

EXTENDING THE UNCONDITIONAL SUPPORT IN AN IWANIEC–LUO–SARNAK FAMILY

LUCILE DEVIN, DANIEL FIORILLI AND ANDERS SÖDERGREN

ABSTRACT. We study the harmonically weighted one-level density of low-lying zeros of L -functions in the family of holomorphic newforms of fixed even weight k and prime level N tending to infinity. For this family, Iwaniec, Luo and Sarnak proved that the Katz–Sarnak prediction for the one-level density holds unconditionally when the support of the Fourier transform of the implied test function is contained in $(-\frac{3}{2}, \frac{3}{2})$. In this paper, we extend this admissible support to $(-\Theta_k, \Theta_k)$, where $\Theta_2 = 1.866\dots$ and Θ_k tends monotonically to 2 as k tends to infinity. This is asymptotically as good as the best known GRH result. The main novelty in our analysis is the use of zero-density estimates for Dirichlet L -functions.

1. INTRODUCTION

Since the seminal paper of Iwaniec, Luo and Sarnak [ILS] on low-lying zeros of families of L -functions attached to holomorphic modular forms, much work has been done towards the Katz–Sarnak heuristics [KS] (see for instance [SST] and the references therein). In the family of cusp forms of fixed weight k and large level N , Iwaniec, Luo and Sarnak have proven that the Katz–Sarnak conjecture for the one-level density holds for test functions ϕ for which $\text{supp}(\hat{\phi}) \subset (-\frac{3}{2}, \frac{3}{2})$ unconditionally, and $\text{supp}(\hat{\phi}) \subset (-2, 2)$ under the relevant Riemann Hypothesis. These results have been refined by Miller [Mil] and Miller–Montague [MM] to estimates containing lower-order terms, under the same assumptions. In the current paper, we extend the admissible support of the test function for the harmonically weighted one-level density: we show unconditionally that the Katz–Sarnak conjecture holds whenever $\text{supp}(\hat{\phi}) \subset (-\Theta_k, \Theta_k)$, where Θ_k is a monotonically increasing function of k such that $\Theta_2 = 1.866\dots$, $\Theta_4 = 1.942\dots$, and $0 < 2 - \Theta_k \ll \frac{1}{k}$ for $k \geq 6$.

Before we state our main result, we need to introduce some notations. We fix a basis $B_k^*(N)$ of Hecke eigenforms of the space $H_k^*(N)$ of newforms of prime level N and weight k . We also renormalize so that for every $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \in B_k^*(N)$, we have $\lambda_f(1) = 1$. We will use the harmonic weights defined as

$$\omega_f(N) := \frac{\Gamma(k-1)}{(4\pi)^{k-1} (f, f)_N}; \quad (f, f)_N := \int_{\Gamma_0(N) \backslash \mathbb{H}} y^{k-2} |f(z)|^2 dx dy.$$

Note that (combine [ILS, Lemma 2.5] and [HL, I]) $(kN)^{-1-\varepsilon} \ll_{\varepsilon} \omega_f(N) \ll_{\varepsilon} (kN)^{-1+\varepsilon}$. Our main object of study is the one-level density, which is defined as

$$\mathcal{D}_{k,N}^*(\phi; X) := \frac{1}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log X}{2\pi}\right),$$

where $\rho_f = \frac{1}{2} + i\gamma_f$ runs through the non-trivial zeros of $L(s, f)$ (note that γ_f might be non-real), ϕ is an even Schwartz function whose Fourier transform is compactly supported, the parameter

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$X = k^2N$ is the analytic conductor of f , and the total weight is given by

$$\Omega_k(N) = \sum_{f \in B_k^*(N)} \omega_f(N).$$

We have the estimate

$$\Omega_k(N) = 1 + O_k(N^{-1})$$

(see Lemma 2.4). We can now state our main result.

Theorem 1.1. *Let ϕ be an even Schwartz function for which $\text{supp}(\widehat{\phi}) \subset (-\Theta_k, \Theta_k)$, where*

$$\Theta_k := \begin{cases} 1 + \frac{\sqrt{3}}{2} & \text{if } k = 2; \\ 2\left(1 - \frac{1}{10k-5}\right) & \text{if } k \geq 4. \end{cases} \quad (1)$$

Then, for N running through the set of prime numbers, we have the estimate

$$\mathcal{D}_{k,N}^*(\phi; X) = \int_{\mathbb{R}} W(O)(x)\phi(x)dx + o_{N \rightarrow \infty}(1), \quad (2)$$

where $W(O)(x) = 1 + \frac{1}{2}\delta_0(x)$.

Remark 1.2.

- In the literature on low-lying zeros, there are already several unconditional results with large admissible supports. We mention the result of Drappeau–Pratt–Radziwiłł [DPR] in the family of Dirichlet L -functions and that of Fouvry–Kowalski–Michel [FKM] in the family of symmetric square L -functions of holomorphic cusp forms of prime level.
- The use of zero-density estimates in order to circumvent the assumption of the Riemann Hypothesis in the context of low-lying zeros is an idea which was first hinted by Brumer [B], and subsequently successfully used by Kowalski–Michel [KM] in families of holomorphic cusp forms and Baier–Zhao [BZ] in families of elliptic curve L -functions. The novelty in our approach is to use such estimates after an application of the Petersson formula, which relates low-lying zeros of cusp form L -functions to sums over zeros of Dirichlet L -functions, for which powerful zero-density estimates have been known for almost a century (see e.g. [L, Mo]).
- Our techniques also yield unconditional extensions of the admissible support in families of Maass forms L -functions in the level aspect. These results will appear in a forthcoming paper.

