ONE-LEVEL DENSITIES IN FAMILIES OF GRÖSSENCHARAKTERS ASSOCIATED TO CM ELLIPTIC CURVES

C. DAVID, L. DEVIN AND E. WAXMAN

Abstract. We study the low-lying zeros of a family of L-functions attached to the CM elliptic curves $E_d \ : \ y^2 = x^3 - dx$, for each odd and square-free integer d. Writing the L-function of E_d as $L(s - \frac{1}{2}, \xi_d)$ for the appropriate Grössencharakter ξ_d of conductor $\mathfrak{f}_d$, the family $\mathcal{F}_d$ is defined as the family of L-functions attached to the Grössencharakters $\xi_{d,k}$, where for each integer $k \geq 1$, $\xi_{d,k}$ denotes the primitive character inducing ξ_d^k. We observe that the average root number over the family $\mathcal{F}_d$ is $\frac{1}{4}$, which makes the symmetry type of the family (unitary, symplectic or orthogonal) somehow mysterious, as none of the symmetry types would lead to this average value. By computing the one-level density, we find that $\mathcal{F}_d$ breaks down into two natural subfamilies, namely a symplectic family ($L(s, \xi_{d,k})$ for k even) and an orthogonal family ($L(s, \xi_{d,k})$ for k odd). For k odd, $\mathcal{F}_d$ is in fact a subfamily of the automorphic forms of fixed level $4N(\mathfrak{f}_d)$, and even weight $k+1$, and this larger family also has orthogonal symmetry. The main term of the one-level density gives the symmetry and we also compute explicit lower order terms for each case.

1. Introduction

Let $d \in \mathbb{Z}$ be a fixed odd square-free integer, and let E_d denote the complex multiplication (CM) elliptic curve with affine equation $E_d \ : \ y^2 = x^3 - dx$. The L-function of E_d is

$$L(s, E_d) := \prod_{p \nmid 2d} \left(1 - \frac{a_p(E_d)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1} \quad \mathrm{Re}(s) > \frac{3}{2},$$

where for each prime $p \nmid 2d$, one writes $a_p(E_d) := p + 1 - \#E_d(\mathbb{F}_p)$. Because of complex multiplication by $\mathbb{Z}[i]$, this L-function may be written in terms of the L-function of a Grössencharakter on $\mathbb{Z}[i]$. More specifically, we have

$$L(s, E_d) = L(s - \tfrac{1}{2}, \xi_d),$$

where ξ_d is the Grössencharakter defined in (2.13). For each fixed d, we consider the family of L-functions

$$\mathcal{F}_d := \{L(s, \xi_{d,k}) \ : \ k \geq 1\},$$

where $\xi_{d,k}$ denotes the primitive character inducing the power ξ_d^k of the Grössencharakter ξ_d. For the particular case $d = 1$, $\mathcal{F}_d$ and its subfamilies have been used to study fine scale statistics of Gaussian primes in sectors [1, 4, 14, 30].

In this paper we study, for fixed square-free odd $d \in \mathbb{Z}$, the low-lying zeros (i.e. the zeros close to the central point $s = \frac{1}{2}$) across the family of L-functions $\mathcal{F}_d$. The Katz–Sarnak density conjecture [21, 22, 31] states that the distribution of low-lying zeros may be predicted by the symmetry type of the family in question. Precisely, the conjecture states that the zero distribution corresponds to the distribution of eigenvalues close to 1 of random matrices in an

Date: May 24, 2024.

appropriate classical compact group (unitary, symplectic or orthogonal). There is a lengthy history of computing one-level densities, and more generally n-level densities, for various L-function families including those of Dirichlet characters [2, 3, 16, 27], Hecke characters [11, 15, 32], elliptic curves [12, 24, 33], and modular forms [5, 20, 29]. We refer the reader to the paper [25] for an extensive survey of the existing results.

The symmetry type is also related to the distribution of the sign of the functional equation. Let $W(\xi_{d,k}) = \pm 1$ denote the root number of $L(s, \xi_{d,k})$ (i.e. the sign of its functional equation). As demonstrated in Lemma 2.6, one has, for any fixed odd square-free $d \in \mathbb{Z}$, that

$$\lim_{K\to\infty} \frac{\#\{1 \le k \le K : W(\xi_{d,k}) = -1\}}{K} = \frac{1}{4}.$$

We remark that this average value cannot be obtained as the average rank of a family with one of the usual symmetry types. Indeed, we show by studying congruence classes of k modulo 8 separately (see Theorem 1.1) that $\mathcal{F}_d$ breaks down into two natural subfamilies with different symmetry types.

In order to state our main result, let us first introduce some notation. Let ϕ be an even Schwartz function such that $\widehat{\phi}(s) := \int_{-\infty}^{\infty} \phi(t)e^{-2\pi its}\,\mathrm{d}t$, the Fourier transform of ϕ, is compactly supported. For each fixed odd square-free $d \in \mathbb{Z}$, we wish to understand the behaviour of the properly normalized low-lying zeros across the family $\mathcal{F}_d$. To this end, we define

$$\mathcal{D}(\phi, \xi_{d,k}) := \sum_{L(\frac{1}{2}+i\gamma,\xi_{d,k})=0} \phi\left(\frac{\gamma \log\left(k^2\mathrm{N}(\mathfrak{f}_{d,k})\right)}{2\pi}\right), \tag{1.1}$$

where $\mathrm{N}(\mathfrak{f}_{d,k})$ is the norm of the conductor of $\xi_{d,k}$, and where the normalisation factor is chosen upon noting that the analytic conductor of $L(s, \xi_{d,k})$ is asymptotic to $k^2\mathrm{N}(\mathfrak{f}_{d,k})$ (for further details concerning this chosen normalisation see the beginning of Section 3 below).

As the root number $W(\xi_{d,k})$ depends on the congruence class of k modulo 8 (see Lemma 2.6), it is natural to study the family $\mathcal{F}_d$ according to the congruence class of k modulo 8. For any $\alpha \in \mathbb{Z}/8\mathbb{Z}$, we thus define

$$\mathcal{F}_d^\alpha := \{L(s, \xi_{d,k}) \ : \ k \ge 1,\ k \equiv \alpha \,(\mathrm{mod}\, 8)\}.$$

The one-level density of the family $\mathcal{F}_d^\alpha$ is then given for any $K \in \mathbb{N}$, by

$$\mathcal{D}(K; \phi, \mathcal{F}_d^\alpha) := \frac{8}{K} \sum_{\substack{1\le k\le K \\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \mathcal{D}(\phi, \xi_{d,k}). \tag{1.2}$$

Our main theorem is as follows.

Theorem 1.1. *Let d be an odd square-free integer, ϕ be an even Schwartz function with $\mathrm{supp}(\widehat{\phi}) \subset (-1, 1)$ and fix $J \in \mathbb{N}$ and $\alpha \in \{1, 2, \dots, 8\}$. Let $M_{d,\alpha} := \mathrm{N}(\mathfrak{f}_{d,\alpha})^{\frac{1}{2}}$, where $\mathfrak{f}_{d,\alpha}$ is the conductor of $\xi_{d,\alpha}$ given by (2.19). When $K \to \infty$ and α is even,*

$$\mathcal{D}(K; \phi, \mathcal{F}_d^\alpha) = \widehat{\phi}(0) - \frac{1}{2}\int_{\mathbb{R}} \widehat{\phi}(u)\mathrm{d}u + \sum_{m=1}^{J} \frac{C_m(d, \alpha, \phi)}{(\log(KM_{d,\alpha}))^m} + O_J\left(\frac{1 + \log|d|}{(\log(KM_{d,\alpha}))^{J+1}}\right),$$

and when $K \to \infty$ and α is odd,

$$\mathcal{D}(K;\phi,\mathcal{F}_d^\alpha) = \widehat{\phi}(0) + \frac{1}{2}\int_{\mathbb{R}} \widehat{\phi}(u)\mathrm{d}u + \sum_{m=1}^{J} \frac{C_m(d,\alpha,\phi)}{(\log(KM_{d,\alpha}))^m} + O_J\left(\frac{1+\log|d|}{(\log(KM_{d,\alpha}))^{J+1}}\right),$$

where the $C_m(d,\alpha,\phi)$ are given by (5.1).

Remark 1.2. Since $\mathrm{supp}(\widehat{\phi}) \subset (-1,1)$, we have $\pm\frac{1}{2}\int_{\mathbb{R}} \widehat{\phi}(u)\mathrm{d}u = \pm\frac{1}{2}\int_{-1}^{1} \widehat{\phi}(u)\mathrm{d}u$, which is what is should for any ϕ by (1.3) and (1.4). Then, the result as stated would be false if the support of $\widehat{\phi}$ was enlarged. But the contribution of the inert primes always gives $\pm\frac{1}{2}\int_{\mathbb{R}} \widehat{\phi}(u)\mathrm{d}u$ by Lemma 3.5, which explains why $\mathrm{supp}(\widehat{\phi}) \subset (-1,1)$ is a difficult barrier to break: one needs to find a contribution from the split primes cancelling the wrong contribution from the inert primes.

In [32], the third author computed the one-level density of the subfamily $k \equiv 0 \pmod 4$ for the particular case $d = 1$, for test functions ϕ such that $\widehat{\phi} \subset (-1,1)$, up to an error term of size $O(1/(\log K)^2)$. In such a case, the root number is identically 1 for each element in the family, and the one-level density has symplectic symmetry. Our work generalizes this in three distinct directions. Specifically, we allow $d \in \mathbb{Z}$ to be any odd and square-free integer; we consider $k \equiv \alpha \pmod 8$ for any $\alpha \in \mathbb{Z}/8\mathbb{Z}$; and we explicitly compute all lower order terms up to an arbitrary negative powers of $\log(KM_{d,\alpha})$. Contrary to the main term, which follows the Katz–Sarnak prediction, the lower-order terms have no universal behaviour, as they contain features which depend on the particular family in question. We refer to [5, 9, 10, 28] for comparisons.

An additional interesting feature of this work is a very explicit description of the lower-order contributions in terms of generalized Euler constants. Such a description enables a concrete understanding of the speed of convergence to the conjectured distribution (see Section 4 and Appendix A).

Note that Theorem 1.1 determines the symmetry type of the families $\mathcal{F}_d^\alpha$. Indeed, by the Katz–Sarnak density conjecture, we expect that for a family $\mathcal{F}$, there exists some compact group $G \in \{U, Sp, O, SO(\text{even}), SO(\text{odd})\}$ (depending on $\mathcal{F}$) such that

$$\lim_{K\to\infty} \mathcal{D}(K;\phi,\mathcal{F}_d^\alpha) = \int_{\mathbb{R}} W_G(x)\phi(x)\mathrm{d}x = \int_{\mathbb{R}} \widehat{W_G}(t)\widehat{\phi}(t)\mathrm{d}t,$$

where

$$W_G(x) = \begin{cases} 1 & \text{if } G = U \\ 1 - \frac{\sin(2\pi x)}{2\pi x} & \text{if } G = Sp \\ 1 + \frac{1}{2}\delta_0(x) & \text{if } G = O \\ 1 + \frac{\sin(2\pi x)}{2\pi x} & \text{if } G = SO(\text{even}) \\ 1 + \delta_0(x) - \frac{\sin(2\pi x)}{2\pi x} & \text{if } G = SO(\text{odd}), \end{cases}$$

and δ_0 is the Dirac function. The Fourier transform of each such density is then given by

$$\widehat{W_G}(t) = \begin{cases} \delta_0(t) & \text{if } G = U \\ \delta_0(t) - \frac{1}{2}\eta(t) & \text{if } G = Sp \\ \delta_0(t) + \frac{1}{2} & \text{if } G = O \\ \delta_0(t) + \frac{1}{2}\eta(t) & \text{if } G = SO(\text{even}) \\ \delta_0(t) + 1 - \frac{1}{2}\eta(t) & \text{if } G = SO(\text{odd}) \end{cases} \tag{1.3}$$

where

$$\eta(t) = \begin{cases} 1 & |t| < 1 \\ \frac{1}{2} & t = 1 \\ 0 & t > 1. \end{cases} \tag{1.4}$$

By Theorem 1.1 and (1.3), $\mathcal{F}_d^\alpha$ has symplectic symmetry when α is even, and orthogonal symmetry when α is odd. In the latter case, the three orthogonal symmetry types cannot be distinguished when $\mathrm{supp}(\widehat{\phi}) \subset (-1, 1)$. By differentiating the functional equation (see (2.24)), we moreover observe that $W(\xi_{d,k}) = -1$ if and only if $\mathrm{ord}_{s=\frac{1}{2}} L(s, \xi_{d,k})$ is odd. Thus, upon defining

$$S_\pm(d) := \{\alpha \in (\mathbb{Z}/8\mathbb{Z})^\times \;:\; W(\xi_d^k) = \pm 1 \text{ when } k \equiv \alpha \pmod 8\}, \tag{1.5}$$

we expect (by heuristically taking $\phi = \delta_0$) that $G = SO(\text{even})$ when $\alpha \in S_+(d)$ and $G = SO(\text{odd})$ when $\alpha \in S_-(d)$. The sets $S_\pm(d)$ are explicitly computed in (2.29).

Observe moreover that when α odd, $\mathcal{F}_d^\alpha$ is in fact a subfamily of the automorphic forms of fixed level $4\mathrm{N}(\mathfrak{f}_{d,1})$ and varying even weight (see Remark 2.4). This larger family also has orthogonal symmetry. This was proven in [20] by computing the corresponding one-level density unconditionally for $\widehat{\phi}$ supported in $(-1, 1)$, and with an extra averaging over the weights, for $\widehat{\phi}$ supported in $(-2, 2)$.

As an application of Theorem 1.1, we obtain a proportion for non-vanishing at the central point in each family $\mathcal{F}_d^\alpha$. This partially answers a question in [4], posed in the cases $d = \pm 1$. The proportion depends upon the symmetry type of the family.

Corollary 1.3. *Let d be an odd square-free integer.*
When $\alpha \in \mathbb{Z}/8\mathbb{Z}$ is even, the proportion of non-vanishing in $\mathcal{F}_d^\alpha$ is at least

$$\lim_{K\to\infty} \frac{8}{K} \,\#\left\{1 \le k \le K,\, k \equiv \alpha \pmod 8 \;:\; L(\tfrac{1}{2}, \xi_{d,k}) \neq 0\right\} \ge 75\%.$$

In the case α is odd and $\alpha \in S_+(d)$, we have

$$\lim_{K\to\infty} \frac{8}{K} \,\#\left\{1 \le k \le K,\, k \equiv \alpha \pmod 8 \;:\; L(\tfrac{1}{2}, \xi_{d,k}) \neq 0\right\} \ge 25\%.$$

Finally, if α is odd and $\alpha \in S_-(d)$, each $L(s, \xi_{d,k})$ vanishes at $s = \frac{1}{2}$, and we have

$$\lim_{K\to\infty} \frac{8}{K} \,\#\left\{1 \le k \le K,\, k \equiv \alpha \pmod 8 \;:\; \mathrm{ord}_{s=\frac{1}{2}} L(s, \xi_{d,k}) = 1\right\} \ge 75\%.$$

The structure of this paper is as follows. In Section 2 we define the Grössencharakters $\xi_{d,k}$, and explicitly compute the root numbers $W(\xi_{d,k})$ for all square-free $d \in \mathbb{Z}$ and $k \ge 1$. This leads to the observation (equation (2.28)) that $W(\xi_{d,k}) = -1$ for $\frac{1}{4}$ of the $k \ge 1$, and depends only on $k \pmod 8$ (with different congruences for different values of d). In Section 3, we then employ the explicit formula to compute the contribution of the Γ-factors, of the ramified primes, and of the inert primes towards the one-level density computation, for each $\xi_{d,k}$. This yields the symmetry of each family $\mathcal{F}_d^\alpha$ (see Proposition 3.6), provided one can show that the average of the contribution of the split primes (i.e. of $U_{\text{split}}(\phi, d, k)$ over $1 \leq k \leq K$) is small. In Section 4 we then offer alternative expressions for $c_{j,\text{inert}}(k)$, the contributions of the inert primes to $C_m(d, \alpha, \phi)$, helpful in order to compute such constants, explicitly. Next, in Section 5 we provide a simple argument showing that Theorem 1.1 holds for $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{2}, \frac{1}{2})$, and in Section 6 we extend this result to the range $\text{supp}(\widehat{\phi}) \subset (-1, 1)$. Finally, we prove Corollary 1.3 in Section 7.

Acknowledgements

The authors are grateful to Anders Södergren for helpful discussions. This paper was written during visits to the authors at Concordia Université, Université du Littoral Côte d'Opale and Institut Henri-Poincaré (Research in Paris program). We thank those institutions for financial support and hospitality. CD is supported by the Natural Sciences and Engineering Research Council of Canada [DG-155635-2019] and by the Fonds de recherche du Québec - Nature et technologies [Projet de recherche en équipe 300951]. LD is supported by the PEPS JCJC 2023 program of the INSMI (CNRS). EW was supported by a Zuckerman Post Doctoral Fellowship and by the Israel Science Foundation (Grant No. 1881/20).

2. Background

2.1. **Quartic Residue Symbol.** We refer to [17, Ch. 9] for the material of this section. To begin, we define the *norm* of a Gaussian integer $\alpha \in \mathbb{Z}[i]$ to be $\mathrm{N}(\alpha) := \alpha\overline{\alpha}$. To any Gaussian prime $\pi \in \mathbb{Z}[i]$ such that $\pi \nmid 2$, we define the *quartic residue symbol* modulo π to be the quartic character $\chi_{(\pi)} : (\mathbb{Z}[i]/(\pi))^\times \longrightarrow \{\pm 1, \pm i\}$ such that

$$\chi_{(\pi)}(\alpha) := \left(\frac{\alpha}{\pi}\right)_4 \equiv \alpha^{(\mathrm{N}(\pi)-1)/4} \pmod \pi, \tag{2.1}$$

and extended to $\mathbb{Z}[i]$ by $\chi_{(\pi)}(\alpha) = 0$ when $(\alpha, \pi) \neq 1$. Consider a non-unit $\beta \in \mathbb{Z}[i]$ with prime decomposition given by $\beta = \pi_1^{e_1} \dots \pi_s^{e_s}$. When $(\beta, 2) = 1$, we may extend by multiplicativity to then define, for any $\alpha \in \mathbb{Z}[i]^\times$,

$$\chi_{(\beta)}(\alpha) := \left(\frac{\alpha}{\beta}\right)_4 = \prod_{i=1}^{s} \left(\frac{\alpha}{\pi_i}\right)_4^{e_i}.$$

In particular, we note [17, Prop. 9.8.5] that for odd $d \in \mathbb{N}$ and $n \in \mathbb{Z}$ such that $(n, d) = 1$, we have

$$\chi_{(d)}(n) = 1. \tag{2.2}$$

Recall that the *conductor* of a character $\chi : (\mathbb{Z}[i]/\mathfrak{m})^\times \longrightarrow S^1$ refers to the smallest divisor $\mathfrak{f}|\mathfrak{m}$ such that χ factors through $(\mathbb{Z}[i]/\mathfrak{f})^\times$ via the projection $(\mathbb{Z}[i]/\mathfrak{m})^\times \twoheadrightarrow (\mathbb{Z}[i]/\mathfrak{f})^\times$. If the conductor of χ is $\mathfrak{m}$, then χ is referred to as a *primitive* character mod $\mathfrak{m}$. In particular, we note that if $\beta \in \mathbb{Z}[i]$ as above is square-free, then $\chi_{(\beta)}$ is a primitive quartic character with conductor (β).

