2, and related results

First recall some notations used in [De, §2].

Definition 1. Let $\mathcal{P}$ be a set of positive integers, the *natural density* $\mathrm{dens}(\mathcal{P})$ of $\mathcal{P}$ is given by

$$\mathrm{dens}(\mathcal{P}) = \lim_{X\to\infty} \frac{1}{X} \sum_{k\leq X} \mathbf{1}_{\mathcal{P}}(k)$$

if the limit exists, where $\mathbf{1}_{\mathcal{P}}$ is the indicator function of the set $\mathcal{P}$.

For $\gamma_1, \dots, \gamma_N \in \mathbf{R}$, we denote by $\langle \gamma_1, \dots, \gamma_N \rangle_{\mathbf{Q}}$ the $\mathbf{Q}$-vector space spanned by these elements.

The following theorem corrects the flawed hypothesis in [De, Th. 2.2].

Theorem 1. *Let $N \geq 1$ and let $\gamma_1, \dots, \gamma_N \in (0, \pi)$ be distinct real numbers such that $\pi \notin \langle \gamma_1, \dots, \gamma_N \rangle_{\mathbf{Q}}$. Let $D \geq 1$ and fix $\mathbf{c}_1, \dots, \mathbf{c}_N \in \mathbf{C}^D$. Let $F = (F_1, F_2, \dots, F_D) : \mathbf{N} \to \mathbf{R}^D$ be the function defined by*

$$F(k) = \sum_{n=1}^{N} \left(\mathbf{c}_n e^{ik\gamma_n} + \overline{\mathbf{c}_n} e^{-ik\gamma_n} \right) = \sum_{n=1}^{N} \left(\mathbf{a}_n \cos(\gamma_n k) + \mathbf{b}_n \sin(\gamma_n k) \right).$$

The image of F is contained in a compact subset of the subspace V_F of $\mathbf{R}^D$ generated by the vectors $\mathbf{a}_1, \dots, \mathbf{a}_N, \mathbf{b}_1, \dots, \mathbf{b}_N$.

Then, for every subspace $H \subset \mathbf{R}^D$ not containing V_F and every vector $\alpha \in \mathbf{R}^D$, one has $\mathrm{dens}(F \in \alpha + H) = 0$.

In particular, if $V_F \not\subset \bigcup_{d=1}^{D} \{x \in \mathbf{R}^D : x_d = 0\}$, then, for every $\alpha_1, \dots, \alpha_D \in \mathbf{R}$, one has $\mathrm{dens}(F_d = \alpha_d) = 0$ for all $1 \leq d \leq D$, and the density

$$\mathrm{dens}(F_1 > \alpha_1, F_2 > \alpha_2, \dots, F_D > \alpha_D)$$

exists.

Date: September 18, 2020.

Remark 1.
(1) Note that the difference with [De, Th. 2.2] lies in the hypothesis on the real numbers $\gamma_1, \dots, \gamma_N$: we need that $\pi \notin \langle \gamma_1, \dots, \gamma_N \rangle_{\mathbf{Q}}$. In particular, the hypothesis is stronger than the one in [De, Th. 2.2], but still weaker than the full Linear Independence.
(2) In the case $\pi \in \langle \gamma_1, \dots, \gamma_N \rangle_{\mathbf{Q}}$, Bailleul observed in [Ba, Th. 1.5] that the subtorus generated by $\gamma_1, \dots, \gamma_N$ over $\mathbf{Z}$ may not be connected, this is the reason of the gap in the proof of [De, Th. 2.2]. This is also the cause of the difference between the continuous case ([De, Th. 1.2]) and the discrete case. Indeed, the subtorus $\overline{\{(y\gamma_1, \dots, y\gamma_N) : y \in \mathbf{R}\}/(2\pi\mathbf{Z})^N}$ is connected unconditionally : it is the continuous image of the connected set $\mathbf{R}$.
This last remark corrects [De, Rem. 2.3.(i) and Rem. 3.7], note also that one should read $\mathbf{R}/2\pi\mathbf{Z}$ in [De, Rem. 2.3.(i)] instead of $\mathbf{Z}/2\pi\mathbf{Z}$.