Here is a brief summary of the proof of Theorem 1.1. In Section 2, we apply the explicit formula and express the one-level density $\mathcal{D}_{k,N}^*(\phi; X)$ as a sum over eigenvalues of Hecke operators at prime power values. Averaging over the family of newforms of prime level, we apply the Petersson formula and turn this last expression into a sum of Kloosterman sums weighted by Bessel functions. In Section 3, we rewrite the Kloosterman sums in terms of Dirichlet characters and Gauss sums using orthogonality. This last expression allows us to apply Mellin inversion and use a variant of the explicit formula for Dirichlet L -functions. Finally, in Section 4 we complete the proof of Theorem 1.1 using zero-density estimates. It is the shape of these zero-density estimates which gives the exact restriction on the support which appears in Theorem 1.1. Note that if one assumes the Grand Density Conjecture [IK, p. 250], then one can prove that the Katz–Sarnak prediction holds in the full range $\text{supp}(\widehat{\phi}) \subset (-2, 2)$, independently of k (see Remark 4.2).

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2. PREREQUISITES AND FIRST ESTIMATES

The goal of this section is to gather a few identities and estimates which will be central in our analysis, including the explicit and Petersson formulas. We will then bound some of the terms in these formulas and reduce our problem to estimates on averages of Kloosterman sums.

We begin with the explicit formula.

Lemma 2.1 (Explicit Formula). *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be an even Schwartz function whose Fourier transform has compact support. Then for $X > 1$ we have the formula*

$$\begin{aligned} \mathcal{D}_{k,N}^*(\phi; X) &= \widehat{\phi}(0) \frac{\log \frac{N}{\pi^2}}{\log X} + \frac{1}{\log X} \int_{\mathbb{R}} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{k+1}{4} + \frac{\pi it}{\log X} \right) + \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{k-1}{4} + \frac{\pi it}{\log X} \right) \right) \phi(t) dt \\ &\quad - \frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{p, \nu} \frac{\alpha_f^\nu(p) + \beta_f^\nu(p)}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log X} \right) \frac{\log p}{\log X}. \end{aligned} \quad (3)$$

Here, $\alpha_f(p)$ and $\beta_f(p)$ are the local coefficients of the L -function

$$L(s, f) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s} \right)^{-1};$$

in particular, for $p \nmid N$ we have that $|\alpha_f(p)| = |\beta_f(p)| = 1$, and for $p \mid N$ we use the convention that $\beta_f(p) = 0$.

Proof. For $f \in B_k^*(N)$, the formula¹ [ILS, (4.11)] reads

$$\begin{aligned} \sum_{\gamma_f} \phi \left(\gamma_f \frac{\log X}{2\pi} \right) &= \widehat{\phi}(0) \frac{\log \frac{N}{\pi^2}}{\log X} + \frac{1}{\log X} \int_{\mathbb{R}} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{k+1}{4} + \frac{\pi it}{\log X} \right) + \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{k-1}{4} + \frac{\pi it}{\log X} \right) \right) \phi(t) dt \\ &\quad - 2 \sum_{p, \nu} \frac{\alpha_f^\nu(p) + \beta_f^\nu(p)}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log X} \right) \frac{\log p}{\log X}. \end{aligned} \quad (4)$$

Next, we sum against the weight $\omega_f(N)$ to obtain the desired formula. \square

We now introduce the Petersson formula, which involves the Kloosterman sum

$$S(m, n; c) := \sum_{x \bmod c}^* e \left(\frac{mx + n\bar{x}}{c} \right)$$

and the Bessel functions J_k .

Lemma 2.2. *Let $k \in \mathbb{N}$. We have the bound*

$$J_{k-1}(x) \ll \min \left(\frac{1}{(k-1)!} \left(\frac{x}{2} \right)^{k-1}, x^{-\frac{1}{4}} (|x - k + 1| + k^{\frac{1}{3}})^{-\frac{1}{4}} \right).$$

Proof. This is [ILS, (2.11)', (2.11)"], which follows immediately from bounds in [W, K], see [DFS, Lemma 3.2]. \square

¹We fixed a minor typo in the arguments of the gamma factors.

Lemma 2.3 (Petersson Formula). *Assuming that N is prime, $(m, N) = 1$, $(n, N^2) \mid N$, and k is an even integer, we have the formula*

$$\begin{aligned} & \sum_{f \in B_k^*(N)} \omega_f(N) \lambda_f(m) \lambda_f(n) \\ &= \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 \pmod N} c^{-1} S(m, n; c) J_{k-1}(4\pi\sqrt{mn}/c) + O_\varepsilon\left((N\nu((n, N)))^{-1}(kmn)^\varepsilon\right), \end{aligned} \quad (5)$$

where $\nu(t) := t \prod_{p \mid t} (1 + \frac{1}{p})$.

Proof. We combine [ILS, Proposition 2.8] with [ILS, Proposition 2.1], in the case where N is prime. In their notation, thanks to [ILS, Lemma 2.5], we have that

$$\begin{aligned} & \sum_{f \in B_k^*(N)} \omega_f(N) \lambda_f(m) \lambda_f(n) = \frac{12}{(k-1)N} \Delta_{k,N}^*(m, n) \\ &= \sum_{LM=N} \frac{\mu(L)}{L\nu((n, L))} \sum_{\ell \mid L^\infty} \ell^{-1} \Delta_{k,M}(\ell^2 m, n) \\ &= \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 \pmod N} c^{-1} S(m, n; c) J_{k-1}(4\pi\sqrt{mn}/c) - \frac{1}{N\nu((n, N))} \sum_{\ell \mid N^\infty} \ell^{-1} \Delta_{k,1}(\ell^2 m, n). \end{aligned} \quad (6)$$

The first equality above is a restatement of the definition of $\Delta_{k,N}^*(m, n)$ [ILS, (2.53), (2.54)], and moreover we recall (see [ILS, (2.7)]) that $\Delta_{k,N}(m, n) = \sum_{f \in B_k(N)} \omega_f(N) \lambda_f(m) \lambda_f(n)$ denotes the sum over all forms in an orthogonal basis $B_k(N)$ of the space of cusp forms of level N and weight k (including the old-forms). To estimate the contribution of the last term on the right-hand side of (6), we use Deligne's bound $\lambda_f(n) \ll_\varepsilon n^\varepsilon$, which combined with the bounds $|B_k(1)| \ll k$ and $\omega_f(N) \ll_\varepsilon (kN)^{-1+\varepsilon}$ yields the bound $\Delta_{k,1}(\ell^2 m, n) = O_\varepsilon((\ell mnk)^\varepsilon)$, resulting in the claimed estimate. \square

We continue by estimating the sum over c in Lemma 2.3.