A non-unit Gaussian integer $\alpha \in \mathbb{Z}[i]$ is said to be *primary* if $\alpha \equiv 1 \,(\mathrm{mod}\,(2+2i))$. As in [17, Lem. 7, Sect. 9.8], we find that any proper ideal $(\alpha) \in \mathbb{Z}[i]$ coprime to (2) has precisely one primary generator $\boldsymbol{\alpha} \in \{\pm\alpha, \pm i\alpha\}$. In what follows, we denote the primary generator $\boldsymbol{\alpha}$ of an ideal (α) by a bold letter. We cite from [17, Ch. 9, Thm. 2] the following reciprocity law:

Proposition 2.1 (Law of Quartic Reciprocity)**.** *Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}[i]$ be primary such that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1$. Then*

$$\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{\beta}}\right)_4 = \left(\frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}}\right)_4 (-1)^{\frac{\mathrm{N}(\boldsymbol{\alpha})-1}{4}\,\frac{\mathrm{N}(\boldsymbol{\beta})-1}{4}}. \tag{2.3}$$

We will also use quadratic and quartic characters modulo even ideals. More precisely, we introduce the character $\chi_{(2)} : (\mathbb{Z}[i]/(2))^\times \longrightarrow \{\pm 1\}$ given by

$$\chi_{(2)}(\alpha) := \begin{cases} 1 & \alpha \equiv 1 \,(\mathrm{mod}\,2) \\ -1 & \alpha \equiv i \,(\mathrm{mod}\,2), \end{cases} \tag{2.4}$$

the character $\chi_{(2+2i)} : (\mathbb{Z}[i]/(2+2i))^\times \longrightarrow \{\pm 1, \pm i\}$ given by

$$\chi_{(2+2i)}(\alpha) := \begin{cases} 1 & \alpha \equiv 1 \,(\mathrm{mod}\,2+2i) \\ -1 & \alpha \equiv -1 \,(\mathrm{mod}\,2+2i) \\ -i & \alpha \equiv i \,(\mathrm{mod}\,2+2i) \\ i & \alpha \equiv -i \,(\mathrm{mod}\,2+2i), \end{cases} \tag{2.5}$$

and the character $\chi_{(4)} : (\mathbb{Z}[i]/(4))^\times \longrightarrow \{\pm 1, \pm i\}$ given by

$$\chi_{(4)}(\alpha) := \begin{cases} 1 & \alpha \equiv 1, -(3+2i) \,(\mathrm{mod}\,4) \\ -1 & \alpha \equiv -1, (3+2i) \,(\mathrm{mod}\,4) \\ i & \alpha \equiv i, -i(3+2i) \,(\mathrm{mod}\,4) \\ -i & \alpha \equiv -i, i(3+2i) \,(\mathrm{mod}\,4). \end{cases} \tag{2.6}$$

Note that $\chi_{(2)}, \chi_{(2+2i)}$ and $\chi_{(4)}$ are primitive characters on $\mathbb{Z}[i]$, with conductors given by (2), $(2+2i)$, and (4), respectively. We further remark that for $\alpha \in \mathbb{Z}[i]$, with $(\alpha, 2) = 1$, the primary generator of (α) is given by

$$\boldsymbol{\alpha} = \chi_{(2+2i)}(\alpha)\ \alpha. \tag{2.7}$$

Finally, by [23, Thm. 6.9] (see also [17, Ch. 9 Ex. 32−33]), we note that for odd $d \in \mathbb{Z} \setminus \{1\}$,

$$\chi_{(d)}(1+i) = i^{(\boldsymbol{d}-1)/4}, \tag{2.8}$$

where $\boldsymbol{d}$ denotes the primary generator of (d).

2.2. ***L*-functions of Elliptic Curves.** Much of the following material on elliptic curves may be found in [17, Ch. 18]. Let E be an elliptic curve over $\mathbb{Q}$ with conductor N_E. For $p \nmid N_E$, let $a_p(E) := p + 1 - \#E(\mathbb{F}_p)$. By the Hasse's bound, $a_p(E) \leq 2\sqrt{p}$. The *L-function* attached to E is then defined by

$$L(s, E) := \prod_p L_p(s, E)^{-1}, \qquad \mathrm{Re}(s) > \frac{3}{2}, \tag{2.9}$$

where

$$L_p(s,E) := \begin{cases} (1-a_p(E)p^{-s}+p^{1-2s}) & \text{if } p \text{ has good reduction at } p \\ (1-p^{-s}) & \text{if } p \text{ has split multiplicative reduction at } p \\ (1+p^{-s}) & \text{if } p \text{ has non-split multiplicative reduction at } p \\ 1 & \text{if } p \text{ has additive reduction at } p. \end{cases}$$

In the case $E_d : y^2 = x^3 - dx$, the conductor is $N_{E_d} = 2^5 d^2$, and all primes $p \mid N_{E_d}$ have additive reduction, which gives

$$L(s,E_d) = \prod_{p \nmid 2d} \left(1 - \frac{a_p(E_d)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}, \quad \mathrm{Re}(s) > \frac{3}{2}. \tag{2.10}$$

The following result will enable us to express $L(s,E_d)$ as the L-function of a Grössencharakter over $\mathbb{Q}(i)$.

Proposition 2.2. *Let d be an odd square-free integer. For $p \equiv 1 \pmod 4$, $p \nmid d$, write $p\mathbb{Z}[i] = \mathfrak{p}\overline{\mathfrak{p}}$ and let $\boldsymbol{\pi}_{\mathfrak{p}}$ be the primary generator of $\mathfrak{p}$. Then on $Re(s) > 3/2$, we have*

$$L(s,E_d) = \prod_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ p\nmid d}} (1+p^{-2s+1})^{-1} \prod_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ (p)=\mathfrak{p}\overline{\mathfrak{p}}\\ p\nmid d}} \left(1 - \frac{\overline{\chi_{\mathfrak{p}}(d)}\boldsymbol{\pi}_{\mathfrak{p}}}{p^s}\right)^{-1} \left(1 - \frac{\chi_{\mathfrak{p}}(d)\overline{\boldsymbol{\pi}}_{\mathfrak{p}}}{p^s}\right)^{-1}. \tag{2.11}$$

Proof. Let us first remark that the contribution of the primes $p \equiv 1 \pmod 4$ is well defined because $\overline{\boldsymbol{\pi}}_{\mathfrak{p}}$ is the primary generator of $\overline{\mathfrak{p}}$. Indeed, since $\boldsymbol{\pi}_{\mathfrak{p}}$ is primary, there exists some $a+bi \in \mathbb{Z}[i]$ such that

$$\boldsymbol{\pi}_{\mathfrak{p}} = 1 + (a+bi)(2+2i),$$

from which it follows that

$$\overline{\boldsymbol{\pi}}_{\mathfrak{p}} = \overline{1+(a+bi)(2+2i)} = 1 + (-a-bi)(2+2i) \equiv 1 \,(\mathrm{mod}\,(2+2i)).$$

Moreover, since $d \in \mathbb{Z}$, we have $\chi_{\overline{\mathfrak{p}}}(d)\chi_{\mathfrak{p}}(d) = \chi_{(p)}(d) = 1$, that is $\chi_{\overline{\mathfrak{p}}}(d) = \overline{\chi_{\mathfrak{p}}(d)}$.

The proof then follows directly from [17, Thm. 5 in Ch. 18] which ensures that for $p \nmid 2d$ and $p \equiv 3 \pmod 4$, we have $\#E_d(\mathbb{F}_p) = p+1$, while for $p \nmid 2d$ and $p \equiv 1 \pmod 4$ we have

$$\#E_d(\mathbb{F}_p) = p + 1 - a_p(E_d) = p + 1 - \overline{\chi_{\mathfrak{p}}(d)}\boldsymbol{\pi}_{\mathfrak{p}} - \chi_{\mathfrak{p}}(d)\overline{\boldsymbol{\pi}}_{\mathfrak{p}}.$$

The definition (2.10) concludes the proof. ∎

We will now describe Grössencharakter (or Hecke character) whose L-function is $L(s,E_d)$.

2.3. **Grössencharakters.** Fix a non-zero integral ideal $\mathfrak{m}$ of $\mathbb{Q}(i)$, and let $J^{\mathfrak{m}}$ denote the group of fractional ideals in $\mathbb{Q}(i)$ coprime to $\mathfrak{m}$. As in [26, VII.6.1], we say that $\xi : J^{\mathfrak{m}} \to S^1$ is a *Grössencharakter* modulo $\mathfrak{m}$ on $\mathbb{Q}(i)$, if there exists a pair of characters

$$\xi_{\mathrm{fin}} : (\mathbb{Z}[i]/\mathfrak{m})^{\times} \longrightarrow S^1, \quad \xi_{\infty} : \mathbb{C}^{\times} \longrightarrow S^1$$

such that

$$\xi((\alpha)) = \xi_{\mathrm{fin}}(\alpha)\xi_{\infty}(\alpha), \tag{2.12}$$

for every $\alpha \in \mathbb{Z}[i]$ coprime to $\mathfrak{m}$. ξ_{fin} is referred to as the *finite* component of ξ, while ξ_∞ is referred to as the *infinite* component of ξ. The infinite component ξ_∞ may moreover be written in the form

$$\xi_\infty(\alpha) = \left(\frac{\alpha}{|\alpha|}\right)^{\ell} |\alpha|^{it}, \qquad \text{with } \ell \in \mathbb{Z}, t \in \mathbb{R},$$

in which case ξ is said to be of *type* (ℓ, t), and we refer to $\ell \in \mathbb{Z}$ as the *frequency* of ξ. The L-function attached to the Grössencharakter ξ is

$$L(s, \xi) := \prod_{\mathfrak{p} \text{ prime}} \left(1 - \frac{\xi(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^s}\right)^{-1}, \qquad \mathrm{Re}(s) > 1.$$

The *conductor* of ξ refers to the conductor of ξ_{fin}, and similarly ξ is said to be a *primitive* Grössencharakter when ξ_{fin} is primitive.

Fix $d \in \mathbb{Z}$ to be odd and square-free. For any prime ideal $\mathfrak{p} \subset \mathbb{Z}[i]$ with primary generator $\boldsymbol{\pi}_\mathfrak{p}$, define

$$\xi_d(\mathfrak{p}) := \begin{cases} \dfrac{\boldsymbol{\pi}_\mathfrak{p}}{|\boldsymbol{\pi}_\mathfrak{p}|} \cdot \overline{\chi}_\mathfrak{p}(d) & \text{when } (\boldsymbol{\pi}_\mathfrak{p}, 2+2i) = 1, \\ 0 & \text{when } (\boldsymbol{\pi}_\mathfrak{p}, 2+2i) \neq 1, \end{cases} \tag{2.13}$$

and extend by multiplicativity to all ideals $\mathfrak{a} \subset \mathbb{Z}[i]$. ξ_d then defines a Grössencharakter character modulo $d(2+2i)$. Indeed, for any ideal $(\alpha) \subset \mathbb{Z}[i]$ coprime to $d(2+2i)$, we may write $\xi_d((\alpha)) = \xi_{d,\text{fin}}(\alpha)\xi_\infty(\alpha)$, where $\xi_\infty : \mathbb{C}^\times \to S^1$ and $\xi_{d,\text{fin}} : (\mathbb{Z}[i]/(d(2+2i)))^\times \to S^1$ are given by

$$\xi_\infty : \alpha \mapsto \frac{\alpha}{|\alpha|}, \quad \xi_{d,\text{fin}} : \alpha \mapsto \overline{\chi}_{(\alpha)}(d)\ \chi_{(2+2i)}(\alpha). \tag{2.14}$$

By (2.7), this matches (2.13) when $(\alpha) \neq (1+i)$ is prime.

As the primary generator of a prime $p \equiv 3 \pmod 4$ is equal to $-p$, we find by (2.2) that $\xi_d((p)) = -1$. Upon comparing Euler factors with (2.11), we see that

$$L(s - \tfrac{1}{2}, \xi_d) = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \frac{\xi_d(\mathfrak{p})}{\mathrm{N}(\mathfrak{p})^{s-\frac{1}{2}}}\right)^{-1} = L(s, E_d). \tag{2.15}$$

2.4. **Computing Conductors.** Let us proceed by providing a more user-friendly expression for $\xi_{d,\text{fin}}$.

Lemma 2.3. *Let $d \in \mathbb{Z}$ be an odd and square-free integer, and $\xi_{d,\text{fin}}$ as in (2.14). One has*

$$\xi_{d,\text{fin}} = \begin{cases} \overline{\chi}_{(d)}\chi_{(2+2i)} & \text{if } d \equiv 1 \pmod 8 \\ \overline{\chi}_{(d)}\chi_{(2)}\overline{\chi}_{(4)} & \text{if } d \equiv 3 \pmod 8 \\ \overline{\chi}_{(d)}\chi_{(2+2i)}\chi_{(2)} & \text{if } d \equiv 5 \pmod 8 \\ \overline{\chi}_{(d)}\overline{\chi}_{(4)} & \text{if } d \equiv 7 \pmod 8. \end{cases} \tag{2.16}$$

In particular the conductor of ξ_d is given by

$$\mathfrak{f}_d = \begin{cases} ((2+2i)d) & \text{if } d \equiv 1 \pmod 4 \\ (4d) & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

Proof. If $d \equiv 1 \pmod 4$, then d is primary, and noting that $\mathrm{N}(d) - 1 \equiv 0 \pmod 8$, it follows from quartic reciprocity (Proposition 2.1) and (2.7) that

$$\overline{\chi}_{(\alpha)}(d) = \overline{\chi}_{(\chi_{(2+2i)}(\alpha)\alpha)}(d) = \overline{\chi}_{(d)}(\chi_{(2+2i)}(\alpha)\alpha).$$

Thus, by multiplicativity,

$$\xi_{d,\mathrm{fin}}(\alpha) = \overline{\chi}_{(d)}(\alpha)\ \overline{\chi}_{(d)}(\chi_{(2+2i)}(\alpha))\ \chi_{(2+2i)}(\alpha) = \overline{\chi}_{(d)}(\alpha)\ \chi_{(2+2i)}(\alpha)^{1-\sum_{\mathfrak{p}|d}\frac{(\mathrm{N}(\mathfrak{p})-1)}{4}}.$$

Moreover, observe that for $p \equiv 3 \pmod 4$, one has $\frac{\mathrm{N}((p))-1}{4} = \begin{cases} 0 \pmod 4 & \text{if } p \equiv 7, 15 \pmod{16} \\ 2 \pmod 4 & \text{if } p \equiv 3, 11 \pmod{16} \end{cases}$.

Similarly for $p \equiv 1 \pmod 4$, one has $\sum_{\mathfrak{p}|p} \frac{\mathrm{N}(\mathfrak{p})-1}{4} = \begin{cases} 0 \pmod 4 & \text{if } p \equiv 1 \pmod 8 \\ 2 \pmod 4 & \text{if } p \equiv -3 \pmod 8 \end{cases}$.

These together imply that

$$\sum_{\mathfrak{p}|d} \frac{\mathrm{N}(\mathfrak{p})-1}{4} \equiv 2\,\#\{p \mid d : p \equiv \pm 3 \pmod 8\} \pmod 4 \equiv \begin{cases} 0 \pmod 4 & \text{if } d \equiv \pm 1 \pmod 8 \\ 2 \pmod 4 & \text{if } d \equiv \pm 3 \pmod 8. \end{cases}$$

Upon noting that $\chi_{(2+2i)}^2 = \chi_{(2)}$, one has

$$\chi_{(2+2i)}^{-\sum_{\mathfrak{p}|d}\frac{\mathrm{N}(\mathfrak{p})-1}{4}} = \begin{cases} \chi_0 & \text{if } d \equiv \pm 1 \pmod 8 \\ \chi_{(2)} & \text{if } d \equiv \pm 3 \pmod 8, \end{cases} \tag{2.17}$$

where χ_0 is the trivial character modulo $(2+2i)$. Regrouping, (2.16) now follows for the case $d \equiv 1 \pmod 4$.

Next, suppose $d \equiv 3 \pmod 4$. Then $-d$ is primary, and similarly to above we find that

$$\begin{aligned} \xi_{d,\mathrm{fin}}(\alpha) &= \overline{\chi}_{(\alpha)}(-1)\ \overline{\chi}_{(\alpha)}(-d)\chi_{(2+2i)}(\alpha) \\ &= \overline{\chi}_{(\alpha)}(-1)\ \overline{\chi}_{(-d)}(\alpha)\ \overline{\chi}_{(-d)}(\chi_{(2+2i)}(\alpha))\chi_{(2+2i)}(\alpha) \\ &= \overline{\chi}_{(\alpha)}(-1)\ \overline{\chi}_{(d)}(\alpha)\chi_{(2+2i)}(\alpha)^{1-\sum_{\mathfrak{p}|d}\frac{(\mathrm{N}(\mathfrak{p})-1)}{4}}. \end{aligned}$$

Since

$$\overline{\chi}_{(\alpha)}(-1) = (-1)^{(\mathrm{N}(\alpha)-1)/4} = \begin{cases} 1 & \alpha \equiv \pm 1, \pm i \pmod 4 \\ -1 & \alpha \equiv \pm(3+2i), \pm i(3+2i) \pmod 4, \end{cases}$$

it follows from (2.5) and (2.6) that

$$\overline{\chi}_{(\alpha)}(-1)\chi_{(2+2i)}(\alpha) = \overline{\chi}_{(4)}(\alpha).$$

Combining the above with (2.17), the lemma follows. □

Finally, for any $k \geq 1$, we consider

$$\xi_d^k : (\alpha) \mapsto \xi_{d,\mathrm{fin}}^k(\alpha) \left(\frac{\alpha}{|\alpha|}\right)^k. \tag{2.18}$$

Let $\xi_{d,k}$ denote the primitive character inducing ξ_d^k. Noting that $\chi^2_{(2+2i)} = \chi_{(2)} = \overline{\chi}^2_{(4)}$ has conductor (2), it follows from (2.16) that $\xi_{d,k}$ is a character with frequency k and conductor

$$\mathfrak{f}_{d,k} = \begin{cases} \mathfrak{f}_d & k \text{ odd} \\ (2d) & k \equiv 2 \pmod 4 \\ (1) & k \equiv 0 \pmod 4. \end{cases} \tag{2.19}$$

Since the same set of primes divide the conductors of both $\xi_{d,k}$ and ξ_d whenever $k \not\equiv 0 \pmod 4$, it follows that

$$\xi_{d,k}(\mathfrak{p}) = \begin{cases} \xi_d^k(\mathfrak{p}) = \left(\dfrac{\boldsymbol{\pi}_\mathfrak{p}}{|\boldsymbol{\pi}_\mathfrak{p}|}\right)^k \cdot \overline{\chi}_\mathfrak{p}(d)^k & \text{if } \mathfrak{p} \nmid \mathfrak{f}_d \\ 0 & \text{if } \mathfrak{p} \mid \mathfrak{f}_d \text{ and } k \not\equiv 0 \pmod 4 \\ \left(\dfrac{\pi_\mathfrak{p}}{|\pi_\mathfrak{p}|}\right)^k & \text{if } \mathfrak{p} = (\pi_\mathfrak{p}) \mid \mathfrak{f}_d \text{ and } k \equiv 0 \pmod 4, \end{cases} \tag{2.20}$$

where, in the last case, we further note that $\xi_{d,k}(\mathfrak{p})$ does not depend on the chosen generator $\pi_\mathfrak{p}$ of $\mathfrak{p}$.