The main application of this result is the analogue of Chebyshev's bias in rings of polynomials with coefficients in finite fields. Let us recall some definitions.

Definition 2. Let p^α be a prime power, and $Q \in \mathbf{F}_{p^\alpha}[t]$. Let $a_0, a_1, \dots, a_D \bmod Q$ be distinct invertible congruence classes, and let $\pi(k; Q, a_i)$ denote the number of irreducible polynomials in $\mathbf{F}_{p^\alpha}[t]$ with degree at most k that are congruent to $a_i \bmod Q$.

If, for each permutation σ, the set $\mathcal{P}_{Q;a_{\sigma(0)},a_{\sigma(1)},\dots,a_{\sigma(D)}} := \{k \in \mathbf{N} : \pi(k; Q, a_{\sigma(0)}) > \pi(k; Q, a_{\sigma(1)}) > \dots > \pi(k; Q, a_{\sigma(D)})\}$ admits a natural density, we say that the irreducible polynomial race is weakly inclusive. Moreover, if every set of the form $\{k \in \mathbf{N} : \pi(k; Q, a_i) = \pi(k; Q, a_j)\}$, $i \neq j$, has natural density equal to zero, we say that the ties have density zero.

The hypothesis in [De, Cor. 2.5] has to be changed accordingly.

Corollary 2. *Let p^α be a prime power, and $Q \in \mathbf{F}_{p^\alpha}[t]$. Suppose that there exists $M \geq 1$ and $\gamma_1, \dots, \gamma_M \in (0, \pi)$ such that $\pi \notin \langle \gamma_1, \dots, \gamma_M \rangle_{\mathbf{Q}}$, and such that for each character $\chi \bmod Q$, there exists $1 \leq m \leq M$ with $L(\frac{1}{2} + i\gamma_m, \chi) = 0$ but $L(\frac{1}{2} + i\gamma_m, \chi') \neq 0$ for $\chi' \neq \chi$.*

Then, every irreducible polynomial race in $\mathbf{F}_{p^\alpha}[t]$ modulo Q is weakly inclusive and the ties have density zero.

Finally, the hypothesis in Theorem 1 is now too strong to deduce [De, Cor. 2.6], we should thus consider this statement as not proved. Indeed, one needs to take into account all the zeros of the Dirichlet L-functions of the quadratic characters modulo $f(T)(T - u)$, but we only have information on the zeros of the Dirichlet L-functions of the primitive quadratic character.

2. Proof of the corrected statement

The key lemma is the following.

Lemma 3. *Let $N \geq 1$ and $\gamma_1, \dots, \gamma_N \in (0, \pi)$ be distinct real numbers such that $\pi \notin \langle \gamma_1, \dots, \gamma_N \rangle_{\mathbf{Q}}$. Then the sub-torus $A = \overline{\{(k\gamma_1, \dots, k\gamma_N) : k \in \mathbf{Z}\}/(2\pi\mathbf{Z})^N}$ is connected.*

Proof. We will show that $A = \overline{\{(y\gamma_1, \dots, y\gamma_N) : y \in \mathbf{R}\}/(2\pi\mathbf{Z})^N}$, then the conclusion follows from the fact that this sub-torus is connected. The first inclusion ($\subset$) is immediate.

Let $(e_1, \dots, e_d)$ be a basis of $\langle \gamma_1, \dots, \gamma_N \rangle_{\mathbf{Q}}$, such that for all $1 \le i \le N$, one has $\gamma_i = \sum_{k=1}^d g_{i,k} e_k$ with $g_{i,k} \in \mathbf{Z}$. By hypothesis, the set $\{2\pi, e_1, \dots, e_d\}$ is linearly independent over $\mathbf{Q}$, thus, by the discrete version of the Kronecker–Weyl Equidistribution Theorem (see e.g. [Ba, Th. 1.2]), one has $\overline{\{(ke_1, \dots, ke_d) : k \in \mathbf{Z}\}} = (\mathbf{R}/2\pi\mathbf{Z})^d$. In particular, for every $y \in \mathbf{R}$ and $\epsilon > 0$, there exists $\ell \in \mathbf{Z}$ and $m_1, \dots, m_d \in \mathbf{Z}$ such that $\max_{1 \le k \le d} |\ell e_k - y e_k - m_k 2\pi| < \epsilon / \max_{1 \le i \le N} \sum_{k=1}^d |g_{i,k}|$. Thus, for all $1 \le i \le N$, we have