Lemma 2.4 (Estimated Petersson Formula). *Let k be a fixed even integer. If N is prime, $N^2 \nmid n$ and $(m, N) = 1$, we have*

$$\sum_{f \in B_k^*(N)} \omega_f(N) \lambda_f(m) \lambda_f(n) = \delta(m, n) + O_{k,\varepsilon}\left((n, N)^{-\frac{1}{2}} N^{-1+\varepsilon} (mn)^{\frac{1}{4}+\varepsilon}\right).$$

If in addition $mn \leq N^2$, then we have the bound

$$\sum_{f \in B_k^*(N)} \omega_f(N) \lambda_f(m) \lambda_f(n) = \delta(m, n) + O_{k,\varepsilon}\left((N(n, N))^{-1} (mn)^\varepsilon + (n, N)^{-\frac{1}{2}} N^{-k+\frac{1}{2}+\varepsilon} (mn)^{\frac{k-1}{2}+\varepsilon}\right).$$

Proof. By Lemma 2.3, as N is prime, we have

$$\begin{aligned} \sum_{f \in B_k^*(N)} \omega_f(N) \lambda_f(m) \lambda_f(n) &= \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 \pmod N} c^{-1} S(m, n; c) J_{k-1}(4\pi\sqrt{mn}/c) \\ &\quad + O_{k,\varepsilon}\left((N(n, N))^{-1} (mn)^\varepsilon\right). \end{aligned}$$

Using Weil's bound in the form [ILS, (2.13)], together with Lemma 2.2, we have

$$\begin{aligned} \sum_{c \equiv 0 \pmod N} c^{-1} S(m, n; c) J_{k-1}(4\pi\sqrt{mn}/c) &\ll_{\varepsilon, k} \sum_{c \equiv 0 \pmod N} c^{-\frac{1}{2}+\varepsilon} \left(\frac{\sqrt{mn}}{c}\right)^{k-1} \frac{(m, n, c)}{(m, c)^{\frac{1}{2}} + (n, c)^{\frac{1}{2}}} \\ &\ll \frac{(m, n, N)}{(m, N)^{\frac{1}{2}} + (n, N)^{\frac{1}{2}}} N^{-k+\frac{1}{2}+\varepsilon} (mn)^{\frac{k-1}{2}} \sum_{b \geq 1} (m, n, b)^{\frac{1}{2}} b^{\frac{1}{2}-k+\varepsilon} \\ &\ll_{\varepsilon} \frac{(m, n, N)}{(m, N)^{\frac{1}{2}} + (n, N)^{\frac{1}{2}}} N^{-k+\frac{1}{2}+\varepsilon} (mn)^{\frac{k-1}{2}+\varepsilon}, \end{aligned}$$

which concludes the proof in the case $mn \leq N^2$. In case $N^2 < mn$, we cut the sum over c and apply Weil's bound:

$$\begin{aligned} &\sum_{c \equiv 0 \pmod N} c^{-1} S(m, n; c) J_{k-1}(4\pi\sqrt{mn}/c) \\ &\ll_{\varepsilon, k} \sum_{\substack{c \equiv 0 \pmod N \\ c > \sqrt{mn}}} c^{-\frac{1}{2}+\varepsilon} \left(\frac{\sqrt{mn}}{c}\right)^{k-1} \frac{(m, n, c)}{(m, c)^{\frac{1}{2}} + (n, c)^{\frac{1}{2}}} + \sum_{\substack{c \equiv 0 \pmod N \\ c < \sqrt{mn}}} c^{-\frac{1}{2}+\varepsilon} \left(\frac{\sqrt{mn}}{c}\right)^{-\frac{1}{2}} \frac{(m, n, c)}{(m, c)^{\frac{1}{2}} + (n, c)^{\frac{1}{2}}} \\ &\ll \frac{(m, n, N)}{(m, N)^{\frac{1}{2}} + (n, N)^{\frac{1}{2}}} N^{-1+\varepsilon} (mn)^{\frac{1}{4}+\varepsilon}, \end{aligned}$$

which gives the desired bound. \square

We now apply the Hecke relations and discard certain prime powers from the expression (3).

Lemma 2.5. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\text{supp}(\widehat{\phi})) < \infty$. For N running through the set of prime numbers, we have that*

$$\begin{aligned} \mathcal{D}_{k, N}^*(\phi; X) &= \widehat{\phi}(0) \frac{\log \frac{N}{\pi^2}}{\log X} + \frac{1}{\log X} \int_{\mathbb{R}} \left(\frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{k+1}{4} + \frac{\pi it}{\log X} \right) + \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{k-1}{4} + \frac{\pi it}{\log X} \right) \right) \phi(t) dt \\ &\quad + 2 \sum_{p \nmid N} \frac{1}{p} \widehat{\phi} \left(\frac{2 \log p}{\log X} \right) \frac{\log p}{\log X} - \frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{\lambda_f(p^\nu)}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log X} \right) \frac{\log p}{\log X} + O_{k, \varepsilon}(N^{-1+\varepsilon}). \end{aligned} \tag{7}$$