Remark 2.4. When k is odd, the Hecke character $\xi_{d,k}$ can be used to build an automorphic cusp form of weight $k+1$ and level $4\mathrm{N}(\mathfrak{f}_d)$, and with nebentypus given by the quadratic character modulo 4 multiplied by the restriction of $\xi_{d,k}$ to $\mathbb{Q}$ (see [18, Theorem 12.5]).

We conclude this section with the following definitions of the angle of the Gaussian primes with respect to the character ξ_d.

Definition 2.5. *Let d be an odd square-free integers, and $p \equiv 1 \pmod 4$. We write $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ and recall that $\xi_d(\overline{\mathfrak{p}}) = \overline{\xi_d(\mathfrak{p})}$. We define $\theta_{d,p}$ and $z_{d,p}$ by*

$$\xi_d(\mathfrak{p}) + \xi_d(\overline{\mathfrak{p}}) := 2\cos\theta_{d,p}, \quad \theta_{d,p} \in (0,\pi) \tag{2.21}$$

$$z_{d,p} := \sqrt{p} e^{i\theta_{d,p}} \in \mathbb{Z}[i]. \tag{2.22}$$

For example, if $d = 1$, then $z_{1,p} = a + 2bi$ is primary with $b \geq 0$, and $p = a^2 + 4b^2$. We note that, in general, for $p \equiv 1 \pmod 4$, $p \nmid d$, with $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, we have that

$$\xi_{d,k}(\mathfrak{p}) + \xi_{d,k}(\overline{\mathfrak{p}}) = 2\cos k\theta_{d,p}, \tag{2.23}$$

where $\theta_{d,p} \in (0,\pi)$ is the angle defined by (2.21).

2.5. **Computing Root Numbers.** From the work of Hecke [13], $L(s,\xi_{d,k})$ has analytic continuation to $\mathbb{C}$. Noting that the discriminant of $\mathbb{Q}(i)$ is equal to -4, we define the *completed L-function*

$$\Lambda(s,\xi_{d,k}) := (4\mathrm{N}(\mathfrak{f}_{d,k}))^{s/2} (2\pi)^{-s} \Gamma\left(s + \frac{k}{2}\right) L(s,\xi_{d,k})$$

where $\mathfrak{f}_{d,k}$ is as in (2.19). As in [19, Theorem 3.8], we note that $\Lambda(s,\xi_{d,k})$ satisfies the functional equation

$$\Lambda(s,\xi_{d,k}) = W(\xi_{d,k})\Lambda(1-s,\overline{\xi}_{d,k}) = W(\xi_{d,k})\Lambda(1-s,\xi_{d,k}), \tag{2.24}$$

where the sign of the functional equation is denoted by the *root number* $W(\xi_{d,k}) = \pm 1$, and where the last equality follows upon noting that $\overline{\xi}_{d,k}(\mathfrak{a}) = \xi_{d,k}(\overline{\mathfrak{a}})$.

We now proceed to compute $W(\xi_{d,k})$ explicitly, for any given $k \in \mathbb{N}$ and square-free odd $d \in \mathbb{Z}$.

Lemma 2.6. *Let d be an odd square-free integer. If k is even, the root number of $\xi_{d,k}$ is*

$$W(\xi_{d,k}) = 1. \tag{2.25}$$

If $d \equiv 1 \pmod 4$, then it satisfies

$$W(\xi_d^k) = \begin{cases} -\mathrm{sgn}(d) & \text{if } d \equiv 5, 9 \pmod{16} \text{ and } k \equiv 1, 3 \pmod 8 \\ \mathrm{sgn}(d) & \text{if } d \equiv 5, 9 \pmod{16} \text{ and } k \equiv 5, 7 \pmod 8 \\ \mathrm{sgn}(d) & \text{if } d \equiv 1, 13 \pmod{16} \text{ and } k \equiv 1, 3 \pmod 8 \\ -\mathrm{sgn}(d) & \text{if } d \equiv 1, 13 \pmod{16} \text{ and } k \equiv 5, 7 \pmod 8, \end{cases} \tag{2.26}$$

and if $d \equiv 3 \pmod 4$, we have

$$W(\xi_d^k) = \begin{cases} \mathrm{sgn}(d) & \text{if } d \equiv 3 \pmod 8 \text{ and } k \equiv 1 \pmod 4 \\ -\mathrm{sgn}(d) & \text{if } d \equiv 3 \pmod 8 \text{ and } k \equiv 3 \pmod 4 \\ -\mathrm{sgn}(d) & \text{if } d \equiv 7 \pmod 8 \text{ and } k \equiv 1 \pmod 4 \\ \mathrm{sgn}(d) & \text{if } d \equiv 7 \pmod 8 \text{ and } k \equiv 3 \pmod 4. \end{cases} \tag{2.27}$$

For any fixed square-free odd $d \in \mathbb{Z}$, we therefore conclude that

$$\lim_{K \to \infty} \frac{\#\{1 \leq k \leq K : W(\xi_{d,k}) = -1\}}{K} = \frac{1}{4}. \tag{2.28}$$

In particular, recalling definition (1.5), we have

$$S_-(d) = \begin{cases} \{5,7\} & \text{if } d \equiv 1, 13 \pmod{16} \text{ and } d > 0. \\ \{1,3\} & \text{if } d \equiv 1, 13 \pmod{16} \text{ and } d < 0. \\ \{1,3\} & \text{if } d \equiv 5, 9 \pmod{16} \text{ and } d > 0. \\ \{5,7\} & \text{if } d \equiv 5, 9 \pmod{16} \text{ and } d < 0. \\ \{3,7\} & \text{if } d \equiv 3 \pmod 8 \text{ and } d > 0. \\ \{1,5\} & \text{if } d \equiv 3 \pmod 8 \text{ and } d < 0. \\ \{1,5\} & \text{if } d \equiv 7 \pmod 8 \text{ and } d > 0. \\ \{3,7\} & \text{if } d \equiv 7 \pmod 8 \text{ and } d < 0. \end{cases} \tag{2.29}$$

Proof of Lemma 2.6. As in [19, (3.85), (3.86)] (see also [4, Section 4]), we will use the formula

$$W(\xi_{d,k}) = i^{-k} \mathrm{N}(\mathfrak{f}_{d,k})^{-\frac{1}{2}} \xi_{d,k,\infty}(\gamma_{d,k}) \sum_{x \in \mathbb{Z}[i]/\mathfrak{f}_{d,k}} \xi_{d,k,\mathrm{fin}}(x) e^{2\pi i \,\mathrm{Tr}\left(\frac{x}{\gamma_{d,k}}\right)}$$

where we take $\gamma_{d,k} \in \mathbb{Z}[i]$ to be any generator of the ideal $2\mathfrak{f}_{d,k}$, and $\mathfrak{c} = \mathbb{Z}[i]$, so that $(\mathfrak{c}, \mathfrak{f}_{d,k}) = 1$ and $(2)\mathfrak{c}\mathfrak{f}_{d,k} = (\gamma_{d,k})$.

First, we treat the case $k \equiv 0 \pmod 4$. By (2.19) we may choose $\gamma_{d,k} = 2$, from which it follows that

$$W(\xi_{d,k}) = e^{2\pi i \,\mathrm{Tr}\left(\frac{1}{2}\right)} = 1.$$

For $k \not\equiv 0 \,(\mathrm{mod}\, 4)$, we note by Lemma 2.3 that

$$\xi_{d,k,\mathrm{fin}}(x) = \overline{\chi}_{(d)}^{k}(x) \cdot \eta_{d,k}(x),$$

where

$$\eta_{d,k} := \begin{cases} \chi_{(2)} & \text{when } k \equiv 2 \,(\mathrm{mod}\, 4) \\ \chi_{(2+2i)}^{k} & \text{when } d \equiv 1 \,(\mathrm{mod}\, 8) \text{ and } k \text{ is odd} \\ \chi_{(2+2i)}^{k}\chi_{(2)} & \text{when } d \equiv 5 \,(\mathrm{mod}\, 8) \text{ and } k \text{ is odd} \\ \overline{\chi}_{(4)}^{k}\chi_{(2)} & \text{when } d \equiv 3 \,(\mathrm{mod}\, 8) \text{ and } k \text{ is odd} \\ \overline{\chi}_{(4)}^{k} & \text{when } d \equiv 7 \,(\mathrm{mod}\, 8) \text{ and } k \text{ is odd,} \end{cases}$$

is a primitive character modulo (g) with

$$g := \begin{cases} 2 & \text{when } k \equiv 2 \,(\mathrm{mod}\, 4) \\ 2+2i & \text{when } k \equiv 1 \,(\mathrm{mod}\, 2) \text{ and } d \equiv 1 \,(\mathrm{mod}\, 4) \\ 4 & \text{when } k \equiv 1 \,(\mathrm{mod}\, 2) \text{ and } d \equiv 3 \,(\mathrm{mod}\, 4). \end{cases}$$

Write $|d| = \prod_j p_j$, where p_j run through the distinct *rational* primes dividing the square-free odd integer d. Since $(g, d) = 1$, by the Chinese remainder theorem there exists a ring isomorphism

$$\mathbb{Z}[i]/(g) \times \prod_{p_j|d} \mathbb{Z}[i]/(p_j) \to \mathbb{Z}[i]/(gd) \quad (x_0, (x_j)_j) \mapsto ux_0 + \sum_j v_j x_j,$$

where $u \equiv 1 \,(\mathrm{mod}\, g)$ and $u \equiv 0 \,(\mathrm{mod}\, d)$, while $v_j \equiv 1 \,(\mathrm{mod}\, p_j)$ and $v_j \equiv 0 \,(\mathrm{mod}\, \frac{gd}{p_j})$ for all j. Upon choosing $\gamma_{d,k} = 2g|d|$, we find that

(2.30)

$$\begin{aligned} \sum_{x \in \mathbb{Z}[i]/\mathfrak{f}_{d,k}} \xi_{d,k,\mathrm{fin}}(x) e^{2\pi i \,\mathrm{Tr}\left(\frac{x}{\gamma_{d,k}}\right)} &= \sum_{x \in \mathbb{Z}[i]/(gd)} \overline{\chi}_{(d)}^{k}(x) \cdot \eta_{d,k}(x) e^{2\pi i \,\mathrm{Tr}\left(\frac{x}{2g|d|}\right)} \\ &= \sum_{x_0 \in \mathbb{Z}[i]/(g)} \prod_{p_j|d} \sum_{x_j \in \mathbb{Z}[i]/(p_j)} \overline{\chi}_{(p_j)}^{k}\Big(ux_0 + \sum_j v_j x_j\Big) \eta_{d,k}\Big(ux_0 + \sum_j v_j x_j\Big) e^{2\pi i \,\mathrm{Tr}\left(\frac{ux_0 + \sum_j v_j x_j}{2g|d|}\right)} \\ &= \sum_{x_0 \in \mathbb{Z}[i]/(g)} \eta_{d,k}\,(ux_0)\, e^{2\pi i \,\mathrm{Tr}\left(\frac{ux_0}{2g|d|}\right)} \prod_{p_j|d} \sum_{x_j \in \mathbb{Z}[i]/(p_j)} \overline{\chi}_{(p_j)}^{k}\,(v_j x_j)\, e^{2\pi i \,\mathrm{Tr}\left(\frac{v_j x_j}{2g|d|}\right)}. \end{aligned}$$

Applying the change of variables $\alpha = ux_0/|d|$ and $\beta = v_j x_j p_j / g|d|$, we then find that

$$
\begin{aligned}
W(\xi_{d,k}) &= \frac{i^{-k}}{|gd|}\left(\frac{g}{|g|}\right)^k \eta_{d,k}(|d|) \sum_{\alpha \in \mathbb{Z}[i]/(g)} \eta_{d,k}(\alpha) e^{2\pi i \operatorname{Tr}\left(\frac{\alpha}{2g}\right)} \\
&\quad \times \prod_{\substack{p_j | d \\ p_j > 0}} \overline{\chi}^k_{(p_j)}\left(\frac{g|d|}{p_j}\right) \sum_{\beta \in \mathbb{Z}[i]/(p_j)} \overline{\chi}^k_{(p_j)}(\beta) e^{2\pi i \operatorname{Tr}\left(\frac{\beta}{2p_j}\right)} \\
&= W(\xi_{d,k}, 2) \times \eta_{d,k}(|d|) \times \overline{\chi}^k_{(d)}(g) \times \prod_{\substack{p_j | d \\ p_j > 0}} W(\xi_{d,k}, p_j)
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
W(\xi_{d,k}, 2) &:= \frac{i^{-k}}{|g|}\left(\frac{g}{|g|}\right)^k \sum_{\alpha \in \mathbb{Z}[i]/(g)} \eta_{d,k}(\alpha) e^{2\pi i \operatorname{Tr}\left(\frac{\alpha}{2g}\right)} \\
W(\xi_{d,k}, p) &:= \frac{1}{p} \sum_{x \in \mathbb{Z}[i]/(p)} \overline{\chi}^k_{(p)}(x) e^{2\pi i \operatorname{Tr}\left(\frac{x}{2p}\right)},
\end{aligned}
$$

and where we have used (2.2).

We first study the contribution of 2. For $k \equiv 2 \pmod 4$, we compute

$$
W(\xi_{d,k}, 2) = -\frac{1}{2} \sum_{x \in \mathbb{Z}[i]/(2)} \eta_{d,k}(x) e^{2\pi i \operatorname{Tr}\left(\frac{x}{4}\right)} = -\frac{1}{2}\left(1 \cdot e^{i\pi} - 1 \cdot 1\right) = 1.
\tag{2.32}
$$

If k is odd and $d \equiv 1 \pmod 4$, then we compute

$$
\begin{aligned}
i^k e^{\frac{-i\pi k}{4}} W(\xi_{d,k}, 2) &= \frac{1}{2\sqrt{2}} \sum_{x \in \mathbb{Z}[i]/(2+2i)} \eta_{d,k}(x) e^{2\pi i \operatorname{Tr}\left(\frac{x}{4+4i}\right)} \\
&= \frac{1}{2\sqrt{2}}\left(1 \cdot e^{\frac{i\pi}{2}} - e^{-\frac{i\pi}{2}} + \eta_{d,k}(i) e^{\frac{i\pi}{2}} + \overline{\eta_{d,k}(i)} e^{\frac{-i\pi}{2}}\right) \\
&= \frac{i}{\sqrt{2}}\left(1 + \eta_{d,k}(i)\right) = \begin{cases} e^{i\left(\frac{\pi}{2} \mp \frac{\pi}{4}\right)} & \text{if } k \equiv \pm 1 \pmod 4 \text{ and } d \equiv 1 \pmod 8 \\ e^{i\left(\frac{\pi}{2} \pm \frac{\pi}{4}\right)} & \text{if } k \equiv \pm 1 \pmod 4 \text{ and } d \equiv -3 \pmod 8. \end{cases}
\end{aligned}
\tag{2.33}
$$

Similarly when k is odd and $d \equiv 3 \pmod 4$, we compute

$$
i^k W(\xi_{d,k}, 2) = \frac{1}{4} \sum_{x \in \{\pm 1, \pm i, \pm(3+2i), \pm(-2+3i)\}} \eta_{d,k}(x) e^{2\pi i \operatorname{Tr}\left(\frac{x}{8}\right)}
\tag{2.34}
$$

$$
= \frac{1}{4}\left(1 \cdot e^{\frac{i\pi}{2}} - e^{-\frac{i\pi}{2}} + \eta_{d,k}(i) + \eta_{d,k}(-i) - e^{\frac{3\pi i}{2}} + e^{-\frac{3\pi i}{2}} - \eta_{d,k}(2-3i) - \eta_{d,k}(-2+3i)\right) = i.
$$

Let us now study the contribution at a rational odd prime p. We have

$$\begin{aligned}\sum_{x\in\mathbb{Z}[i]/(p)} \overline{\chi}^k_{(p)}(x)e^{2\pi i\,\mathrm{Tr}\left(\frac{x}{2p}\right)} &= \sum_{a\in\mathbb{Z}/(p)}\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi)e^{2\pi i\,\mathrm{Tr}\left(\frac{a+bi}{2p}\right)} \\ &= \sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(bi) + \sum_{a\in(\mathbb{Z}/(p))^\times}\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi)e^{2\pi i\,\mathrm{Tr}\left(\frac{a+bi}{2p}\right)} \\ &= (p-1)\overline{\chi}^k_{(p)}(i) + \sum_{a\in(\mathbb{Z}/(p))^\times} e^{\frac{2\pi i a}{p}} \sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi)\end{aligned}$$

by (2.2). Note that the sum

$$\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi) = \overline{\chi}^k_{(p)}(a)\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(1+bi) = \sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(1+bi)$$

is independent of $a\in(\mathbb{Z}/(p))^\times$. Moreover, by orthogonality, we find that

$$0 = \sum_{a\in(\mathbb{Z}/(p))}\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi) = \sum_{a\in(\mathbb{Z}/(p))^\times}\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi) + (p-1)\overline{\chi}^k_{(p)}(i).$$

It follows that for all $a\in(\mathbb{Z}/(p))^\times$,

$$\sum_{b\in\mathbb{Z}/(p)} \overline{\chi}^k_{(p)}(a+bi) = -\overline{\chi}^k_{(p)}(i).$$

Hence

$$\begin{aligned} W(\xi_{d,k},p) = \frac{1}{p}\sum_{x\in\mathbb{Z}[i]/(p)} \overline{\chi}^k_{(p)}(x)e^{2\pi i\,\mathrm{Tr}\left(\frac{x}{2p}\right)} &= \frac{\overline{\chi}^k_{(p)}(i)}{p}\left((p-1) - \sum_{a\in(\mathbb{Z}/(p))^\times} e^{\frac{2\pi i a}{p}}\right) \\ &= \overline{\chi}^k_{(p)}(i) = \begin{cases}(-1)^k & \text{when } p\equiv\pm3\ (\mathrm{mod}\ 8)\\ 1 & \text{when } p\equiv\pm1\ (\mathrm{mod}\ 8),\end{cases}\end{aligned} \tag{2.35}$$

and therefore for odd square-free $d\in\mathbb{Z}$,

$$\prod_{\substack{p_j|d\\ p_j>0}} W(\xi_{d,k},p_j) = \begin{cases}(-1)^k & \text{when } d\equiv\pm3\ (\mathrm{mod}\ 8)\\ 1 & \text{when } d\equiv\pm1\ (\mathrm{mod}\ 8).\end{cases} \tag{2.36}$$