$$|\ell\gamma_i - y\gamma_i - \sum_{k=1}^d g_{i,k} m_k 2\pi| \le \sum_{k=1}^d |g_{i,k}| \cdot |\ell e_k - y e_k - m_k 2\pi| < \epsilon.$$

Using the fact that $g_{i,k} \in \mathbf{Z}$, this shows that $y(\gamma_1, \dots, \gamma_N) \in A$, for all $y \in \mathbf{R}$, which concludes the proof. □

We can now give the proof of Theorem 1.

Proof of Theorem 1. We follow the proof from [De], the only flaw is in the proof of [De, Lem. 3.5].

Let $\mathbf{S}$ be the unit sphere of $\mathbf{R}^D$ and, let $f : (V \cap \mathbf{S}) \times A$ be the function defined by

$$f(\mathbf{u}, \theta) := \sum_{n=1}^N 2\,\mathrm{Re}(\langle \mathbf{u}, \mathbf{c}_n \rangle e^{i\theta_n}) = \sum_{n=1}^N \big(\langle \mathbf{u}, \mathbf{a}_n \rangle \cos\theta_n + \langle \mathbf{u}, \mathbf{b}_n \rangle \sin\theta_n\big), \tag{1}$$

this function is analytic in the variable θ.

Since the vectors $\mathbf{a}_1, \dots, \mathbf{a}_N, \mathbf{b}_1, \dots, \mathbf{b}_N$ span V, there exists at least one of them, say $\mathbf{v}_j$, such that $\langle \mathbf{u}, \mathbf{v}_j \rangle \ne 0$. So, the function $f(\mathbf{u}, \cdot)$ is a linear combination of the $2N$ characters of A defined by $(\theta_1, \dots, \theta_N) \mapsto e^{i\theta_n}$, $(\theta_1, \dots, \theta_N) \mapsto e^{-i\theta_n}$, $1 \le n \le N$, with at least one non-zero coefficient. Note that these $2N$ characters are all distinct, and also distinct from the trivial character $(\theta_1, \dots, \theta_N) \mapsto 1$. This follows from the fact that the values of the derivative of these functions restricted to $\{(y\gamma_1, \dots, y\gamma_N) : y \in \mathbf{R}\}/(2\pi\mathbf{Z})^N \subset A$ at $y = 0$ are respectively $\gamma_n, -\gamma_n$, $1 \le n \le N$, and 0 which are distinct. Thus by a result of Dedekind–Artin ([La, VI, Th. 4.1]), those characters are linearly independent and the function $f(\mathbf{u}, \cdot)$ is not constant on A. Moreover, Lemma 3 implies that A is connected, and the fact that $N > 0$ and $\gamma_1 \ne 0$ ensures that it is not restricted to $\{0\}$, so that a non-constant analytic function on A is indeed non-locally constant.

The rest of the proof follows similarly to [De]. □

Acknowledgments

I thank Alexandre Bailleul for his careful reading of [De] and for helpful discussions that led to this correction, as well as the referee for their useful remarks that improved the text.

References

[Ba] Alexandre Bailleul, *Explicit Kronecker-Weyl theorems and applications to prime number races.* preprint, July 2020. http://arxiv.org/abs/2007.05763.

[De] Lucile Devin, *Limiting Properties of the Distribution of Primes in an Arbitrarily Large Number of Residue Classes*. Canadian Mathematical Bulletin, 2020, 1–13.
[La] Serge Lang, *Algebra*, Third ed., Graduate Texts in Mathematics, Vol. 211, Springer-Verlag, New York, 2002. https://doi.org/10.1007/978-1-4613-0041-0

Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, SE-412 96 Gothenburg, Sweden
Email address: devin@chalmers.se