Proof. Note that if $p = N$ then $\lambda_f(p^\nu) = \lambda_f(p)^\nu$, and moreover $\lambda_f(p) = O(p^{-\frac{1}{2}})$ [Miy, Theorem 4.6.17], so the contribution of this term to the prime sum in (3) is $O_k(N^{-1})$. Next, by the Hecke relations, the sum over prime numbers not dividing N in (3) is equal to

$$\begin{aligned} &-\frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p, \nu \\ p \nmid N}} \frac{\lambda_f(p^\nu)}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log X} \right) \frac{\log p}{\log X} \\ &+ \frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p, \nu \geq 2 \\ p \nmid N}} \frac{\lambda_f(p^{\nu-2})}{p^{\nu/2}} \widehat{\phi} \left(\frac{\nu \log p}{\log X} \right) \frac{\log p}{\log X}. \end{aligned}$$

The contribution of the summands with $\nu = 2$ in the second term are given by

$$\frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{p \nmid N} \frac{\lambda_f(1)}{p} \widehat{\phi} \left(\frac{2 \log p}{\log X} \right) \frac{\log p}{\log X} = 2 \sum_{p \nmid N} \frac{1}{p} \widehat{\phi} \left(\frac{2 \log p}{\log X} \right) \frac{\log p}{\log X}.$$

The goal is now to estimate the contribution of the remaining terms, that is the terms with $\nu \geq 3$ and $p \neq N$. From Lemma 2.4 and the prime number theorem, we see that

$$\begin{aligned}
\frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p \nmid N \\ \nu \geq 3}} \frac{\lambda_f(p^{\nu-2})}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} \\
\ll_{k,\varepsilon} N^{-1+\varepsilon} \sum_{p \leq X^{\sigma/3}} \frac{\log p}{p^{\frac{1}{2}+2\varepsilon}} \sum_{3 \leq \nu \leq \sigma \frac{\log X}{\log p}} p^{(-\frac{1}{4}+\varepsilon)\nu} \\
\ll N^{-1+\varepsilon} \sum_{p \leq X^{\sigma/3}} \frac{\log p}{p^{\frac{5}{4}-\varepsilon}} \ll N^{-1+\varepsilon},
\end{aligned}$$

which gives the claimed result. \square

We are now ready to apply the Petersson formula.

Corollary 2.6. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\text{supp}(\widehat{\phi})) < 2$. Then, for N running through the set of prime numbers, we have that*

$$\begin{aligned}
\mathcal{D}_{k,N}^*(\phi; X) &= \widehat{\phi}(0) + \frac{\phi(0)}{2} \\
&\quad - \frac{4\pi i^k}{\Omega_k(N)} \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{1}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} \sum_{c \equiv 0 \pmod N} c^{-1} S(p^\nu, 1; c) J_{k-1}(4\pi\sqrt{p^\nu}/c) + O_k\left(\frac{1}{\log X}\right). \quad (8)
\end{aligned}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
& - \frac{2}{\Omega_k(N)} \sum_{f \in B_k^*(N)} \omega_f(N) \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{\lambda_f(p^\nu)}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} \\
&= - \frac{4\pi i^k}{\Omega_k(N)} \sum_{\substack{p \nmid N \\ \nu \geq 1}} \frac{1}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} \sum_{c \equiv 0 \pmod N} c^{-1} S(p^\nu, 1; c) J_{k-1}(4\pi\sqrt{p^\nu}/c) + O_{k,\varepsilon}(N^{\frac{\sigma}{2}-1+\varepsilon}),
\end{aligned}$$

which, combined with Lemma 2.5 and [DFS, Lemmas 2.2 and 4.1], gives the desired result. \square

3. THE PRIME SUM IN TERMS OF ZEROS OF DIRICHLET L -FUNCTIONS

In the previous section we reduced the problem of estimating the one-level density to that of bounding sums over primes containing averages of Kloosterman sums. We will now evaluate these averages using Dirichlet characters, which will allow us to relate our problem to zeros of Dirichlet L -functions.

Lemma 3.1. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\text{supp}(\widehat{\phi})) < 2$. For N running through the set of prime numbers, we have that*

$$\begin{aligned}
\mathcal{D}_{k,N}^*(\phi; X) &= \widehat{\phi}(0) + \frac{\phi(0)}{2} \\
& - \frac{4\pi i^k}{\Omega_k(N) \log X} \sum_{\substack{c \equiv 0 \pmod N \\ c < N^{1+\frac{1}{2k-3}}}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \pmod c \\ \chi \neq \chi_0}} \tau(\bar{\chi})^2 \sum_{\substack{p \nmid c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c) + O_k\left(\frac{1}{\log X}\right). \quad (9)
\end{aligned}$$

Proof. For $p \nmid c$, the Kloosterman sums satisfy

$$S(p^\nu, 1; c) = \frac{1}{\varphi(c)} \sum_{a \bmod c} S(a, 1; c) \sum_{\chi \bmod c} \bar{\chi}(a) \chi(p^\nu); \quad \sum_{a \bmod c} \bar{\chi}(a) S(a, 1; c) = \tau(\bar{\chi})^2.$$

We will substitute these expressions in Corollary 2.6. Let $C = N^{1+\frac{1}{2k-3}}$. Bounding the contribution of the primes dividing c using [ILS, (2.13)] and Lemma 2.2, the sum on the right-hand side of (8) is equal to

$$\begin{aligned} & -\frac{4\pi i^k}{\Omega_k(N)} \sum_{c \equiv 0 \pmod N} c^{-1} \sum_{\substack{p|c \\ \nu \geq 1}} \frac{1}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} S(p^\nu, 1; c) J_{k-1}(4\pi\sqrt{p^\nu}/c) + O_{k,\varepsilon}(N^{(\sigma-2)\frac{k-2}{2}-\frac{3}{2}+\varepsilon}) \\ & = -\frac{4\pi i^k}{\Omega_k(N)} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} c^{-1} \sum_{\substack{p|c \\ \nu \geq 1}} \frac{1}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \frac{\log p}{\log X} S(p^\nu, 1; c) J_{k-1}(4\pi\sqrt{p^\nu}/c) + O_{k,\varepsilon}(N^{-\frac{3}{2}+\varepsilon} + N^{\frac{k}{2}(\sigma-2)+\varepsilon}) \\ & = -\frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{a \bmod c} \sum_{\chi \bmod c} \bar{\chi}(a) S(a, 1; c) \sum_{\substack{p|c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c) \\ & \quad + O_{k,\varepsilon}(N^{-\frac{3}{2}+\varepsilon} + N^{\frac{k}{2}(\sigma-2)+\varepsilon}) \\ & = -\frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \bmod c \\ \chi \neq \chi_0}} \tau(\bar{\chi})^2 \sum_{\substack{p|c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c) \\ & \quad + O_{k,\varepsilon}(N^{-\frac{3}{2}+\varepsilon} + N^{\frac{k}{2}(\sigma-2)+\varepsilon} + N^{k(\frac{\sigma}{2}-1)-1+\varepsilon}), \end{aligned}$$