By (2.31), (2.32) and (2.36), it follows that when $k\equiv 2\ (\mathrm{mod}\ 4)$,

$$W(\xi_{d,k}) = \eta_{d,k}(|d|) = 1.$$

This proves (2.25). When $k\equiv\pm1\ (\mathrm{mod}\ 4)$ and $d\equiv 1\ (\mathrm{mod}\ 8)$, it follows from (2.8), (2.31), (2.33) and (2.36), that

$$\begin{aligned} W(\xi_{d,k}) &= W(\xi_{d,k},2)\times\eta_{d,k}(|d|)\times\overline{\chi}^k_{(d)}(2+2i)\times\prod_{\substack{p_j|d\\ p_j>0}} W(\xi_{d,k},p_j) \\ &= \mp i e^{\frac{i\pi k}{4}}\, e^{i\left(\frac{\pi}{2}\mp\frac{\pi}{4}\right)}\, \chi^k_{(2+2i)}(|d|)\,\overline{\chi}^k_{(d)}(1+i)\overline{\chi}^k_{(d)}(2) \\ &= \pm e^{\frac{i\pi(k\mp1)}{4}}\mathrm{sgn}(d)(-i)^{\frac{k(d-1)}{4}},\end{aligned}$$

where in the last line we note that $\chi^k_{(2+2i)}(|d|) = \mathrm{sgn}(d)$. Similarly, when $k \equiv \pm 1 \,(\mathrm{mod}\, 4)$ and $d \equiv 5 \,(\mathrm{mod}\, 8)$, we find that

$$\begin{aligned} W(\xi_{d,k}) &= \pm i e^{\frac{i\pi k}{4}}\; e^{i\left(\frac{\pi}{2} \pm \frac{\pi}{4}\right)}\; \eta_{d,k}(|d|)\; \overline{\chi}^k_{(d)}(2+2i) \\ &= \mp e^{\frac{i\pi(k\pm 1)}{4}} \mathrm{sgn}(d)(-i)^{\frac{k(d-1)}{4}}. \end{aligned}$$

This proves (2.26). Finally, when k is odd and $d \equiv 3 \,(\mathrm{mod}\, 8)$, we note by (2.31), (2.34), (2.36), that

$$\begin{aligned} W(\xi_{d,k}) &= W(\xi_{d,k}, 2) \times \eta_{d,k}(|d|) \times \overline{\chi}^k_{(d)}(4) \times \prod_{\substack{p_j | d \\ p_j > 0}} W(\xi_{d,k}, p_j) \\ &= -i^{1-k} \overline{\chi}^k_{(4)}(|d|) \; = (-1)^{\frac{k-1}{2}} \mathrm{sgn}(d), \end{aligned}$$

and if $d \equiv 7 \,(\mathrm{mod}\, 8)$, we have

$$W(\xi_{d,k}) = i^{1-k} \overline{\chi}^k_{(4)}(|d|) = -(-1)^{\frac{k-1}{2}} \mathrm{sgn}(d),$$

which proves (2.27). □

3. Computing $\mathcal{D}(\phi, \xi_{d,k})$

To begin our analysis of $\mathcal{D}(\phi, \xi_{d,k})$, let us first explain the scaling parameter in (1.1). Let

$$\mathcal{N}_{d,k}(T) := \#\{s \in \mathbb{C} : L(s, \xi_{d,k}) = 0, 0 \le \mathrm{Re}(s) \le 1, -T \le \mathrm{Im}(s) \le T\}$$

denote the number of zeros of $L(s, \xi_{d,k})$ on the critical strip up to height T. Using the functional equation (2.24) as in [15], we find that in the limit as $k \to \infty$,

$$\frac{1}{2T} \mathcal{N}_{d,k}(T) \sim \frac{1}{2T} \frac{T \log(T k^2 \mathrm{N}(\mathfrak{f}_{d,k}))}{\pi} \sim \frac{\log\left(k^2 N_{d,k}\right)}{2\pi} = \frac{\log\left(k M_{d,k}\right)}{\pi}, \tag{3.1}$$

under the assumption of GRH (see also [19, Thm. 5.8]), and where we use the notation

$$N_{d,k} := \mathrm{N}(\mathfrak{f}_{d,k}), \quad M_{d,k} := \mathrm{N}(\mathfrak{f}_{d,k})^{\frac{1}{2}}.$$

To see that the analytic conductor of $L(s, \xi_{d,k})$ is asymptotic to $k^2 \mathrm{N}(\mathfrak{f}_{d,k})$ according to the definition [19, p. 95], we use the duplication formula to write

$$\Lambda(s, \xi_{d,k}) = \frac{2^{\frac{k}{2}}}{2\sqrt{\pi}} \left(4\mathrm{N}(\mathfrak{f}_{d,k})\right)^{s/2} \pi^{-s} \Gamma\Big(\frac{s + \frac{k}{2}}{2}\Big) \Gamma\Big(\frac{s + \frac{k}{2} + 1}{2}\Big) L(s, \xi_{d,k}).$$

This justifies our definition of the quantity (1.1) since the scaling parameter has been appropriately chosen so that the average spacing between the scaled zeros is 1. Several different normalizations are used in the literature; we found this one more natural to compute the lower order terms in descending powers of $(\log k M_{d,k})^{-1}$ since it guarantees that the main term in Lemma 3.1 is exactly $\widehat{\phi}(0)$.

For each character $\xi_{d,k}$, we rewrite the functional equation (2.24) as

$$L(s, \xi_{d,k}) = X_{d,k}(s) L(1-s, \xi_{d,k}),$$

where

$$X_{d,k}(s) := W(\xi_{d,k}) \left(\mathrm{N}(\mathfrak{f}_{d,k})\right)^{\frac{1}{2}-s} \pi^{2s-1} \frac{\Gamma\left(1 - s + \frac{k}{2}\right)}{\Gamma\left(s + \frac{k}{2}\right)}.$$

Upon taking logarithmic derivatives, we find that

$$\frac{L'}{L}(s,\xi_{d,k}) = \frac{X'_{d,k}}{X_{d,k}}(s) - \frac{L'}{L}(1-s,\xi_{d,k}) \tag{3.2}$$

$$\frac{X'_{d,k}}{X_{d,k}}(s) = -\log \mathrm{N}(\mathfrak{f}_{d,k}) + 2\log\pi - \frac{\Gamma'}{\Gamma}\Big(1-s+\frac{k}{2}\Big) - \frac{\Gamma'}{\Gamma}\Big(s+\frac{k}{2}\Big). \tag{3.3}$$

Since ϕ is an even Schwartz function, and $L(s,\xi_{d,k})$ has no trivial zeros for $-\frac{1}{4} < \mathrm{Re}(s) < \frac{5}{4}$, we see from (3.2) that

$$\begin{aligned}
\mathcal{D}(\phi,\xi_{d,k}) &= \frac{1}{2\pi i}\left(\int_{(\frac{5}{4})} - \int_{(-\frac{1}{4})}\right)\frac{L'}{L}(s,\xi_{d,k})\phi\left(\frac{\log(k^2N_{d,k})}{2\pi}\ \frac{s-\frac{1}{2}}{i}\right)\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{(\frac{5}{4})}\left(\frac{L'}{L}(s,\xi_{d,k}) - \frac{L'}{L}(1-s,\xi_{d,k})\right)\phi\left(\frac{\log(k^2N_{d,k})}{2\pi}\ \frac{s-\frac{1}{2}}{i}\right)\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{(\frac{5}{4})}\left(2\frac{L'}{L}(s,\xi_{d,k}) - \frac{X'_{d,k}}{X_{d,k}}(s)\right)\phi\left(\frac{\log(k^2N_{d,k})}{2\pi}\ \frac{s-\frac{1}{2}}{i}\right)\mathrm{d}s \\
&= U_L(\phi,d,k) + U_\Gamma(\phi,d,k), \qquad (3.4)
\end{aligned}$$

where

$$U_L(\phi,d,k) := \frac{1}{\pi i}\int_{(\frac{5}{4})}\frac{L'}{L}(s,\xi_{d,k})\phi\left(\frac{\log(k^2N_{d,k})}{2\pi}\ \frac{s-\frac{1}{2}}{i}\right)\mathrm{d}s \tag{3.5}$$

and

$$U_\Gamma(\phi,d,k) := -\frac{1}{2\pi i}\int_{(\frac{5}{4})}\frac{X'_{d,k}}{X_{d,k}}(s)\phi\left(\frac{\log(k^2N_{d,k})}{2\pi}\ \frac{s-\frac{1}{2}}{i}\right)\mathrm{d}s. \tag{3.6}$$

Lemma 3.1. *Let k be a positive integer, d be a square-free integer, and ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then as $k\to\infty$,*

$$U_\Gamma(\phi,d,k) = \widehat{\phi}(0) - \frac{\log 2\pi}{\log(kM_{d,k})}\widehat{\phi}(0) + O\left(\frac{1}{k\log(kM_{d,k})}\right).$$

Proof. Upon applying the change of variables $r = s-\frac{1}{2}$ to (3.6) and using (3.3), we have

$$\begin{aligned}
U_\Gamma(\phi,d,k) = \frac{1}{2\pi i}\int_{(\frac{3}{4})}\left(\log N_{d,k} - 2\log\pi + \frac{\Gamma'}{\Gamma}\Big(\frac{k+1}{2}-r\Big) + \frac{\Gamma'}{\Gamma}\Big(\frac{k+1}{2}+r\Big)\right) \\
\times\,\phi\left(\frac{\log(k^2N_{d,k})}{2\pi i}\ r\right)\mathrm{d}r.
\end{aligned}$$

Noting that $\Gamma'(s)/\Gamma(s)$ is holomorphic on the half-plane $\mathrm{Re}(s)>0$, we shift the contour to the imaginary line $\mathrm{Re}(r)=0$, and upon applying the change of variables

$$\tau := \frac{\log(k^2N_{d,k})}{2\pi i}\ r,$$

we find that

$$U_\Gamma(\phi,d,k)=\frac{1}{\log(k^2N_{d,k})}\int_{\mathbb{R}}\Big(\log N_{d,k}-2\log\pi+\frac{\Gamma'}{\Gamma}\Big(\frac{k+1}{2}-\frac{2\pi i\tau}{\log(k^2N_{d,k})}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{k+1}{2}+\frac{2\pi i\tau}{\log(k^2N_{d,k})}\Big)\Big)\phi(\tau)\,\mathrm{d}\tau.$$

As in the proof of [32, Lemma 5.1], it follows that

$$\begin{aligned}U_\Gamma(\phi,d,k)&=\frac{1}{\log(k^2N_{d,k})}\Big((\log N_{d,k}-2\log\pi)\widehat{\phi}(0)+2\int_{\mathbb{R}}\phi(\tau)\Big(\log\Big|\tfrac{k+1}{2}+\tfrac{2\pi i\tau}{\log(k^2N_{d,k})}\Big|+O\left(\tfrac{1}{k}\right)\Big)\mathrm{d}\tau\Big)\\&=\frac{\widehat{\phi}(0)}{\log(k^2N_{d,k})}\Big(\log N_{d,k}-\log\pi^2+2\log\left(\tfrac{k+1}{2}\right)+O\big(\tfrac{1}{k}\big)\Big)\\&=\widehat{\phi}(0)-\frac{\log(4\pi^2)}{\log(k^2N_{d,k})}\widehat{\phi}(0)+O\Big(\frac{1}{k\log(k^2N_{d,k})}\Big)\end{aligned}$$

as desired. □

We now compute the contribution of $U_L(\phi,d,k)$. Separating according to the splitting properties of $(p)\subseteq\mathbb{Z}[i]$, we write

$$-\frac{L'(s,\xi_{d,k})}{L(s,\xi_{d,k})}=\sum_{\substack{p\equiv1\,(\mathrm{mod}\,4)\\(p)=\mathfrak{p}\overline{\mathfrak{p}}\\n\geq1}}\frac{\big(\xi_{d,k}^n(\mathfrak{p})+\overline{\xi}_{d,k}^n(\mathfrak{p})\big)\log p}{p^{ns}}+\sum_{\substack{p\equiv3\,(\mathrm{mod}\,4)\\n\geq1}}\frac{2\xi_{d,k}^n((p))\log p}{p^{2ns}}+\sum_{n\geq1}\frac{\xi_{d,k}^n((1+i))\log 2}{2^{ns}},\tag{3.7}$$

and

$$U_L(\phi,d,k)=U_{\mathrm{split}}(\phi,d,k)+U_{\mathrm{inert}}(\phi,d,k)+U_{\mathrm{ram}}(\phi,k)\tag{3.8}$$

where

$$U_{\mathrm{split}}(\phi,d,k):=-\frac{1}{\pi i}\int_{(\frac{5}{4})}\sum_{\substack{p\equiv1\,(\mathrm{mod}\,4)\\(p)=\mathfrak{p}\overline{\mathfrak{p}}\\n\geq1}}\frac{\big(\xi_{d,k}^n(\mathfrak{p})+\overline{\xi}_{d,k}^n(\mathfrak{p})\big)\log p}{p^{ns}}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)\mathrm{d}s\tag{3.9}$$

$$U_{\mathrm{inert}}(\phi,d,k):=-\frac{1}{\pi i}\int_{(\frac{5}{4})}\sum_{\substack{p\equiv3\,(\mathrm{mod}\,4)\\(p)\nmid\mathfrak{f}_{d,k}\\n\geq1}}\frac{2(-1)^{kn}\log p}{p^{2ns}}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)\mathrm{d}s\tag{3.10}$$

$$U_{\mathrm{ram}}(\phi,k):=-\frac{1}{\pi i}\int_{(\frac{5}{4})}\sum_{n\geq1}\frac{\xi_{d,k}^n((1+i))\log 2}{2^{ns}}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)\mathrm{d}s.\tag{3.11}$$

Note that by (2.20), one has

$$\xi_{d,k}((1+i))=\begin{cases}0 & \text{if } k\not\equiv0\,(\mathrm{mod}\,4)\\(-1)^{\frac{k}{4}} & \text{if } k\equiv0\,(\mathrm{mod}\,4),\end{cases}$$

so that $U_{\mathrm{ram}}(\phi, k)$ is indeed independent of d.

Lemma 3.2. *Let k be a positive integer, d be a square-free integer, and ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Let $U_{\mathrm{split}}(\phi, d, k)$, $U_{\mathrm{inert}}(\phi, d, k)$, and $U_{\mathrm{ram}}(\phi, k)$ be given by (3.9), (3.10) and (3.11), respectively. Then*

$$(3.12)\quad U_{\mathrm{split}}(\phi, d, k) = -\frac{1}{\log(kM_{d,k})} \sum_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ (p)=\mathfrak{p}\overline{\mathfrak{p}}\\ n\geq 1}} \frac{\left(\xi_{d,k}^n(\mathfrak{p}) + \overline{\xi}_{d,k}^n(\mathfrak{p})\right)\log p}{p^{n/2}} \widehat{\phi}\left(\frac{n\log p}{2\log(kM_{d,k})}\right)$$

$$(3.13)\quad U_{\mathrm{inert}}(\phi, d, k) = -\frac{1}{\log(kM_{d,k})} \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ (p)\nmid\mathfrak{f}_{d,k}\\ n\geq 1}} \frac{2(-1)^{kn}\log p}{p^n}\, \widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right)$$

$$(3.14)\quad U_{\mathrm{ram}}(\phi, k) = -\frac{1}{\log(kM_{d,k})} \sum_{n\geq 1} \frac{(-1)^{\frac{kn}{4}}\log 2}{2^{\frac{n}{2}}} \widehat{\phi}\left(\frac{n\log 2}{2\log(kM_{d,k})}\right) \qquad \text{if } k\equiv 0 \,(\mathrm{mod}\, 4),$$

and $U_{\mathrm{ram}}(\phi, k) = 0$ if $k \not\equiv 0 \,(\mathrm{mod}\, 4)$.

Proof. For the inert primes, we compute

$$\begin{aligned} -U_{\mathrm{inert}}(\phi, d, k) &= \frac{1}{\pi i}\int_{(\frac{5}{4})} \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ (p)\nmid\mathfrak{f}_{d,k}\\ n\geq 1}} \frac{2(-1)^{kn}\log p}{p^{2ns}}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right) \mathrm{d}s \\ &= \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ (p)\nmid\mathfrak{f}_{d,k}\\ n\geq 1}} \frac{2(-1)^{kn}\log p}{\pi i}\int_{(\frac{1}{2})} p^{-2ns}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right) \mathrm{d}s. \end{aligned}$$

Note that switching the order of summation and integration is justified upon noting that at $\mathrm{Re}(s) = \frac{5}{4}$,

$$\begin{aligned} &\int_{(\frac{5}{4})} \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ (p)\nmid\mathfrak{f}_{d,k}\\ n\geq 1}} \left|\frac{2(-1)^{kn}\log p}{p^{2ns}}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)\right| \mathrm{d}s \\ &\qquad\ll \sum_{m} \frac{\log m}{m^4}\int_{(\frac{5}{4})} \left|\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)\right| \mathrm{d}s \ll \infty, \end{aligned}$$

since ϕ is a Schwartz function (e.g. [32, Lemma 3.7]). The shift from $\mathrm{Re}(s) = \frac{5}{4}$ to $\mathrm{Re}(s) = \frac{1}{2}$ is then further justified upon noting that $s \mapsto p^{-2ns}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)$ is holomorphic. Applying the change of variables $t = (s-\frac{1}{2})\frac{\log(k^2N_{d,k})}{2\pi i}$, and noting that $p^{-2ns} = p^{-2n(s-\frac{1}{2})}p^{-n}$, we obtain

$$\frac{1}{\pi i}\int_{(\frac{1}{2})} p^{-2ns}\phi\left(\frac{(s-\frac{1}{2})\log(k^2N_{d,k})}{2\pi i}\right)\,\mathrm{d}s$$
$$=\frac{2}{p^n\log(k^2N_{d,k})}\int_{-\infty}^{\infty}\phi(t)e^{-2\pi i\frac{2n\log p}{\log(k^2N_{d,k})}t}\mathrm{d}t=\frac{1}{p^n\log(kM_{d,k})}\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right).$$

The proofs for $U_{\mathrm{split}}(\phi,d,k)$ and $U_{\mathrm{ram}}(\phi,k)$ follow similarly. □

We now compute $U_{\mathrm{ram}}(\phi,k)$, including lower-order terms in descending powers of $\log(kM_{d,k})$.

Lemma 3.3. *Let k be a positive integer, d be a square-free integer, and ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then for any $J\in\mathbb{N}$ and $k\to\infty$, we have*

$$U_{\mathrm{ram}}(\phi,k)=\sum_{\substack{j=0\\ j\text{ even}}}^{J-1}\frac{c_{j,\mathrm{ram}}(k)\ \widehat{\phi}^{(j)}(0)}{(\log(kM_{d,k}))^{j+1}}+O_J\left((\log(kM_{d,k}))^{-J-1}\right),$$

where for $j\geq 0$,

$$c_{j,\mathrm{ram}}(k):=\begin{cases}-\frac{2}{j!}\left(\frac{\log 2}{2}\right)^{j+1}\mathrm{Li}_{-j}\left(\frac{(-1)^{\frac{k}{4}}}{\sqrt{2}}\right) & \text{if } k\equiv 0 \pmod 4\\ 0 & \text{if } k\not\equiv 0 \pmod 4,\end{cases}\tag{3.15}$$

and

$$\mathrm{Li}_{-j}(z):=\sum_{n=1}^{\infty}n^jz^n.\tag{3.16}$$

Proof. Suppose $k\equiv 0\pmod 4$. Then by (3.14), we have

$$U_{\mathrm{ram}}(\phi,k)=-\frac{1}{\log(kM_{d,k})}\sum_{n=1}^{N}\frac{(-1)^{\frac{kn}{4}}\log 2}{2^{\frac{n}{2}}}\widehat{\phi}\left(\frac{n\log 2}{2\log(kM_{d,k})}\right)$$
$$-\frac{1}{\log(kM_{d,k})}\sum_{n\geq N+1}\frac{(-1)^{\frac{kn}{4}}\log 2}{2^{\frac{n}{2}}}\widehat{\phi}\left(\frac{n\log 2}{2\log(kM_{d,k})}\right),$$

for any $N\in\mathbb{N}$. Choosing $N=2J\log\log(kM_{d,k})/\log 2$, we then bound

$$\frac{1}{\log(kM_{d,k})}\sum_{n\geq N+1}\frac{(-1)^{\frac{kn}{4}}\log 2}{2^{\frac{n}{2}}}\widehat{\phi}\left(\frac{n\log 2}{2\log(kM_{d,k})}\right)\ll\frac{2^{-\frac{N}{2}}}{\log(kM_{d,k})}=\frac{2^{-\frac{J\log\log(kM_{d,k})}{\log 2}}}{\log(kM_{d,k})}=(\log(kM_{d,k}))^{-J-1}.$$

Moreover, by Taylor expansion one has

$$\widehat{\phi}\left(\frac{n\log 2}{2\log(kM_{d,k})}\right)=\sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j)}(0)}{j!}\left(\frac{n\log 2}{2\log(kM_{d,k})}\right)^j+O_J\left(\left(\frac{n\log 2}{2\log(kM_{d,k})}\right)^J\right),\tag{3.17}$$

so that

$$\sum_{n=1}^{N} \frac{(-1)^{\frac{kn}{4}} \log 2}{2^{\frac{n}{2}}} \widehat{\phi}\left(\frac{n \log 2}{2\log(kM_{d,k})}\right)$$
$$= \sum_{n=1}^{N} \frac{(-1)^{\frac{kn}{4}} \log 2}{2^{\frac{n}{2}}} \left(\sum_{j=0}^{J-1} \frac{\widehat{\phi}^{(j)}(0)}{j!} \left(\frac{n \log 2}{2\log(kM_{d,k})}\right)^{j} + O_J\left(\left(\frac{n \log 2}{2\log(kM_{d,k})}\right)^{J}\right)\right).$$

Since

$$\sum_{n=1}^{N} \frac{\log 2}{2^{\frac{n}{2}}} (n\log 2)^J \ll_J \sum_{n=1}^{\infty} \frac{n^J}{2^{\frac{n}{2}}} < \infty,$$

we find that

$$U_{\mathrm{ram}}(\phi, k) = -2\sum_{j=0}^{J-1} \frac{\widehat{\phi}^{(j)}(0)}{j!} \left(\frac{\log 2}{2\log(kM_{d,k})}\right)^{j+1} \sum_{n=1}^{N} \frac{(-1)^{\frac{kn}{4}} n^j}{2^{\frac{n}{2}}} + O_J\left((\log(kM_{d,k}))^{-J-1}\right).$$

Since $n^j/2^{n/2}$ is ultimately decreasing, one has

$$\sum_{n=N+1}^{\infty} \frac{(-1)^{\frac{kn}{4}} n^j}{2^{\frac{n}{2}}} \ll \sum_{n=N+1}^{\infty} \frac{n^j}{2^{\frac{n}{2}}} \ll_j \int_N^\infty \frac{x^j}{2^{\frac{x}{2}}} \mathrm{d}x = \int_N^\infty x^j e^{-\frac{x}{2}\log 2} \mathrm{d}x,$$

so that upon repeated application of integration by parts, we find that

$$\int_N^\infty x^j e^{-\frac{x}{2}\log 2}\mathrm{d}x = \frac{N^j}{2^{N/2}} \frac{2}{\log 2} + \frac{2j}{\log 2}\int_N^\infty x^{j-1} e^{-x\frac{\log 2}{2}}\mathrm{d}x = O_j\left(\frac{N^j}{2^{N/2}}\right) = O_J\left(\frac{(\log\log(kM_{d,k}))^j}{(\log(kM_{d,k}))^J}\right).$$

We thus may write

$$\begin{aligned} U_{\mathrm{ram}}(\phi, k) =& -2\sum_{j=0}^{J-1} \frac{\widehat{\phi}^{(j)}(0)}{j!} \left(\frac{\log 2}{2\log(kM_{d,k})}\right)^{j+1} \left(\sum_{n=1}^{\infty} \frac{(-1)^{\frac{kn}{4}} n^j}{2^{\frac{n}{2}}} + O_J\left(\frac{(\log\log(kM_{d,k}))^j}{(\log(kM_{d,k}))^J}\right)\right) \\ &+ O_J\left((\log(kM_{d,k}))^{-J-1}\right) \\ =& -2\sum_{j=0}^{J-1} \frac{\widehat{\phi}^{(j)}(0)}{j!} \left(\frac{\log 2}{2\log(kM_{d,k})}\right)^{j+1} \sum_{n=1}^{\infty} \frac{(-1)^{\frac{kn}{4}} n^j}{2^{\frac{n}{2}}} + O_J\left((\log(kM_{d,k}))^{-J-1}\right), \end{aligned}$$

as desired, where we note that since ϕ is even, we have $\widehat{\phi}^{(j)}(0) = 0$ for all odd j. □

Next, we write

$$U_{\mathrm{inert}}(\phi, d, k) = U_{\mathrm{inert}}(\phi, k) + U_{\mathrm{inert},d}(\phi, k),$$

where

$$U_{\mathrm{inert}}(\phi, k) := -\frac{1}{\log(kM_{d,k})} \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4) \\ n\geq 1}} \frac{2(-1)^{kn}\log p}{p^n} \,\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right) \tag{3.18}$$

and

$$U_{\mathrm{inert},d}(\phi, k) := \frac{1}{\log(kM_{d,k})} \sum_{\substack{p\equiv 3 \bmod 4 \\ (p)|\mathfrak{f}_{d,k} \\ n\geq 1}} \frac{2(-1)^{kn}\log p}{p^n} \,\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right).$$

We proceed to compute $U_{\mathrm{inert},d}(\phi,k)$ including the lower-order terms in descending powers of $\log(kM_{d,k})$.

Lemma 3.4. *Let k be a positive integer, d be a square-free integer, and ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then for any $J\in\mathbb{N}$ and $k\to\infty$, we have*

$$U_{\mathrm{inert},d}(\phi,k)=\sum_{\substack{j=0\\ j\text{ even}}}^{J-1}\frac{c_{j,\mathrm{inert},d}(k)\ \widehat{\phi}^{(j)}(0)}{(\log(kM_{d,k}))^{j+1}}+O_J\left(\frac{\log|d|}{(\log(kM_{d,k}))^{J+1}}\right),$$

where for $j\geq 0$, the constants $c_{j,\mathrm{inert},d}(k)$ are

$$c_{j,\mathrm{inert},d}(k):=\begin{cases}\frac{2}{j!}\sum_{\substack{p\equiv 3\ (\mathrm{mod}\ 4)\\ p\mid d}}(\log p)^{j+1}\mathrm{Li}_{-j}\left(\frac{(-1)^k}{p}\right) & \text{if } k\not\equiv 0\ (\mathrm{mod}\ 4)\\ 0 & \text{if } k\equiv 0\ (\mathrm{mod}\ 4).\end{cases}\tag{3.19}$$

and $\mathrm{Li}_{-j}(z)$ is defined by (3.16).

Proof. We get for $k\not\equiv 0\ (\mathrm{mod}\ 4)$

$$U_{\mathrm{inert},d}(\phi,k)=\frac{1}{\log(kM_{d,k})}\sum_{\substack{p\equiv 3\ (\mathrm{mod}\ 4)\\ p\mid d}}\sum_{n\geq 1}\frac{2(-1)^{kn}\log p}{p^n}\,\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right).$$

For each such $p|d$, we cut the inner sum at $N_p=J\log\log(kM_{d,k})/\log p$, to obtain

$$\frac{1}{\log(kM_{d,k})}\sum_{n\geq N_p+1}\frac{2(-1)^{kn}\log p}{p^n}\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right)\ll\frac{p^{-(N_p+1)}\log p}{\log(kM_{d,k})}=\frac{\log p}{p}(\log(kM_{d,k}))^{-J-1}.$$

Taylor expanding as in (3.17), we find that

$$\begin{aligned}&\frac{1}{\log(kM_{d,k})}\sum_{n=1}^{\infty}\frac{2(-1)^{kn}\log p}{p^n}\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right)\\ &=\frac{1}{\log(kM_{d,k})}\sum_{1\leq n\leq N_p}\frac{2(-1)^{kn}\log p}{p^n}\sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j)}(0)}{j!}\left(\frac{n\log p}{\log(kM_{d,k})}\right)^j+O_J\left(\frac{1}{p}\left(\frac{\log p}{\log(kM_{d,k})}\right)^{J+1}\right),\end{aligned}$$

where the bound on the error term is computed upon noting that

$$\frac{1}{\log(kM_{d,k})}\sum_{1\leq n\leq N_p}\frac{\log p}{p^n}\left(\frac{n\log p}{\log(kM_{d,k})}\right)^J\ll\sum_{n=1}^{\infty}\frac{n^J}{p^n}\left(\frac{\log p}{\log(kM_{d,k})}\right)^{J+1}\ll_J\frac{1}{p}\left(\frac{\log p}{\log(kM_{d,k})}\right)^{J+1}.$$

Upon applying the trivial bound

$$\sum_{p\mid d}\frac{(\log p)^{J+1}}{p}\ll_J\sum_{p\mid d}\log p\leq\log|d|,$$

we thus obtain

$$U_{\mathrm{inert},d}(\phi,k)=\sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j)}(0)}{j!}\sum_{\substack{p\equiv 3\ (\mathrm{mod}\ 4)\\ p\mid d}}\sum_{1\leq n\leq N_p}\frac{2(-1)^{kn}n^j}{p^n}\left(\frac{\log p}{\log(kM_{d,k})}\right)^{j+1}+O_J\left(\frac{\log|d|}{(\log(kM_{d,k}))^{J+1}}\right).$$

Upon repeated application of integration by parts, we find similarly to as above that

$$\sum_{n\geq N_p+1}\frac{n^j(\log p)^{j+1}}{p^n}\ll_j\frac{N_p^j(\log p)^{j+1}}{p^{N_p}}=O_J\left(\frac{(\log\log(kM_{d,k}))^j\log p}{(\log(kM_{d,k}))^J}\right),$$

and therefore we have

$$U_{\text{inert},d}(\phi,k)=\sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j)}(0)}{j!(\log(kM_{d,k}))^{j+1}}\sum_{\substack{p\equiv 3\ (\mathrm{mod}\ 4)\\ p\mid d}}\sum_{n\geq 1}\frac{2(-1)^{kn}n^j(\log p)^{j+1}}{p^n}+O_J\left(\frac{\log|d|}{(\log(kM_{d,k}))^{J+1}}\right),$$

as desired. □

Next we compute $U_{\text{inert}}(\phi,k)$. To this end, we define

$$\psi_k(t;3,4):=(-1)^k\sum_{\substack{p^n\leq t\\ p\equiv 3\ (\mathrm{mod}\ 4)\\ n\geq 1}}(-1)^{kn}\log p=\frac{t}{2}+O_A\left(\frac{t}{(\log t)^A}\right), \tag{3.20}$$

for any $A>0$, by the prime number theorem in arithmetic progressions.

Lemma 3.5. *Let k be a positive integer, d be a square-free integer, and ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then for any $J\in\mathbb{N}$ and $k\to\infty$, we have*

$$U_{\text{inert}}(\phi,k)=\frac{(-1)^{k+1}}{2}\int_{\mathbb{R}}\widehat{\phi}(u)du+\sum_{\substack{j=0\\ j\text{ even}}}^{J-1}\frac{c_{j,\text{inert}}(k)\ \widehat{\phi}^{(j)}(0)}{(\log(kM_{d,k}))^{j+1}}+O_J\left((\log(kM_{d,k}))^{-J-1}\right),$$

where

$$c_{0,\text{inert}}(k):=(-1)^{k+1}\left(1+2\int_1^\infty\frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}\mathrm{d}t\right) \tag{3.21}$$

and for $j\geq 1$,

$$c_{j,\text{inert}}(k):=2(-1)^k\int_1^\infty\frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t. \tag{3.22}$$

Proof. Recalling (3.18), we write

$$U_{\text{inert}}(\phi,k)=-\frac{1}{\log(kM_{d,k})}\sum_{\substack{p\equiv 3\ (\mathrm{mod}\ 4)\\ n\geq 1}}\frac{2(-1)^{kn}\log p}{p^n}\widehat{\phi}\left(\frac{n\log p}{\log(kM_{d,k})}\right)$$

and compute

$$\sum_{\substack{p\equiv 3 \ (\mathrm{mod}\ 4)\\ n\geq 1}} \frac{2(-1)^{kn}\log p}{p^n}\,\widehat{\phi}\left(\frac{n\log p}{\log\left(kM_{d,k}\right)}\right)$$

$$= 2(-1)^k\int_1^\infty \widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)\frac{\mathrm{d}\psi_k(t;3,4)}{t} = 2(-1)^{k+1}\int_1^\infty \psi_k(t;3,4)\mathrm{d}\left(\frac{\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)}{t}\right)$$

$$= (-1)^k\left(\widehat{\phi}(0)+\int_1^\infty \widehat{\phi}\left(\frac{\log t}{\log\left(kM_{d,k}\right)}\right)\frac{\mathrm{d}t}{t}\right) - 2(-1)^k\int_1^\infty\left(\psi_k(t;3,4)-\frac{t}{2}\right)\mathrm{d}\left(\frac{\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)}{t}\right)$$

(3.23)

$$= (-1)^k\left(\widehat{\phi}(0)+\frac{\log\left(kM_{d,k}\right)}{2}\int_{\mathbb{R}}\widehat{\phi}(u)\mathrm{d}u - 2\int_1^\infty\left(\psi_k(t;3,4)-\frac{t}{2}\right)\mathrm{d}\left(\frac{\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)}{t}\right)\right),$$

since $\widehat{\phi}(u)$ is even. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)}{t}\right) = \frac{1}{t^2}\left(\frac{1}{\log(kM_{d,k})}\cdot\widehat{\phi}^{(1)}\left(\frac{\log t}{\log(kM_{d,k})}\right)-\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)\right),$$

and that

$$\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right) = \sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j)}(0)}{j!}\left(\frac{\log t}{\log(kM_{d,k})}\right)^j + O_J\left(\left(\frac{\log t}{\log(kM_{d,k})}\right)^J\right),$$

we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)}{t}\right)$$

$$= \frac{1}{t^2}\left(\left(\sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j+1)}(0)}{j!}\left(\frac{(\log t)^j}{(\log(kM_{d,k}))^{j+1}}\right)-\sum_{j=0}^{J-1}\frac{\widehat{\phi}^{(j)}(0)}{j!}\left(\frac{\log t}{\log(kM_{d,k})}\right)^j\right)+O_J\left(\left(\frac{\log t}{\log(kM_{d,k})}\right)^J\right)\right)$$

$$= \frac{1}{t^2}\left(-\widehat{\phi}(0)+\left(\sum_{j=1}^{J-1}\frac{\widehat{\phi}^{(j)}(0)(\log t)^{j-1}}{(\log(kM_{d,k}))^j(j-1)!}\left(1-\frac{\log t}{j}\right)\right)+O_J\left(\left(\frac{\log t}{\log(kM_{d,k})}\right)^J\right)\right)$$

Thus

$$
\begin{aligned}
&-2\int_1^\infty \left(\psi_k(t;3,4)-\frac{t}{2}\right)\mathrm{d}\left(\frac{\widehat{\phi}\left(\frac{\log t}{\log(kM_{d,k})}\right)}{t}\right)\\
&=2\int_1^\infty \frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}\left(\widehat{\phi}(0)-\left(\sum_{j=1}^{J-1}\frac{\widehat{\phi}^{(j)}(0)(\log t)^{j-1}}{(\log(kM_{d,k}))^j(j-1)!}\left(1-\frac{\log t}{j}\right)\right)+O_J\left(\left(\frac{\log t}{\log(kM_{d,k})}\right)^J\right)\right)\mathrm{d}t\\
&=2\widehat{\phi}(0)\int_1^\infty \frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}\mathrm{d}t-(-1)^k\sum_{j=1}^{J-1}\frac{\widehat{\phi}^{(j)}(0)c_{j,\mathrm{inert}}(k)}{(\log(kM_{d,k}))^j}+O_J\left((\log(kM_{d,k}))^{-J}\right),
\end{aligned}
$$

where the error term in the last step is bounded upon noting that

$$\int_1^\infty \frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}(\log t)^J\mathrm{d}t \ll \int_1^\infty \frac{1}{t\log t}\mathrm{d}t \ll 1,$$

by (3.20) with $A > J+1$. Replacing above, and then in (3.23), we get

$$U_{\mathrm{inert}}(\phi,k)=\frac{(-1)^{k+1}}{2}\int_{\mathbb{R}}\widehat{\phi}(u)\mathrm{d}u+\sum_{j=0}^{J-1}\frac{c_{j,\mathrm{inert}}(k)\widehat{\phi}^{(j)}(0)}{(\log(kM_{d,k}))^{j+1}}+O_J\left((\log(kM_{d,k}))^{-J-1}\right),$$

as desired. □

By (3.4) and (3.8), we see that upon combining Lemma 3.1, Lemma 3.3, Lemma 3.4, and Lemma 3.5, we obtain the following.

Proposition 3.6. *Let k be a positive integer, d be a square-free integer, and ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then for any $J \in \mathbb{N}$ and $k \to \infty$, we have*

$$
\begin{aligned}
\mathcal{D}(\phi,\xi_{d,k}) &= \widehat{\phi}(0)+\frac{(-1)^{k+1}}{2}\int_{\mathbb{R}}\widehat{\phi}(u)\mathrm{d}u\\
&\quad+\sum_{\substack{j=0\\ j \text{ even}}}^{J-1}\frac{c_j(d,k)\widehat{\phi}^{(j)}(0)}{(\log(kM_{d,k}))^{j+1}}+U_{\mathrm{split}}(\phi,d,k)+O_J\left(\frac{1+\log|d|}{(\log(kM_{d,k}))^{J+1}}\right),
\end{aligned}
$$

where

$$c_0(d,k) := -\log 2\pi + c_{0,\mathrm{ram}}(k)+c_{0,\mathrm{inert}}(k)+c_{0,\mathrm{inert},d}(k) \tag{3.24}$$

and for $j \geq 1$,

$$c_j(d,k) := c_{j,\mathrm{ram}}(k)+c_{j,\mathrm{inert}}(k)+c_{j,\mathrm{inert},d}(k). \tag{3.25}$$

We moreover observe that $c_j(d,k)$ depends only on the congruence class of k modulo 4.

4. Alternative Expressions for $c_{j,\mathrm{inert}}(k)$

We now provide a few alternative expressions for the lower order constants, $c_{j,\mathrm{inert}}(k)$, that are more amenable to numerical analysis.