where the error terms account for the contributions of terms with $p \mid c$, terms with $c \geq C$, and the principal character. \square

In the next lemma we express the third term on the right-hand side of (9) as a contour integral.

Lemma 3.2. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\text{supp}(\widehat{\phi})) < 2$. For N running through the set of prime numbers, we have that*

$$\begin{aligned} \mathcal{D}_{k,N}^*(\phi; X) &= \widehat{\phi}(0) + \frac{\phi(0)}{2} + O_k\left(\frac{1}{\log X}\right) \\ & \quad + O_k\left(\sum_{\substack{c \equiv 0 \pmod N \\ c < N^{1+\frac{1}{2k-3}}} \frac{1}{c\varphi(c)} \sum_{1 \neq d|c} d \sum_{\chi \bmod d}^* \left| \int_{(2)} \frac{L'(s+\frac{1}{2}, \chi)}{L(s+\frac{1}{2}, \chi)} \Psi_{\phi, X, c, k}(s) ds \right| \right), \end{aligned} \quad (10)$$

where the star over the sum means that we are summing over primitive characters, and where

$$\Psi_{\phi, X, c, k}(s) = \frac{1}{\log X} \int_0^\infty x^{s-1} \widehat{\phi}\left(\frac{\log x}{\log X}\right) J_{k-1}(4\pi\sqrt{x}/c) dx. \quad (11)$$

Proof. We will once more use the shorthand $C = N^{1+\frac{1}{2k-3}}$. Note that the sum on the right-hand side of (9) is equal to

$$-\frac{4\pi i^k}{\Omega_k(N)} \frac{1}{\log X} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \bmod c \\ \chi \neq \chi_0}} \tau(\bar{\chi})^2 \sum_{\substack{p|c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi^*(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c),$$

where χ^* is the primitive character modulo c^* inducing χ . Hence, this sum is

$$\begin{aligned}
&\ll_k \frac{1}{\log X} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{\substack{\chi \pmod c \\ \chi \neq \chi_0}} c^* \left| \sum_{\substack{p|c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi^*(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c) \right| \\
&= \frac{1}{\log X} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{1 \neq d|c} d \sum_{\chi \pmod d}^* \left| \sum_{\substack{p|c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c) \right|.
\end{aligned}$$

The next step will be to apply Mellin inversion to the sum inside the absolute values. Doing so and interchanging the sum and integral thanks to absolute convergence (see Lemma 3.3 below), we obtain the identity

$$\begin{aligned}
\frac{1}{\log X} \sum_{\substack{p|c \\ \nu \geq 1}} \frac{\log p}{p^{\nu/2}} \widehat{\phi}\left(\frac{\nu \log p}{\log X}\right) \chi(p^\nu) J_{k-1}(4\pi\sqrt{p^\nu}/c) &= -\frac{1}{2\pi i} \int_{(2)} \frac{L'(s + \frac{1}{2}, \chi)}{L(s + \frac{1}{2}, \chi)} \Psi_{\phi, X, c, k}(s) ds \\
&+ O_\varepsilon\left(\frac{N^{\sigma(\frac{k}{2}-1)+\varepsilon}}{c^{k-1}}\right).
\end{aligned}$$

The claimed estimate follows. \square

We now establish analytic properties of the function $\Psi_{\phi, X, c, k}(s)$.

Lemma 3.3. *The function $\Psi_{\phi, X, c, k}(s)$ defined in (11) is entire and satisfies the bound*

$$\Psi_{\phi, X, c, k}(s) \ll_k \frac{X^{\sigma|\Re(s) + \frac{k-1}{2}|}}{(|s| + 1)^2 c^{k-1}}.$$

Proof. The fact that $\widehat{\phi}$ has compact support immediately implies that $\Psi_{\phi, X, c, k}(s)$ is entire. Assume now that $|s| > 1$. We change variables $u = \frac{\log x}{\log X}$ in the definition of $\Psi_{\phi, X, c, k}(s)$ and then integrate by parts twice:

$$\begin{aligned}
\Psi_{\phi, X, c, k}(s) &= \int_{\mathbb{R}} X^{us} \widehat{\phi}(u) J_{k-1}(4\pi X^{\frac{u}{2}}/c) du \\
&= - \int_{\mathbb{R}} \frac{X^{us}}{s \log X} \left(\widehat{\phi}'(u) J_{k-1}(4\pi X^{\frac{u}{2}}/c) + \widehat{\phi}(u) \frac{2\pi}{c} X^{\frac{u}{2}} \log X J'_{k-1}(4\pi X^{\frac{u}{2}}/c) \right) du \\
&= \int_{\mathbb{R}} \frac{X^{us}}{(s \log X)^2} \left(\widehat{\phi}''(u) J_{k-1}(4\pi X^{\frac{u}{2}}/c) + \widehat{\phi}'(u) \frac{4\pi}{c} X^{\frac{u}{2}} \log X J'_{k-1}(4\pi X^{\frac{u}{2}}/c) \right. \\
&\quad \left. + \widehat{\phi}(u) \frac{\pi}{c} X^{\frac{u}{2}} (\log X)^2 J'_{k-1}(4\pi X^{\frac{u}{2}}/c) + \widehat{\phi}(u) \frac{4\pi^2}{c^2} X^u (\log X)^2 J''_{k-1}(4\pi X^{\frac{u}{2}}/c) \right) du.
\end{aligned}$$

By the identity

$$2J'_k(x) = J_{k-1}(x) - J_{k+1}(x)$$

(see [W, equation (2) page 17]) and Lemma 2.2, we deduce the bound²

$$\Psi_{\phi, X, c, k}(s) \ll_k \int_{-\sigma}^{\sigma} \frac{X^{u\Re(s)}}{|s|^2} \left(\frac{X^{\frac{u}{2}}}{c}\right)^{k-1} du \ll \frac{X^{\sigma|\Re(s) + \frac{k-1}{2}|}}{|s|^2 c^{k-1}}.$$

Finally, for $|s| \leq 1$ the bound $\Psi_{\phi, X, c, k}(s) \ll \frac{X^{\sigma|\Re(s) + \frac{k-1}{2}|}}{c^{k-1}}$ follows directly from the definition. \square

The next step will be to move the contour of integration to the left in the integral appearing in (10).

²Note that in the case $k = 2$, we also use the identity $J_{-1}(x) = -J_1(x)$.

Proposition 3.4. *Let ϕ be an even Schwartz test function for which $\sigma = \sup(\text{supp}(\widehat{\phi})) < 2$. For $k \geq 2$ a fixed even integer, and for N running through the set of prime numbers, we have*

$$\begin{aligned} \mathcal{D}_{k,N}^*(\phi; X) &= \widehat{\phi}(0) + \frac{\phi(0)}{2} + O_k\left(\frac{1}{\log X}\right) \\ &+ O_{k,\varepsilon}\left(N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N \leq D < N^{1+\frac{1}{2k-3}}} \sup_{\frac{1}{2} \leq \beta < 1} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} \sum_{\substack{d \sim D \\ d \equiv 0 \pmod N}} \sum_{\chi \pmod d}^* \sum_{\substack{\rho_\chi \\ \beta \leq \Re(\rho_\chi) < \beta + \frac{1}{\log N} \\ T-1 \leq |\Im(\rho_\chi)| < 2T}} 1\right), \end{aligned} \quad (12)$$

where ρ_χ is running over the non-trivial zeros of the Dirichlet L -function $L(s, \chi)$ and $d \sim D$ means $\frac{D}{2} < d \leq D$.

Proof. For an integer $d > 1$, we let $\chi \pmod d$ be a primitive character. We will pull the contour of integration to the left in the integral

$$\frac{1}{2\pi i} \int_{(2)} \frac{L'(s + \frac{1}{2}, \chi)}{L(s + \frac{1}{2}, \chi)} \Psi_{\phi, X, c, k}(s) ds,$$

which appears in Lemma 3.2. Moreover, following [D, §19] and applying Lemma 3.3, this integral is easily shown to be equal to

$$\sum_{\rho_\chi} \Psi_{\phi, X, c, k}(\rho_\chi - \frac{1}{2}) + \Psi_{\phi, X, c, k}(-\frac{1}{2}) 1_{\chi(-1)=1} + \frac{1}{2\pi i} \int_{(-1)} \frac{L'(s + \frac{1}{2}, \chi)}{L(s + \frac{1}{2}, \chi)} \Psi_{\phi, X, c, k}(s) ds.$$

Arguing as in [D, §19], we can show that on the line $\Re(s) = -1$ we have the bound

$$\frac{L'(s + \frac{1}{2}, \chi)}{L(s + \frac{1}{2}, \chi)} \ll \log(d|s|).$$

Combining this with Lemma 3.3, we deduce that

$$\frac{1}{2\pi i} \int_{(-1)} \frac{L'(s + \frac{1}{2}, \chi)}{L(s + \frac{1}{2}, \chi)} \Psi_{\phi, X, c, k}(s) ds \ll_k \int_{(-1)} \log(d|s|) \frac{X^{\sigma|\frac{k-3}{2}|}}{|s|^{2c^{k-1}}} ds \ll \frac{X^{\sigma|\frac{k-3}{2}|}}{c^{k-1}} \log d.$$

The total contribution of this term to the second error term in (10) is $\ll_{k,\varepsilon} N^{\sigma|\frac{k-3}{2}|-k+1+\varepsilon}$ (recall that $X = k^2 N$), which is admissible for $\sigma < 2$ and any even $k \geq 2$. Using Lemma 3.3, we see that the contribution of the terms $\Psi_{\phi, X, c, k}(-\frac{1}{2})$ is $O_{k,\varepsilon}(N^{\sigma(\frac{k}{2}-1)-k+1+\varepsilon})$ which is also admissible. We are left with the term involving non-trivial zeros of $L(s, \chi)$ whose contribution to the second error term in (10) is (we set $C = N^{1+\frac{1}{2k-3}}$)

$$\begin{aligned} &\ll_k \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c\varphi(c)} \sum_{1 \neq d|c} d \sum_{\chi \pmod d}^* \sum_{\rho_\chi} |\Psi_{\phi, X, c, k}(\rho_\chi - \frac{1}{2})| \\ &\ll_k N^{\sigma(\frac{k}{2}-1)} \sum_{\substack{c \equiv 0 \pmod N \\ c < C}} \frac{1}{c^k \varphi(c)} \sum_{1 \neq d|c} d \sum_{\chi \pmod d}^* \sum_{\rho_\chi} \frac{N^{\sigma\Re(\rho_\chi)}}{(|\rho_\chi| + 1)^2}, \end{aligned}$$