Lemma 4.1. *Let k be a positive integer. Then, we have*

$$c_{0,\mathrm{inert}}(k) = (-1)^k\left(\gamma_0 - \frac{L'}{L}(1,\chi_4) + \log 2\right) - \sum_{p\equiv 3 \,(\mathrm{mod}\, 4)} \frac{2\log p}{p^2-1}. \tag{4.1}$$

Proof. To begin, we write

$$2\int_1^\infty \frac{\psi_k(t;3,4) - \frac{t}{2}}{t^2}\mathrm{d}t = \int_1^\infty \frac{2\psi_k(t;3,4) - \psi(t) + \psi(t) - t}{t^2}\mathrm{d}t. \tag{4.2}$$

Then, denoting the *Chebyshev* ψ-functions

$$\psi(t) := \sum_{p^n\le t}\log p \quad \text{and} \quad \psi(t,\chi_4) := \sum_{p^n \le t}\chi_4(p^n)\log p,$$

where χ_4 is the quadratic Dirichlet character modulo 4 over $\mathbb{Z}$, we note the identity

$$2\psi_k(t;3,4) - (-1)^k 2\psi_0(\sqrt{t};3,4) = 2\sum_{\substack{p\equiv 3\,(\mathrm{mod}\,4)\\ n \text{ odd}\\ p^n\le t}} \log p = \psi(t) - \psi(t,\chi_4) - \sum_{2^n\le t}\log 2. \tag{4.3}$$

Using (4.2) and (4.3), we get

$$2\int_1^\infty \frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}\mathrm{d}t = \int_1^\infty \frac{-\psi(t,\chi_4) + \psi(t) - t + 2(-1)^k\psi_0(\sqrt{t};3,4) - \log 2\lfloor\frac{\log t}{\log 2}\rfloor}{t^2}\,\mathrm{d}t. \tag{4.4}$$

We separate this integral into four components. We first compute the contribution of the prime 2 by noting that

$$-\log 2\int_1^\infty \lfloor\tfrac{\log t}{\log 2}\rfloor\frac{\mathrm{d}t}{t^2} = -\log 2\sum_{n=0}^\infty n\int_{2^n}^{2^{n+1}}\frac{\mathrm{d}t}{t^2} = -\log 2\sum_{n=0}^\infty \frac{n}{2^{n+1}} = -\log 2. \tag{4.5}$$

Next, by Perron's formula we write

$$-\int_1^\infty \frac{\psi(t,\chi_4)}{t^2}\mathrm{d}t = \frac{1}{2\pi i}\int_1^\infty\left(\int_C \frac{L'}{L}(s,\chi_4)\frac{t^{s-2}}{s}\mathrm{d}s\right)\mathrm{d}t,$$

where C is a vertical path to the left of the line $\mathrm{Re}(s)=1$ taken within a zero-free region of $L(s,\chi_4)$. Upon noting appropriate bounds on the growth of L'/L within the critical strip, we apply Fubini's theorem to flip the order of integration. We then compute

$$\int_1^\infty t^{s-2}\,\mathrm{d}t = -\frac{1}{s-1},$$

and then

$$-\int_1^\infty \frac{\psi(t,\chi_4)}{t^2}\mathrm{d}t = \frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi_4)\frac{\mathrm{d}s}{s(1-s)}.$$

We move the integral to the line $\mathrm{Re}(s) = R > 1$, picking the pole at $s=1$, and where we note that $L'/L(s)$ is bounded on the vertical line $\mathrm{Re}(s)=R$, so that

$$\int_{(R)}\frac{L'}{L}(s,\chi_4)\frac{\mathrm{d}s}{s(1-s)} = o\left(\frac{1}{R}\right),$$

which goes to zero as $R \to \infty$. This gives

$$-\int_1^\infty \frac{\psi(t,\chi_4)}{t^2}\mathrm{d}t = \frac{L'}{L}(1,\chi_4). \tag{4.6}$$

Similarly, with C as defined above, we find that

$$\begin{aligned}\int_1^\infty \frac{\psi(t)-t}{t^2}\mathrm{d}t &= \frac{1}{2\pi i}\int_1^\infty \left(\int_{(2)} -\frac{\zeta'}{\zeta}(s)\frac{t^{s-2}}{s}\mathrm{d}s - \frac{1}{t}\right)\mathrm{d}t \\ &= \frac{1}{2\pi i}\int_1^\infty \left(\int_C -\frac{\zeta'}{\zeta}(s)\frac{t^{s-2}}{s}\mathrm{d}s + \frac{1}{t} - \frac{1}{t}\right)\mathrm{d}t = \frac{1}{2\pi i}\int_C \frac{\zeta'}{\zeta}(s)\frac{\mathrm{d}s}{s(s-1)}.\end{aligned}$$

Recall that

$$\zeta(1+s) = \frac{1}{s} + \sum_{n=0}^\infty \frac{(-1)^n}{n!}\gamma_n s^n,$$

where γ_n are known as the *Stieltjes constants* (and γ_0 is the *Euler–Mascheroni constant*), so that

$$\begin{aligned}\frac{1}{s(s-1)}\frac{\zeta'}{\zeta}(s) &= \left(\frac{1}{s-1} - 1 + O(s-1)\right)\left(-\frac{1}{s-1} + \gamma_0 + O(s-1)\right) \\ &= -\frac{1}{(s-1)^2} + \frac{\gamma_0+1}{s-1} + O(1).\end{aligned}$$

Upon shifting the contour C to the far right, we find by Cauchy's residue theorem that

$$\int_1^\infty \frac{\psi(t)-t}{t^2}\mathrm{d}t = \frac{1}{2\pi i}\int_C \frac{\zeta'}{\zeta}(s)\frac{\mathrm{d}s}{s(s-1)} = -\gamma_0 - 1. \tag{4.7}$$

Finally, we note that

$$\int_1^\infty \frac{2\psi_0(\sqrt{t};3,4)}{t^2}\mathrm{d}t = \frac{1}{2\pi i}\int_1^\infty \left(\int_C A(s)\frac{t^{s-2}}{s}\mathrm{d}s\right)\mathrm{d}t = A(1),$$

where

$$A(s) := 2\sum_{p\equiv 3\,(\mathrm{mod}\,4)}\sum_{n=1}^\infty \frac{\log p}{p^{2ns}}, \tag{4.8}$$

so that

$$(-1)^k \int_1^\infty \frac{2\psi_0(\sqrt{t};3,4)}{t^2}\mathrm{d}t = (-1)^k \sum_{p\equiv 3\,(\mathrm{mod}\,4)} \frac{2\log p}{p^2-1}. \tag{4.9}$$

Inserting (4.5), (4.6), (4.7), and (4.9) into (4.4), we have

$$2\int_1^\infty \frac{\psi_k(t;3,4)-\frac{t}{2}}{t^2}\mathrm{d}t = \frac{L'}{L}(1,\chi_4) - \gamma_0 - 1 + (-1)^k \sum_{p\equiv 3\,(\mathrm{mod}\,4)} \frac{2\log p}{p^2-1} - \log 2, \tag{4.10}$$

from which the lemma follows by (3.21). $\square$

Lemma 4.2. *Let k, j be positive integers. Then, we have*

$$c_{j,\mathrm{inert}}(k) = \frac{(-1)^{j+k}}{j!}\left(\left(\frac{f'}{f}\right)^{(j)}(1) - \left(\frac{L'}{L}\right)^{(j)}(1,\chi_4)\right) - \sum_{p\equiv 3 \,(\mathrm{mod}\, 4)} \frac{(2\log p)^{j+1}}{j!}\mathrm{Li}_{-j}(\tfrac{1}{p^2}) + (-1)^k\frac{(\log 2)^{j+1}}{j!}\mathrm{Li}_{-j}(\tfrac{1}{2}), \tag{4.11}$$

where $f(s) := (s-1)\zeta(s)$ and where $\mathrm{Li}_{-j}(z)$ is as in (3.16).

Proof. To begin, recall that for $j \geq 1$,

$$c_{j,\mathrm{inert}}(k) := 2(-1)^k \int_1^\infty \frac{(\psi_k(t;3,4) - \frac{t}{2})}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\mathrm{d}t.$$

As above, we note that

$$\begin{aligned} &2\int_1^\infty \frac{(\psi_k(t;3,4) - \frac{t}{2})}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\mathrm{d}t \\ &= \int_1^\infty \frac{(-\psi(t,\chi_4) + \psi(t) - t + 2(-1)^k\psi_0(\sqrt{t};3,4) - \log 2\lfloor\frac{\log t}{\log 2}\rfloor)}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\mathrm{d}t. \end{aligned} \tag{4.12}$$

We again treat the four terms separately. First, by Perron's formula, we write

$$-\int_1^\infty \frac{\psi(t,\chi_4)}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\mathrm{d}t = \frac{1}{2\pi i}\int_1^\infty\left(\int_C \frac{L'}{L}(s,\chi_4)\frac{t^{s-2}}{s}\mathrm{d}s\right)\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\mathrm{d}t,$$

where C is again a vertical path to the left of the line $\mathrm{Re}(s) = 1$ taken within a zero-free region of $L(s,\chi_4)$. Flipping the order of integration as above, let us first compute, for $j = 1$,

$$\begin{aligned} I_1(s) &:= \int_1^\infty t^{s-2}(1 - \log t)\ \mathrm{d}t = \int_0^\infty e^{(s-1)u}(1-u)\mathrm{d}u \\ &= \left[(1-u)\frac{e^{u(s-1)}}{s-1}\right]_0^\infty + \int_0^\infty \frac{e^{u(s-1)}}{s-1}\mathrm{d}u = -\frac{1}{s-1} + \frac{1}{s-1}\left[\frac{e^{u(s-1)}}{s-1}\right]_0^\infty \\ &= -\frac{1}{s-1} - \frac{1}{(s-1)^2} = -\frac{s}{(s-1)^2}. \end{aligned}$$

For $j \geq 2$, we have

$$\begin{aligned} I_j(s) &:= \int_1^\infty t^{s-2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\ \mathrm{d}t = \int_0^\infty e^{(s-1)u}\frac{u^{j-1}}{(j-1)!}\left(1 - \frac{u}{j}\right)\ du \\ &= \left[\frac{e^{(s-1)u}}{s-1}\left(\frac{u^{j-1}}{(j-1)!} - \frac{u^j}{j!}\right)\right]_0^\infty - \frac{1}{s-1}\int_0^\infty e^{(s-1)u}\left(\frac{u^{j-2}}{(j-2)!} - \frac{u^{j-1}}{(j-1)!}\right)du \\ &= -\frac{1}{s-1}I_{j-1}(s) = \left(-\frac{1}{s-1}\right)^{j-1}I_1(s) = \frac{(-1)^j s}{(s-1)^{j+1}} \end{aligned}$$

by induction. It follows that

$$-\int_1^\infty \frac{\psi(t,\chi_4)}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1 - \frac{\log t}{j}\right)\mathrm{d}t = \frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi_4)\frac{I_j(s)}{s}\mathrm{d}s = \frac{1}{2\pi i}\int_C \frac{L'}{L}(s,\chi_4)\frac{(-1)^j}{(s-1)^{j+1}}\mathrm{d}s,$$

so that upon shifting the integral and picking up the pole at $s=1$, we find that

$$-\int_1^\infty \frac{\psi(t,\chi_4)}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t = \frac{(-1)^{j+1}}{j!}\left(\frac{L'}{L}\right)^{(j)}(1,\chi_4). \tag{4.13}$$

Similarly, with C as defined above, we find that

$$\begin{aligned}\int_1^\infty \frac{\psi(t)-t}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t &= \frac{1}{2\pi i}\int_1^\infty \left(\int_{(2)} -\frac{\zeta'}{\zeta}(s)\frac{t^{s-2}}{s}\mathrm{d}s - \frac{1}{t}\right)\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t\\ &= \frac{1}{2\pi i}\int_1^\infty \left(\int_C -\frac{\zeta'}{\zeta}(s)\frac{t^{s-2}}{s}\mathrm{d}s + \frac{1}{t} - \frac{1}{t}\right)\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t\\ &= -\frac{1}{2\pi i}\int_C \frac{\zeta'}{\zeta}(s)I_j(s)\frac{\mathrm{d}s}{s} = -\frac{1}{2\pi i}\int_C \frac{\zeta'}{\zeta}(s)\frac{(-1)^j}{(s-1)^{j+1}}\mathrm{d}s.\end{aligned}$$

Upon writing

$$\zeta(s) = \frac{f(s)}{s-1}$$

where

$$f(s) := 1 + \sum_{n=0}^\infty \frac{(-1)^n}{n!}\gamma_n (s-1)^{n+1} = \sum_{k=0}^\infty \frac{f^{(k)}(1)}{k!}(s-1)^k \tag{4.14}$$

is a holomorphic function, we find that

$$\frac{\zeta'}{\zeta}(s) = \frac{f'}{f}(s) - \frac{1}{s-1} = -\frac{1}{s-1} + \sum_{k=0}^\infty \left(\frac{f'}{f}\right)^{(k)}(1)\frac{(s-1)^k}{k!}$$

and therefore that

$$\begin{aligned}-\frac{1}{2\pi i}\int_C \frac{\zeta'}{\zeta}(s)\frac{(-1)^j}{(s-1)^{j+1}}\mathrm{d}s &= -\frac{1}{2\pi i}\int_C \left(-\frac{1}{(s-1)^{j+2}} + \sum_{k=0}^\infty \left(\frac{f'}{f}\right)^{(k)}(1)\frac{(s-1)^{k-j-1}}{k!}\right)(-1)^j\mathrm{d}s\\ &= \frac{(-1)^j}{j!}\left(\frac{f'}{f}\right)^{(j)}(1).\end{aligned}$$

With A as in (4.8), we next note that

$$\begin{aligned}2(-1)^k\int_1^\infty \frac{\psi_0(\sqrt{t};3,4)}{t^2}\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t &= \frac{(-1)^k}{2\pi i}\int_1^\infty \left(\int_C A(s)\frac{t^{s-2}}{s}\mathrm{d}s\right)\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t\\ &= \frac{(-1)^k}{2\pi i}\int_C \frac{A(s)I_j(s)}{s}\mathrm{d}s = (-1)^{k+j+1}\frac{A^{(j)}(1)}{j!}\\ &= \frac{(-1)^{k+1}}{j!}\sum_{p\equiv 3\ (\mathrm{mod}\ 4)} (2\log p)^{j+1}\mathrm{Li}_{-j}(\tfrac{1}{p^2}).\end{aligned}$$

Finally, upon writing $B(s) := \sum_{n\geq 0}\frac{\log 2}{2^{ns}}$, we similarly find that

$$\begin{aligned}-\log 2\int_1^\infty \lfloor\tfrac{\log t}{\log 2}\rfloor\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\frac{\mathrm{d}t}{t^2} &= -\frac{1}{2\pi i}\int_1^\infty \left(\int_C B(s)\frac{t^{s-2}}{s}\mathrm{d}s\right)\frac{(\log t)^{j-1}}{(j-1)!}\left(1-\frac{\log t}{j}\right)\mathrm{d}t\\ &= -\frac{1}{2\pi i}\int_C \frac{B(s)I_j(s)}{s}\mathrm{d}s = (-1)^j\frac{B^{(j)}(1)}{j!} = \frac{(\log 2)^{j+1}}{j!}\mathrm{Li}_{-j}(\tfrac{1}{2}),\end{aligned}$$

from which the lemma follows. ∎

We conclude this section with one final expression for $c_{j,\mathrm{inert}}(k)$, obtained by rewriting the above logarithmic derivatives in terms of Laurent–Stieltjes constants (see e.g. [7]):

Lemma 4.3. *Let k, j be positive integers. Then, we have*

(4.15)
$$c_{j,\mathrm{inert}}(k) = \sum_{\substack{m_1,\dots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(-1)^k (j+1)}{m_1+\cdots+m_{j+1}} \binom{m_1+\cdots+m_{j+1}}{m_1,\dots,m_{j+1}} \left(\prod_{\ell=0}^{j} \left(\frac{\gamma_\ell}{\ell!}\right)^{m_{\ell+1}} - \prod_{\ell=1}^{j+1} \left(\frac{-\gamma_\ell(\chi_4)}{\ell!\gamma_0(\chi_4)}\right)^{m_\ell} \right)$$
$$- \sum_{p\equiv 3\,(\mathrm{mod}\,4)} \frac{(2\log p)^{j+1}}{j!} \mathrm{Li}_{-j}(\tfrac{1}{p^2}) + (-1)^k \frac{(\log 2)^{j+1}}{j!} \mathrm{Li}_{-j}(\tfrac{1}{2}),$$

where
$$\binom{m_1+\cdots+m_{j+1}}{m_1,\dots,m_{j+1}} := \frac{(m_1+\cdots+m_{j+1})!}{m_1!\dots m_{j+1}!}$$
are multinomial coefficients, γ_ℓ are Stieltjes constants, and $\gamma_n(\chi_4)$ are the Laurent–Stieltjes constant for $L(s,\chi_4)$ defined by

$$L^{(n)}(1,\chi_4) := (-1)^n \gamma_n(\chi_4). \tag{4.16}$$

Proof. Upon applying Faà di Bruno's formula [8] for higher derivatives to $\log L(s,\chi_4)$, we find that

$$\left(\frac{L'}{L}\right)^{(j)}(1,\chi_4) = (\log L(s,\chi_4))^{(j+1)}\,|_{s=1}$$
$$= \sum_{\substack{m_1,\dots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(j+1)!}{m_1!\dots m_{j+1}!} \frac{(-1)^{m_1+\cdots+m_{j+1}-1}(m_1+\cdots+m_{j+1}-1)!}{L(1,\chi_4)^{m_1+\cdots+m_{j+1}}} \prod_{\ell=1}^{j+1} \left(\frac{L^{(\ell)}(1,\chi_4)}{\ell!}\right)^{m_\ell}$$
$$= \sum_{\substack{m_1,\dots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(j+1)!(-1)^{j+1}}{m_1+\cdots+m_{j+1}} \frac{(-1)^{m_1+\cdots+m_{j+1}-1}}{\gamma_0(\chi_4)^{m_1+\cdots+m_{j+1}}} \binom{m_1+\cdots+m_{j+1}}{m_1,\dots,m_{j+1}} \prod_{\ell=1}^{j+1} \left(\frac{\gamma_\ell(\chi_4)}{\ell!}\right)^{m_\ell}$$
$$= \sum_{\substack{m_1,\dots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(j+1)!(-1)^{j}}{m_1+\cdots+m_{j+1}} \binom{m_1+\cdots+m_{j+1}}{m_1,\dots,m_{j+1}} \prod_{\ell=1}^{j+1} \left(\frac{-\gamma_\ell(\chi_4)}{\ell!\gamma_0(\chi_4)}\right)^{m_\ell}.$$

It thus follows that

(4.17)
$$\frac{(-1)^{j+1}}{j!}\left(\frac{L'}{L}\right)^{(j)}(1,\chi_4) = \sum_{\substack{m_1,\dots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{-(j+1)}{m_1+\cdots+m_{j+1}} \binom{m_1+\cdots+m_{j+1}}{m_1,\dots,m_{j+1}} \prod_{\ell=1}^{j+1} \left(\frac{-\gamma_\ell(\chi_4)}{\ell!\gamma_0(\chi_4)}\right)^{m_\ell}.$$

Similarly, by comparing terms with f as defined by (4.14), we find that for $\ell \geq 1$,

$$f^{(\ell)}(1) = \frac{\ell!(-1)^{\ell-1}}{(\ell-1)!}\gamma_{\ell-1} = \ell(-1)^{\ell-1}\gamma_{\ell-1}, \tag{4.18}$$

and therefore that

$$\begin{aligned}\left(\frac{f'}{f}\right)^{(j)}(1) &= \sum_{\substack{m_1,\ldots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(j+1)!}{m_1!\ldots m_{j+1}!}\frac{(-1)^{m_1+\cdots+m_{j+1}-1}(m_1+\cdots+m_{j+1}-1)!}{f(1)^{m_1+\cdots+m_{j+1}}}\prod_{\ell=1}^{j+1}\left(\frac{f^{(\ell)}(1)}{\ell!}\right)^{m_\ell}\\ &= \sum_{\substack{m_1,\ldots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(j+1)!}{m_1+\cdots+m_{j+1}}(-1)^{m_1+\cdots+m_{j+1}-1}\binom{m_1+\cdots+m_{j+1}}{m_1,\ldots,m_{j+1}}\prod_{\ell=1}^{j+1}\left(\frac{(-1)^{\ell-1}\gamma_{\ell-1}}{(\ell-1)!}\right)^{m_\ell}\\ &= \sum_{\substack{m_1,\ldots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{(j+1)!(-1)^j}{m_1+\cdots+m_{j+1}}\binom{m_1+\cdots+m_{j+1}}{m_1,\ldots,m_{j+1}}\prod_{\ell=0}^{j}\left(\frac{\gamma_\ell}{\ell!}\right)^{m_{\ell+1}}.\end{aligned}$$

We thus furthermore conclude that

$$\frac{(-1)^j}{j!}\left(\frac{f'}{f}\right)^{(j)}(1) = \sum_{\substack{m_1,\ldots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{j+1}{m_1+\cdots+m_{j+1}}\binom{m_1+\cdots+m_{j+1}}{m_1,\ldots,m_{j+1}}\prod_{\ell=0}^{j}\left(\frac{\gamma_\ell}{\ell!}\right)^{m_{\ell+1}}. \tag{4.19}$$

The lemma now follows from Lemma 4.2, together with (4.17) and (4.19). □

5. Computing $\mathcal{D}(K;\phi,\mathcal{F}_d^\alpha)$

We now compute $\mathcal{D}(K;\phi,\mathcal{F}_d^\alpha)$ by averaging $\mathcal{D}(\phi,\xi_{d,k})$ over appropriate $1 \leq k \leq K$.

Proposition 5.1. *Let d be a square-free integer, $\alpha \in \mathbb{Z}/8\mathbb{Z}$, and let ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then for any $J \geq 1$, one has as $K \to \infty$,*

$$\begin{aligned}\mathcal{D}(K;\phi,\mathcal{F}_d^\alpha) = \widehat{\phi}(0) + \frac{(-1)^{\alpha+1}}{2}\int_{\mathbb{R}}\widehat{\phi}(u)du + \sum_{m=1}^{J}\frac{C_m(d,\alpha,\phi)}{(\log KM_{d,\alpha})^m} + \frac{8}{K}\sum_{\substack{1\leq k\leq K\\ k\equiv\alpha \pmod 8}} U_{\text{split}}(\phi,d,k)\\ + O_J\left(\frac{1+\log|d|}{(\log KM_{d,\alpha})^{J+1}}\right),\end{aligned}$$

with

$$C_m(d,\alpha,\phi) := (m-1)!\sum_{\substack{j=0\\ j \text{ even}}}^{m-1} c_j(d,\alpha)\frac{\widehat{\phi}^{(j)}(0)}{j!}, \tag{5.1}$$

where for $\alpha \equiv 0 \bmod 4$ *and* $j \geq 0$*,*

$$c_j(d,\alpha) = -\,\delta_0(j)\log 2\pi + B_j + \frac{(\log 2)^{j+1}}{j!}\mathrm{Li}_{-j}(\tfrac{1}{2}) - \frac{2^{j+1}}{j!}\sum_{p\equiv 3\,(\mathrm{mod}\,4)}(\log p)^{j+1}\mathrm{Li}_{-j}\left(\frac{1}{p^2}\right)$$
$$-\frac{2}{j!}\left(\frac{\log 2}{2}\right)^{j+1}\mathrm{Li}_{-j}\left(\frac{(-1)^{\frac{\alpha}{4}}}{\sqrt{2}}\right),$$

and for $\alpha \not\equiv 0 \bmod 4$ *and* $j \geq 0$*,*

$$c_j(d,\alpha) = -\delta_0(j)\log 2\pi + (-1)^{\alpha}\left(B_j + \frac{(\log 2)^{j+1}}{j!}\mathrm{Li}_{-j}(\tfrac{1}{2})\right) - \frac{2^{j+1}}{j!}\sum_{p\equiv 3\,(\mathrm{mod}\,4)}(\log p)^{j+1}\mathrm{Li}_{-j}\left(\frac{1}{p^2}\right)$$
$$+\frac{2}{j!}\sum_{\substack{p\equiv 3\,(\mathrm{mod}\,4)\\ p\mid d}}(\log p)^{j+1}\mathrm{Li}_{-j}\left(\frac{(-1)^{\alpha}}{p}\right).$$

Here

$$B_j := \frac{(-1)^j}{j!}\left(\left(\frac{f'}{f}\right)^{(j)}(1) - \left(\frac{L'}{L}\right)^{(j)}(1,\chi_4)\right)$$
$$= \sum_{\substack{m_1,\ldots,m_{j+1}\in\mathbb{Z}_{\geq 0}\\ \sum \ell m_\ell = j+1}} \frac{j+1}{m_1+\cdots+m_{j+1}}\binom{m_1+\cdots+m_{j+1}}{m_1,\ldots,m_{j+1}}\left(\prod_{\ell=0}^{j}\left(\frac{\gamma_\ell}{\ell!}\right)^{m_{\ell+1}} - \prod_{\ell=1}^{j+1}\left(\frac{-\gamma_\ell(\chi_4)}{\ell!\gamma_0(\chi_4)}\right)^{m_\ell}\right),$$

where $f(s) := (s-1)\zeta(s)$*, the* γ_ℓ *are Stieltjes constants, and the* $\gamma_n(\chi_4)$ *are the Laurent–Stieltjes constant for* $L(s,\chi_4)$*.*

Proof. Recall by (1.2) that

$$\mathcal{D}(K;\phi,\mathcal{F}_d^{\alpha}) = \frac{8}{K}\sum_{\substack{1\leq k\leq K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\mathcal{D}(\phi,\xi_{d,k}),$$

where the estimation for $\mathcal{D}(\phi,\xi_{d,k})$ is given in Proposition 3.6. Let $j\in\mathbb{N}$. Since $c_j(d,k)$ and $M_{d,k}$ depend only on the congruence class of k modulo 8, fixing an appropriate representative for $\alpha\in\{1,\ldots 8\}$, we first compute, for any $c>1$,

$$\sum_{\substack{1\leq k\leq K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\frac{1}{(\log(ck))^{j+1}} = \sum_{0\leq\ell\leq\lfloor\frac{K-\alpha}{8}\rfloor}\frac{1}{(\log(c(8\ell+\alpha)))^{j+1}} = \sum_{1\leq\ell\leq\lfloor\frac{K-\alpha}{8}\rfloor}\frac{1}{(\log(8c\ell))^{j+1}} + O(1),$$

since

$$\frac{1}{(\log(8c\ell))^{j+1}} - \frac{1}{(\log(8c\ell+c\alpha))^{j+1}} = \frac{(\log(8c\ell+c\alpha))^{j+1}-(\log(8c\ell))^{j+1}}{(\log(8c\ell+c\alpha))^{j+1}(\log(8c\ell))^{j+1}}$$
$$\ll \frac{(\log(8c\ell+c\alpha)-\log(8c\ell))\,(\log(8c\ell))^j}{(\log(8c\ell))^{2j+2}} \ll \frac{1}{\ell\,(\log(8c\ell))^{j+2}}$$

and the sum over ℓ converges. By the Euler–Maclaurin formula,

$$\sum_{1\le \ell \le \lfloor \frac{K-\alpha}{8} \rfloor} \frac{1}{(\log(8c\ell))^{j+1}} = \int_1^{\frac{K}{8}} \frac{\mathrm{d}t}{(\log(8ct))^{j+1}} + O(1) = \frac{1}{8c}\int_2^{cK} \frac{\mathrm{d}t}{(\log t)^{j+1}} + O(1)$$

$$= \frac{1}{8c}\frac{cK}{(\log(cK))^{j+1}} \sum_{\ell=0}^{J-j} \frac{(j+\ell)!}{j!(\log(cK))^{\ell}} + O_J\Big(\frac{K}{(\log(cK))^{J+2}}\Big).$$

By Proposition 3.6, where we note in particular that $c_j(d,k) = c_j(d,\alpha)$ for any $k \equiv \alpha \pmod 8$, we find that

$$\mathcal{D}(K;\phi,\mathcal{F}_d^{\alpha}) = \widehat{\phi}(0) + \frac{(-1)^{\alpha+1}}{2}\int_{\mathbb{R}} \widehat{\phi}(u)du + \sum_{j=0}^{J-1} \frac{c_j(d,\alpha)\widehat{\phi}^{(j)}(0)}{(\log(KM_{d,\alpha}))^{j+1}} \sum_{\ell=0}^{J-j} \frac{(j+\ell)!}{j!(\log(KM_{d,\alpha}))^{\ell}}$$
$$+ \frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv \alpha \,(\mathrm{mod}\, 8)}} U_{\mathrm{split}}(\phi,d,k) + O_J\left(\frac{1+\log|d|}{(\log(KM_{d,\alpha}))^{J+1}}\right).$$

Setting $m = j+\ell+1$, we write the j,ℓ-sum as

$$\sum_{j=0}^{J-1} \frac{c_j(d,\alpha)\widehat{\phi}^{(j)}(0)}{j!} \sum_{m=j+1}^{J+1} \frac{(m-1)!}{(\log(KM_{d,\alpha}))^m} = \sum_{m=1}^{J+1} \frac{(m-1)!}{(\log(KM_{d,\alpha}))^m} \sum_{j=0}^{m-1} \frac{c_j(d,\alpha)\widehat{\phi}^{(j)}(0)}{j!}$$

and the formula for $\mathcal{D}(K;\phi,\mathcal{F}_d^{\alpha})$ follows by defining $C_m(d,\alpha,\phi)$ as in (5.1), where we note that since ϕ is even, we have $\widehat{\phi}^{(j)}(0) = 0$ for all odd j.

We recall that for $j \ge 1$,

$$c_0(d,k) = -\log 2\pi + c_{0,\mathrm{ram}}(k) + c_{0,\mathrm{inert}}(k) + c_{0,\mathrm{inert},d}(k)$$
$$c_j(d,k) = c_{j,\mathrm{ram}}(k) + c_{j,\mathrm{inert}}(k) + c_{j,\mathrm{inert},d}(k),$$

and replacing (3.15), (3.19), (4.1), (4.11) and (4.15) above, we find the formulas in the statement of the Proposition. □

Theorem 1.1 now follows from Proposition 5.1 provided that we can bound the contribution of the split primes for test-functions ϕ such that $\mathrm{supp}(\widehat{\phi}) \subset (-1,1)$. We begin by first proving the following weaker result, which, combined with Proposition 5.1, provides an asymptotic for $\mathcal{D}(K;\phi,\mathcal{F}_d^{\alpha})$ when $\mathrm{supp}(\widehat{\phi}) \subset (-\frac{1}{2},\frac{1}{2})$.

Lemma 5.2. *Let d be a square-free integer, $\alpha \in \mathbb{Z}/8\mathbb{Z}$, and let ϕ be an even Schwartz function such that $\widehat{\phi}$ is compactly supported. Then as $K \to \infty$, we have*

$$\frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv \alpha \,(\mathrm{mod}\, 8)}} U_{\mathrm{split}}(\phi,d,k) = O(d^{2\nu}K^{2\nu-1}).$$

Proof. Summing (2.23) over k, we get

$$\begin{aligned}&\frac{8}{K}\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} U_{\mathrm{split}}(\phi,d,k)\\ &= -\frac{8}{K}\sum_{\substack{p\equiv 1\,(\mathrm{mod}\,4)\\ (p)\nmid\mathfrak{f}_{d,\alpha}\\ n\ge 1}}\frac{\log p}{p^{\frac{n}{2}}}\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\left(\xi_{d,k}^{n}(\mathfrak{p})+\overline{\xi}_{d,k}^{n}(\mathfrak{p})\right)\widehat{\phi}\left(\frac{n\log p}{2\log(kM_{d,\alpha})}\right)\frac{1}{\log(kM_{d,\alpha})}.\end{aligned}$$

We have

$$\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\left(\xi_{d,k}^{n}(\mathfrak{p})+\overline{\xi}_{d,k}^{n}(\mathfrak{p})\right)=\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\left(e^{ikn\theta_{d,p}}+e^{-ikn\theta_{d,p}}\right)$$

with $\theta_{d,p}$ as defined by (2.21). We define

$$D_{K,\alpha}(x):=\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\left(e^{ikx}+e^{-ikx}\right)=2\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}}\cos(kx).$$

We note that

$$1+\sum_{\alpha\in\mathbb{Z}/8\mathbb{Z}}D_{K,\alpha}(x)=D_K(x)$$

where

$$D_K(x):=\sum_{k=-K}^{K}e^{ikx}=\frac{\sin\left((K+\frac{1}{2})x\right)}{\sin(\frac{x}{2})}$$

is the *Dirichlet kernel* at x. Fixing an appropriate representative for $\alpha\in\{1,\dots 8\}$, we observe that

$$\begin{aligned}D_{K,\alpha}(x)&=\sum_{0\le\ell\le\lfloor\frac{K-\alpha}{8}\rfloor}\left(e^{ix(8\ell+\alpha)}+e^{-ix(8\ell+\alpha)}\right)\\ &=\frac{e^{i\alpha x}-e^{ix(8\lfloor\frac{K-\alpha}{8}\rfloor+\alpha+8)}}{1-e^{8ix}}+\frac{e^{-i\alpha x}-e^{-ix(8\lfloor\frac{K-\alpha}{8}\rfloor+\alpha+8)}}{1-e^{-8ix}}\\ &=\frac{-e^{i(\alpha-4)x}+e^{ix(8\lfloor\frac{K-\alpha}{8}\rfloor+\alpha+4)}+e^{-i(\alpha-4)x}-e^{-ix(8\lfloor\frac{K-\alpha}{8}\rfloor+\alpha+4)}}{2i\sin(4x)}\\ &=\frac{\sin((8\lfloor\frac{K-\alpha}{8}\rfloor+\alpha+4)x)-\sin((\alpha-4)x)}{\sin(4x)}\ll\frac{1}{|\sin(4x)|}.\end{aligned}\tag{5.2}$$

Using the bound (5.2) and using partial summation, we get

$$\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \left(\xi_{d,k}^n(\mathfrak{p})+\overline{\xi}_{d,k}^n(\mathfrak{p})\right)\widehat{\phi}\left(\frac{n\log p}{2\log(kM_{d,\alpha})}\right)\frac{1}{\log\left(kM_{d,\alpha}\right)}$$

$$= D_{K,\alpha}(n\theta_{d,p})\widehat{\phi}\left(\frac{n\log p}{2\log(KM_{d,\alpha})}\right)\frac{1}{\log\left(KM_{d,\alpha}\right)} - \int_1^K D_{t,\alpha}(n\theta_{d,p})\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widehat{\phi}\left(\frac{n\log p}{2\log(tM_{d,\alpha})}\right)}{\log\left(tM_{d,\alpha}\right)}\right)\mathrm{d}t + O(1)$$

$$\ll \frac{1}{|\sin(4n\theta_{d,p})|}+\frac{1}{|\sin(4n\theta_{d,p})|}\int_1^K \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\widehat{\phi}\left(\frac{n\log p}{2\log(tM_{d,\alpha})}\right)}{\log\left(tM_{d,\alpha}\right)}\right)\mathrm{d}t \ll \frac{1}{|\sin(4n\theta_{d,p})|},$$

since the integral converges. This gives

$$\frac{8}{K}\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} U_{\mathrm{split}}(\phi,d,k) \ll \frac{1}{K}\sum_{\substack{p^n\le(KM_{d,\alpha})^{2\nu}\\ p\equiv1\,(\mathrm{mod}\,4)}} \frac{\log p}{p^{n/2}|\sin(4n\theta_{d,p})|}. \tag{5.3}$$

In order to give a lower bound on $|\sin(4n\theta_{d,p})|$, let us first recall that $\tan(\theta_{d,p})\in\mathbb{Q}$. By Niven's theorem, the only rational multiples $a\pi$ such that $\tan(a\pi)$ is a rational number are given by the set $\{m\frac{\pi}{4} : m\in\mathbb{Z}\}\subset\pi\mathbb{Q}$. It follows that, as $p\neq 2$, $p\not\equiv 3\,(\mathrm{mod}\,4)$, the point $z_{d,p}^n = p^{n/2}e^{in\theta_{d,p}}\in\mathbb{Z}[i]$ is a lattice point off the lines $x=0$, $y=0$ and $x=\pm y$ of the complex plane, i.e. that

$$|\mathrm{Im}(z_{d,p}^n)|,\ |\mathrm{Re}(z_{d,p}^n)|,\ |\mathrm{Im}(z_{d,p}^n)\pm\mathrm{Re}(z_{d,p}^n)|\ge 1.$$

Using

$$\sin(4x) = 4\sin x\cos x(\cos x-\sin x)(\cos x+\sin x),$$

we find that for, say, $0\le n\theta_{d,p}\le\frac{\pi}{6}$ and $\frac{5\pi}{6}\le n\theta_{d,p}\le\pi$,

$$|\sin(4n\theta_{d,p})|\ge 4\frac{|\mathrm{Im}(z_{d,p}^n)|}{|z_{d,p}^n|}(\cos\tfrac{\pi}{6})^2\left(\cos\tfrac{\pi}{6}-\sin\tfrac{\pi}{6}\right)\ge p^{-\frac{n}{2}}.$$

For $\frac{\pi}{6}\le n\theta_{d,p}\le\frac{\pi}{3}$ and $\frac{2\pi}{3}\le n\theta_{d,p}\le\frac{5\pi}{6}$ we similarly find that

$$|\sin(4n\theta_{d,p})|\ge 4\sin\tfrac{\pi}{6}\cos\tfrac{\pi}{3}\frac{|\mathrm{Re}(z_{d,p}^n)\pm\mathrm{Im}(z_{d,p}^n)|}{|z_{d,p}^n|}\left(\cos\tfrac{\pi}{3}+\sin\tfrac{\pi}{6}\right)\ge p^{-\frac{n}{2}},$$

while for $\frac{\pi}{3}\le n\theta_{d,p}\le\frac{2\pi}{3}$,

$$|\sin(4n\theta_{d,p})|\ge 4\sin\tfrac{\pi}{3}\frac{|\mathrm{Re}(z_{d,p}^n)|}{|z_{d,p}^n|}(\sin^2\tfrac{\pi}{3}-\cos^2\tfrac{\pi}{3})\ge p^{-\frac{n}{2}},$$

so that for any p,

$$|\sin(4n\theta_{d,p})|\ge p^{-\frac{n}{2}}. \tag{5.4}$$

Inserting into (5.3) and using the prime number theorem in arithmetic progressions, we have

$$\frac{8}{K}\sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} U_{\mathrm{split}}(\phi,d,k)\ll d^{2\nu}K^{2\nu-1},$$

as desired. □

Remark 5.3. In the following, we will use the function $\|\cdot\|_{2\pi}$ to denote the distance to the nearest integer multiple of 2π. Observe that since $\frac{1}{2}\|2x\|_{2\pi} \geq |\sin x|$, we have by (5.4) that

$$\|8n\theta_{d,p}\|_{2\pi} \geq 2p^{-\frac{n}{2}}. \tag{5.5}$$

We will use it often in the next section.