by Lemma 3.3. This expression is

$$\begin{aligned}
&\ll_{\varepsilon} N^{\sigma(\frac{k}{2}-1)-k-1+\varepsilon} \sum_{c' < C/N} \frac{1}{c'^{k+1}} \sum_{\substack{d|c'N \\ d \neq 1}} d \sum_{\chi \bmod d}^* \sum_{\rho_{\chi}} \frac{N^{\sigma \Re(\rho_{\chi})}}{(|\rho_{\chi}|+1)^2} \\
&\ll N^{\sigma(\frac{k}{2}-1)-k-1+\varepsilon} \sum_{1 < d < C} d \sum_{\substack{c' < C/N \\ c' \equiv 0 \pmod{\frac{d}{(d,N)}}}} \frac{1}{c'^{k+1}} \sum_{\chi \bmod d}^* \sum_{\rho_{\chi}} \frac{N^{\sigma \Re(\rho_{\chi})}}{(|\rho_{\chi}|+1)^2} \\
&\ll N^{\sigma(\frac{k}{2}-1)-k-1+\varepsilon} \sum_{1 < d < C} d \left(\frac{d}{(d,N)} \right)^{-k-1} \sum_{\chi \bmod d}^* \sum_{\rho_{\chi}} \frac{N^{\sigma \Re(\rho_{\chi})}}{(|\rho_{\chi}|+1)^2} \\
&\ll N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sum_{\substack{d < C \\ d \equiv 0 \pmod{N}}} d^{-k} \sum_{\chi \bmod d}^* \sum_{\rho_{\chi}} \frac{N^{\sigma \Re(\rho_{\chi})}}{(|\rho_{\chi}|+1)^2} + O_{\varepsilon}(N^{\sigma \frac{k}{2}-k-1+\varepsilon}).
\end{aligned}$$

Truncating the sum over ρ_{χ} at $|\Im(\rho_{\chi})| \leq N^5$, we obtain the error term

$$O_{\varepsilon}(N^{k(\frac{\sigma}{2}-1)-4+\varepsilon}),$$

which is admissible. The final step is to decompose the sum over zeros into subsums in which $\Re(\rho_{\chi})$ lies in a short vertical band of size $\frac{1}{\log N}$ (on which the function $N^{\sigma \Re(\rho_{\chi})}$ is essentially constant), and in which $\Im(\rho_{\chi})$ lies in a dyadic interval. We also cut the sum over d dyadically. This yields the subsums

$$\sum_{\substack{d \sim D \\ d \equiv 0 \pmod{N}}} d^{-k} \sum_{\chi \bmod d}^* \sum_{\substack{\rho_{\chi} \\ \beta \leq \Re(\rho_{\chi}) < \beta + \frac{1}{\log N} \\ T-1 \leq |\Im(\rho_{\chi})| < 2T}} \frac{N^{\sigma \Re(\rho_{\chi})}}{(|\rho_{\chi}|+1)^2},$$

where the parameters D, β, T are such that $N \leq D < C$, $0 < \beta < 1$, and $1 \leq T \leq N^5$. Note that by monotonicity and by symmetry of the zeros about the line $\Re(s) = \frac{1}{2}$, we may add the further restriction $\beta \geq \frac{1}{2}$. There are $\ll (\log N)^3$ such subsums, in each of which

$$d^{-k} \frac{N^{\sigma \Re(\rho_{\chi})}}{(|\rho_{\chi}|+1)^2} \ll \frac{N^{\sigma \beta}}{D^k T^2}.$$

The claimed result follows. \square

4. PROOF OF THEOREM 1.1

The object of this section is to state and apply a zero-density estimate on Dirichlet L -functions, which will be the last step in the proof of Theorem 1.1. For a Dirichlet character χ , we denote

$$N(\beta, T, \chi) := \#\{\rho_{\chi} : \Re(\rho_{\chi}) \geq \beta, |\Im(\rho_{\chi})| < T, L(\rho_{\chi}, \chi) = 0\}.$$

Theorem 4.1. *Fix $\varepsilon > 0$. In the range $\frac{1}{2} + \varepsilon \leq \beta \leq 1$ and for $h \in \mathbb{N}$, $Q \geq 1$, we have the bound*

$$\sum_{\substack{q \leq Q \\ (q,h)=1}} \sum_{\psi \bmod q}^* \sum_{\xi \bmod h} N(\beta, T, \psi \xi) \ll_{\varepsilon} ((hQT)^{(2+\varepsilon)(1-\beta)} + (hQ^2T)^{(1-\beta) \min(\frac{3}{2-\beta}, \frac{3}{3\beta-1})}) (\log hQT)^{O_{\varepsilon}(1)}.$$

Proof. This statement is essentially contained in the proof of [IK, Theorem 10.4]. The difference between our statement and theirs is that we require β to be at a positive distance from $\frac{1}{2}$. Looking at the second half of [IK, page 262], we note that the power of $\log hQT$ comes from a bound on the average of the divisor function and therefore depends only on the exponent to which the appropriate Dirichlet polynomial is raised. Moreover, near the end of the same page, it is noted that this exponent is bounded in terms of ε .

Therefore, going through the second paragraph of [IK, page 263], one can see that in the range $\frac{1}{2} + \varepsilon \leq \beta \leq \frac{3}{4}$ the implied constant in \ll and that in the power of $\log hQT$ in [IK, equation (10.90)] depend only on ε . We conclude by noting that in the range $\frac{3}{4} \leq \beta \leq 1$, the implied constant and that in the power of $\log hQT$ are absolute: they depend on ε that can, in this situation, be taken to be equal to $\frac{1}{4}$. \square

We are now ready to prove our main theorem.

Proof of Theorem 1.1. From Proposition 3.4, we see that it is sufficient to show that (we set $C = N^{1+\frac{1}{2k-3}}$)

$$N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N \leq D < C} \sup_{\frac{1}{2} \leq \beta < 1} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} \sum_{\substack{d \sim D \\ d \equiv 0 \pmod{N}}} \sum_{\chi \pmod{d}}^* \sum_{\substack{\rho_\chi \\ \beta \leq \Re(\rho_\chi) < \beta + \frac{1}{\log N} \\ T-1 \leq |\Im(\rho_\chi)| < 2T}} 1 = o_{N \rightarrow \infty}(1). \quad (13)$$

Note that since $D < N^2$, N^2 does not divide d and thus we may write $d = fN$ with $(f, N) = 1$. Now, by the Chinese Remainder Theorem, every primitive character χ modulo d where d is a multiple of N can be decomposed as $\chi = \psi\xi$ where ψ is primitive modulo f and ξ is primitive modulo N . As a result, the left-hand side of (13) is

$$\ll_{\varepsilon} N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N \leq D < C} \sup_{\frac{1}{2} \leq \beta < 1} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} \sum_{\substack{f \sim D/N \\ (f, N)=1}} \sum_{\psi \pmod{f}}^* \sum_{\xi \pmod{N}}^* N(\beta, 2T, \psi\xi).$$

We first apply the Riemann–von Mangoldt theorem in the range $\frac{1}{2} \leq \beta < \frac{1}{2} + \varepsilon$ and deduce that

$$N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N \leq D < C} \sup_{\frac{1}{2} \leq \beta < \frac{1}{2} + \varepsilon} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} \sum_{\substack{f \sim D/N \\ (f, N)=1}} \sum_{\psi \pmod{f}}^* \sum_{\xi \pmod{N}}^* N(\beta, 2T, \psi\xi) \ll N^{\sigma(\frac{k}{2}-\frac{1}{2}+\varepsilon)+1-k+2\varepsilon},$$

which is admissible for $\sigma < 2 - 9\varepsilon$. Moreover, Theorem 4.1 implies that

$$\begin{aligned} & N^{\sigma(\frac{k}{2}-1)+\varepsilon} \sup_{N \leq D < C} \sup_{\frac{1}{2} + \varepsilon \leq \beta < 1} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} \sum_{\substack{f \sim D/N \\ (f, N)=1}} \sum_{\psi \pmod{f}}^* \sum_{\xi \pmod{N}}^* N(\beta, 2T, \psi\xi) \\ & \ll_{\varepsilon} N^{\sigma(\frac{k}{2}-1)+3\varepsilon} \sup_{N \leq D < C} \sup_{\frac{1}{2} + \varepsilon \leq \beta < 1} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} \left((DT)^{2(1-\beta)} + (N^{-1}D^2T)^{(1-\beta)\min(\frac{3}{2-\beta}, \frac{3}{3\beta-1})} \right). \end{aligned} \quad (14)$$

As $k \geq 2$, the supremum over D is attained at $D = N$ and the supremum over T is attained at $T = 1$, thus, the error term above is

$$\begin{aligned} & \ll N^{\sigma(\frac{k}{2}-1)+3\varepsilon} \sup_{\frac{1}{2} + \varepsilon \leq \beta < 1} N^{\sigma\beta-k} \left(N^{2(1-\beta)} + N^{(1-\beta)\min(\frac{3}{2-\beta}, \frac{3}{3\beta-1})} \right) \\ & \leq \sup_{\frac{1}{2} \leq \beta < 1} \left(N^{\sigma(\beta+\frac{k}{2}-1)-k+2(1-\beta)+3\varepsilon} + N^{\sigma(\beta+\frac{k}{2}-1)-k+(1-\beta)\min(\frac{3}{2-\beta}, \frac{3}{3\beta-1})+3\varepsilon} \right). \end{aligned}$$

The first error term is admissible whenever

$$\sigma < \inf_{\frac{1}{2} \leq \beta < 1} \frac{2\beta + k - 2}{\beta + \frac{k}{2} - 1} = 2.$$

As for the second, it is admissible whenever

$$\sigma < \inf_{\frac{1}{2} \leq \beta < 1} \frac{k - (1 - \beta) \min(\frac{3}{2-\beta}, \frac{3}{3\beta-1})}{\beta + \frac{k}{2} - 1} = \Theta_k,$$

where we recall that Θ_k is defined in (1). The last equality follows from optimizing the two cases in the minimum separately. Note that for $k = 2$, the infimum is attained at $\beta = \sqrt{3} - 1$, and for $k \geq 4$ it is attained at $\beta = \frac{3}{4}$. \square

Remark 4.2. As mentioned in the introduction, the Grand Density Conjecture

$$\sum_{\substack{q \leq Q \\ (q,k)=1}} \sum_{\psi \bmod q}^* \sum_{\xi \bmod k} N(\beta, T, \psi\xi) \ll (kQ^2T)^{2(1-\beta)} (\log kQT)^{O(1)}$$

(see [IK, p.250]) implies the Katz–Sarnak conjecture in the full range $\text{supp}(\widehat{\phi}) \subset (-2, 2)$. Indeed, under this conjecture we can replace (14) by the expression

$$N^{\sigma(\frac{k}{2}-1)+2\epsilon} \sup_{N \leq D < N^{1+\frac{1}{2k-3}}} \sup_{\frac{1}{2} \leq \beta < 1} \sup_{1 \leq T \leq N^5} \frac{N^{\sigma\beta}}{D^k T^2} (N^{-1} D^2 T)^{2(1-\beta)} \ll \sup_{\frac{1}{2} \leq \beta < 1} N^{(\sigma-2)(\beta+\frac{k}{2}-1)+2\epsilon},$$

which is clearly admissible when $\sigma < 2$.

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UNIV. LITTORAL CÔTE D'OPALE, UR 2597 LMPA, LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES
JOSEPH LIOUVILLE, F-62100 CALAIS, FRANCE
E-mail address: `lucile.devin@univ-littoral.fr`

UNIV. PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, 91405, ORSAY, FRANCE
E-mail address: `daniel.fiorilli@universite-paris-saclay.fr`

DEPARTMENT OF MATHEMATICAL SCIENCES, CHALMERS UNIVERSITY OF TECHNOLOGY AND THE UNIVERSITY
OF GOTHENBURG, SE-412 96 GOTHENBURG, SWEDEN
E-mail address: `andesod@chalmers.se`