6. Proof of Theorem 1.1

In order to extend the admissible support in Lemma 5.2 to $(-1,1)$ (and prove Theorem 1.1), we improve upon our bound of the contribution of the split primes by averaging the possible values of $\theta_{d,p}$, and not simply bounding by the worst value. Let

$$S_{p^n} := \Big\{ z = r(z)e^{i\theta(z)} \in \mathbb{C}^\times \quad : n\theta_{d,p} - \tfrac{1}{8}p^{-\frac{n}{2}} \leq \theta(z) \leq n\theta_{d,p} + \tfrac{1}{8}p^{-\frac{n}{2}} \text{ and } p^{\frac{n}{2}} - \tfrac{1}{4} \leq r(z) \leq p^{\frac{n}{2}} + \tfrac{1}{4} \Big\}. \tag{6.1}$$

Note that the area of S_{p^n} is equal to

$$\begin{aligned} A(S_{p^n}) := \iint_{S_{p^n}} r\mathrm{d}r\mathrm{d}\theta : &= \int_{\theta_{\min}}^{\theta_{\max}} \int_{r_{\min}}^{r_{\max}} r\mathrm{d}r\mathrm{d}\theta = \frac{1}{2}\left(r_{\max}^2 - r_{\min}^2\right)\left(\theta_{\max} - \theta_{\min}\right) \\ &= \frac{1}{2}\left(\left(p^{\frac{n}{2}} + \tfrac{1}{4}\right)^2 - \left(p^{\frac{n}{2}} - \tfrac{1}{4}\right)^2\right)\left(\tfrac{1}{4}p^{-\frac{n}{2}}\right) = \frac{1}{8}. \end{aligned}$$

Lemma 6.1. *Let d an odd square-free integer, $p \equiv 1 \pmod 4$ and n a positive integer. Then if $p^{-\frac{n}{2}} = o(\|8n\theta_{d,p}\|_{2\pi})$, we have*

$$\frac{\log p}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}} = \frac{2}{n}\frac{1}{A(S_{p^n})}\iint_{S_{p^n}} \frac{\log r}{\|8\theta\|_{2\pi}}\mathrm{d}r\mathrm{d}\theta\left(1 + O\left(\frac{1}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}}\right)\right).$$

Unconditionally, we moreover find that

$$\frac{\log p}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}} \ll \frac{1}{n}\iint_{S_{p^n}} \frac{\log r}{\|8\theta\|_{2\pi}}\mathrm{d}r\mathrm{d}\theta.$$

Remark 6.2. Note that by (5.5), for any $re^{i\theta} \in S_{p^n}$, and for all $m \in \mathbb{Z}$,

$$\begin{aligned} |8\theta - 2m\pi| &\geq |8n\theta_{d,p} - 2m\pi| - |8n\theta_{d,p} - 8\theta| \\ &\geq \min_{m\in\mathbb{Z}} |8n\theta_{d,p} - 2m\pi| - |8n\theta_{d,p} - 8\theta| = \|8n\theta_{d,p}\|_{2\pi} - |8n\theta_{d,p} - 8\theta| \\ &\geq 2p^{-\frac{n}{2}} - p^{-\frac{n}{2}} = p^{-\frac{n}{2}}. \end{aligned}$$

Thus

$$\|8\theta\|_{2\pi} \geq p^{-\frac{n}{2}} > 0, \tag{6.2}$$

and therefore the integral in Lemma 6.1 is indeed well-defined.

Proof of Lemma 6.1. Let

$$f(x) := \frac{\log x}{x} \quad \text{and} \quad f'(x) = \frac{1 - \log x}{x^2},$$

so that for $re^{i\theta} \in S_{p^n}$, we have

$$\left| \frac{\log r}{r} - \frac{n}{2} \frac{\log p}{p^{\frac{n}{2}}} \right| \ll |r - p^{\frac{n}{2}}| \cdot \|f'\|_{\max} \ll \left| \frac{\log(p^{\frac{n}{2}} - \frac{1}{4})}{(p^{\frac{n}{2}} - \frac{1}{4})^2} \right| \ll \frac{n \log p}{p^n},$$

and

$$\iint_{S_{p^n}} \frac{1}{\|8\theta\|_{2\pi}} \frac{\log r}{r} r \mathrm{d}r \mathrm{d}\theta = \int_{\theta_{\min}}^{\theta_{\max}} \left(\frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}} \right) \int_{r_{\min}}^{r_{\max}} \left(\frac{n}{2} \frac{\log p}{p^{\frac{n}{2}}} + O\left(\frac{n \log p}{p^n} \right) \right) r \mathrm{d}r.$$

We begin by computing the r-integral. Note that

$$\begin{aligned} \int_{r_{\min}}^{r_{\max}} \frac{\log r}{r} r \mathrm{d}r &= \int_{r_{\min}}^{r_{\max}} \left(\frac{n}{2} \frac{\log p}{p^{\frac{n}{2}}} + O\left(\frac{n \log p}{p^n} \right) \right) r \mathrm{d}r \\ &= \left(\frac{n}{2} \frac{\log p}{p^{\frac{n}{2}}} + O\left(\frac{n \log p}{p^n} \right) \right) \int_{p^{\frac{n}{2}} - \frac{1}{4}}^{p^{\frac{n}{2}} + \frac{1}{4}} r \mathrm{d}r = \frac{n \log p}{4} + O\left(\frac{n \log p}{p^{\frac{n}{2}}} \right), \end{aligned}$$

and, in particular, that

$$\frac{n \log p}{4} \ll \int_{r_{\min}}^{r_{\max}} \frac{\log r}{r} r \mathrm{d}r. \tag{6.3}$$

We now compute the θ-integral. Up to translation by $\mathbb{Z}\frac{\pi}{4}$, we may assume that $[\theta_{\min}, \theta_{\max}] \subset \left(0, \frac{\pi}{4}\right)$. We consider several cases. First, suppose $\theta_{\max} < \frac{\pi}{8}$. Then

$$\begin{aligned} \int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}} &= \int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{8\theta} = \frac{1}{8} \left(\log\left(n\theta_{d,p} + \frac{1}{8p^{\frac{n}{2}}} \right) - \log\left(n\theta_{d,p} - \frac{1}{8p^{\frac{n}{2}}} \right) \right) \\ &= \frac{1}{8} \left(\log\left(1 + \frac{1}{8n\theta_{d,p}p^{\frac{n}{2}}} \right) - \log\left(1 - \frac{1}{8n\theta_{d,p}p^{\frac{n}{2}}} \right) \right) \\ &= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} + 1}{8\ell} (8n\theta_{d,p}p^{\frac{n}{2}})^{-\ell} = \frac{1}{32n\theta_{d,p}p^{\frac{n}{2}}} \left(1 + O\left((n\theta_{d,p}p^{\frac{n}{2}})^{-2} \right) \right) \\ &= \frac{1}{4p^{\frac{n}{2}} \|8n\theta_{d,p}\|_{2\pi}} \left(1 + O\left((p^{\frac{n}{2}} \|8n\theta_{d,p}\|_{2\pi})^{-2} \right) \right). \end{aligned}$$

Since the coefficients of the Taylor series are all non-negative, the implied constant is positive, enabling us to conclude that

$$\frac{1}{4p^{\frac{n}{2}} \|8n\theta_{d,p}\|_{2\pi}} \ll \int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}}. \tag{6.4}$$

Next, suppose $\theta_{\min} > \frac{\pi}{8}$. Then

$$\begin{aligned} \int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}} &= \int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{2\pi - 8\theta} = \frac{1}{8} \left(\log\left(\pi - 4n\theta_{d,p} + \frac{1}{2p^{\frac{n}{2}}} \right) - \log\left(\pi - 4n\theta_{d,p} - \frac{1}{2p^{\frac{n}{2}}} \right) \right) \\ &= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1} + 1}{8\ell} (2(\pi - 4n\theta_{d,p})p^{\frac{n}{2}})^{-\ell} = \frac{1}{4p^{\frac{n}{2}} \|8n\theta_{d,p}\|_{2\pi}} \left(1 + O\left((p^{\frac{n}{2}} \|8n\theta_{d,p}\|_{2\pi})^{-2} \right) \right), \end{aligned}$$

and where again we may conclude (6.4), upon noting that the implicit constant is positive. Finally, we consider the case $\theta_{\min} < \frac{\pi}{8} < \theta_{\max}$, in which case $|n\theta_{d,p} - \frac{\pi}{8}| < \frac{1}{8}p^{-\frac{n}{2}}$. Then

$$\begin{aligned}\int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}} &= \int_{\theta_{\min}}^{\frac{\pi}{8}} \frac{\mathrm{d}\theta}{8\theta} + \int_{\frac{\pi}{8}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{2\pi - 8\theta} \\ &= \frac{1}{8}\left(\log\left(\frac{\pi}{8}\right) - \log\left(n\theta_{d,p} - \frac{1}{8p^{\frac{n}{2}}}\right) + \log\left(\frac{\pi}{8}\right) - \log\left(\frac{\pi}{4} - n\theta_{d,p} - \frac{1}{8p^{\frac{n}{2}}}\right)\right) \\ &= \frac{1}{8}\Bigg(2\log\left(\frac{\pi}{2}\right) - \log\left(4n\theta_{d,p}\right) - \log\left(\pi - 4n\theta_{d,p}\right) \\ &\qquad - \log\left(1 - \frac{1}{8n\theta_{d,p}p^{\frac{n}{2}}}\right) - \log\left(1 - \frac{1}{2(\pi - 4n\theta_{d,p})p^{\frac{n}{2}}}\right)\Bigg).\end{aligned}$$

Note that

$$\begin{aligned}2\log\left(\frac{\pi}{2}\right) - \log\left(4n\theta_{d,p}\right) - \log\left(\pi - 4n\theta_{d,p}\right) &= -\log\left(1 - \frac{\pi/2 - 4n\theta_{d,p}}{\pi/2}\right) - \log\left(1 + \frac{\pi/2 - 4n\theta_{d,p}}{\pi/2}\right) \\ &= O\left(\left(\frac{\pi}{2} - 4n\theta_{d,p}\right)^2\right) = O\left(p^{-n}\right),\end{aligned}$$

as well as that

$$\begin{aligned}&-\log\left(1 - \frac{1}{8n\theta_{d,p}p^{\frac{n}{2}}}\right) - \log\left(1 - \frac{1}{2(\pi - 4n\theta_{d,p})p^{\frac{n}{2}}}\right) \\ &= \sum_{\ell=1}^{\infty} \frac{1}{\ell}(8n\theta_{d,p}p^{\frac{n}{2}})^{-\ell} + \sum_{\ell=1}^{\infty} \frac{1}{\ell}(2(\pi - 4n\theta_{d,p})p^{\frac{n}{2}})^{-\ell} \\ &= \frac{1}{p^{\frac{n}{2}}}\left(\frac{1}{8n\theta_{d,p}} + \frac{1}{2\pi - 8n\theta_{d,p}}\right) + O(p^{-n}).\end{aligned}$$

Upon noting that

$$\begin{aligned}&\frac{1}{8n\theta_{d,p}} + \frac{1}{2\pi - 8n\theta_{d,p}} = \frac{1}{\|8n\theta_{d,p}\|_{2\pi} + 8n\theta_{d,p} - \|8n\theta_{d,p}\|_{2\pi}} + \frac{1}{\|8n\theta_{d,p}\|_{2\pi} + 2\pi - 8n\theta_{d,p} - \|8n\theta_{d,p}\|_{2\pi}} \\ &= \frac{2}{\|8n\theta_{d,p}\|_{2\pi}}\Big(1 + O\Big(|2\pi - 8n\theta_{d,p} - \|8n\theta_{d,p}\|_{2\pi}| + |8n\theta_{d,p} - \|8n\theta_{d,p}\|_{2\pi}|\Big)\Big) \\ &= \frac{2}{\|8n\theta_{d,p}\|_{2\pi}}\Big(1 + O\Big(p^{-\frac{n}{2}}\Big)\Big).\end{aligned}$$

To summarize,

$$\int_{\theta_{\min}}^{\theta_{\max}} \frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}} = \frac{1}{4p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}}\left(1 + O\left(p^{-\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}^{-1}\right)\right),$$

and again we conclude (6.4). It follows that

$$\iint_{S_{p^n}} \frac{1}{\|8\theta\|_{2\pi}} \frac{\log r}{r} r\mathrm{d}r\mathrm{d}\theta = \left(\frac{n\log p}{4} + O\left(\frac{n\log p}{p^{\frac{n}{2}}}\right)\right) \frac{1}{4p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}} \left(1 + O\left(\frac{1}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}}\right)\right)$$
$$= A(S_{p^n})\frac{1}{2}\frac{n\log p}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}} \left(1 + O\left(\frac{1}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}}\right)\right).$$

In particular, by (6.3) and (6.4), we conclude that

$$\frac{n\log p}{p^{\frac{n}{2}}\|8n\theta_{d,p}\|_{2\pi}} \ll \frac{1}{A(S_{p^n})} \iint_{S_{p^n}} \frac{1}{\|8\theta\|_{2\pi}} \frac{\log r}{r} r\mathrm{d}r\mathrm{d}\theta,$$

as desired. □

Proposition 6.3. *Let d be an odd square-free integer, $\alpha \in \mathbb{Z}/8\mathbb{Z}$ and ϕ be an even Schwartz function such that supp$(\widehat{\phi}) \subseteq (-\nu,\nu)$. Then as $K \to \infty$, we have*

$$\frac{8}{K} \sum_{\substack{1\le k\le K \\ k\equiv\alpha \,(\mathrm{mod}\, 8)}} U_{\mathrm{split}}(\phi,d,k) = O(d^\nu K^{\nu-1}(\log(dK))^3).$$

Proof. Noting that $\frac{1}{\pi}\|2x\|_{2\pi} \le |\sin x|$, it follows from (5.3) and Lemma 6.1, that

$$\frac{8}{K} \sum_{\substack{1\le k\le K \\ k\equiv\alpha \,(\mathrm{mod}\, 8)}} U_{\mathrm{split}}(\phi,d,k) \ll \frac{1}{K} \sum_{\substack{p^n\le(KM_{d,\alpha})^{2\nu} \\ p\equiv 1 \,(\mathrm{mod}\, 4)}} \frac{\log p}{p^{n/2}|\sin(4n\theta_{d,p})|}.$$

$$\ll \frac{1}{K} \sum_{n=1}^{\frac{2\nu\log K}{\log 5}} \frac{1}{n} \sum_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4) \\ p^n\le(KM_{d,\alpha})^{2\nu}}} \iint_{S_{p^n}} \frac{\log r}{\|8\theta\|_{2\pi}} \,\mathrm{d}r d\theta. \tag{6.5}$$

By (6.1), we note that $|z_{d,p}^n - z| < 1/2$ for any $z \in S_{p^n}$, and thus since $z_{d,p}^n \in \mathbb{Z}[i]$, it follows that for $p \neq q$, $S_{p^n} \cap S_{q^n} = \emptyset$. By (6.2), we find that for any $re^{i\theta} \in S_{p^n}$,

$$\frac{1}{2r} \le \frac{1}{2p^{\frac{n}{2}} - \frac{1}{2}} \le p^{-\frac{n}{2}} \le \|8\theta\|_{2\pi}.$$

It follows that for any $n \ge 1$,

$$\bigcup_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4) \\ p^n<K^{2\nu}}} S_{p^n} \subseteq R_\nu := \left\{z = re^{i\theta} \in \mathbb{C}^\times \;:\; \|8\theta\|_{2\pi} \ge \frac{1}{2r} \text{ and } 1 \le r \le (KM_{d,\alpha})^\nu + \frac{1}{4}\right\}.$$

Thus, as $\theta \mapsto \|8\theta\|_{2\pi}$ is $\frac{\pi}{4}$-periodic, we have

$$\sum_{\substack{p\equiv 1 \,(\mathrm{mod}\, 4)\\ p^n \le (KM_{d,\alpha})^{2\nu}}} \iint_{S_{p^n}} \frac{1}{\|8\theta\|_{2\pi}} \log r \mathrm{d}r\mathrm{d}\theta \ll \iint_{R_\nu} \frac{1}{\|8\theta\|_{2\pi}} \log r \,\mathrm{d}r\mathrm{d}\theta$$

$$\ll \int_1^{(KM_{d,\alpha})^\nu + \frac14} \int_{\frac{1}{16r}}^{\frac{\pi}{4}-\frac{1}{16r}} \frac{\mathrm{d}\theta}{\|8\theta\|_{2\pi}} \log r \mathrm{d}r \ll \int_1^{(KM_{d,\alpha})^\nu + \frac14} \log r \left(\int_{\frac{1}{16r}}^{\frac{\pi}{8}} \frac{\mathrm{d}\theta}{8\theta} + \int_{\frac{\pi}{8}}^{\frac{\pi}{4}-\frac{1}{16r}} \frac{\mathrm{d}\theta}{2\pi - 8\theta} \right) \mathrm{d}r$$

$$\ll \int_1^{(KM_{d,\alpha})^\nu + \frac14} \log r \left(\log \frac{\pi}{8} + \log 16r \right) \mathrm{d}r \ll_\nu (KM_{d,\alpha})^\nu (\log(KM_{d,\alpha}))^2.$$

Thus by (6.5) we conclude that

$$\frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha \,(\mathrm{mod}\, 8)}} U_{\mathrm{split}}(\phi, d, k) \ll_\nu \sum_{n=1}^{\frac{2\nu \log(KM_{d,\alpha})}{\log 2}} \frac{1}{n} K^{-1} (KM_{d,\alpha})^\nu (\log(KM_{d,\alpha}))^2$$

$$\ll_\nu d^\nu K^{\nu-1} (\log(KM_{d,\alpha}))^2 \log\log(KM_{d,\alpha}),$$

as desired. □

Combining Proposition 5.1 and Proposition 6.3, this proves Theorem 1.1.

7. Non-vanishing results

In this section we prove Corollary 1.3. We consider the smooth test function

$$\phi(x) = \phi_\nu(x) := \left(\frac{\sin(\pi\nu x)}{\pi\nu x} \right)^2.$$

We note that $\phi_\nu(0) = 1$, $\phi_\nu(0) \ge 0$ for all x,

$$\widehat{\phi}_\nu(t) = \begin{cases} \frac{\nu - |t|}{\nu^2} & \text{if } |t| < \nu \\ 0 & \text{otherwise,} \end{cases}$$

and that $\mathrm{supp}(\widehat{\phi}) \subset (-\nu, \nu)$. Then, for α even,

$$\frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha \,(\mathrm{mod}\, 8)\\ L(\frac12, \xi_{d,k})=0}} 1 \le \frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha \,(\mathrm{mod}\, 8)}} \frac{\mathrm{ord}_{s=\frac12} L(s, \xi_{d,k})}{2} \le \frac{4}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha \,(\mathrm{mod}\, 8)}} \mathcal{D}_K(\phi_\nu, \xi_{d,k}) \tag{7.1}$$

where we have used the fact that for α even, the order of vanishing of $L(s, \xi_{d,k})$ at $s = \frac12$ is even, since $W(\xi_{d,k}) = 1$ by Lemma 2.6. Using Theorem 1.1 (and then $\nu < 1$), by approximating the test function ϕ_ν by a series of Schwartz functions, we compute that

$$\lim_{K\to\infty} \frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha \,(\mathrm{mod}\, 8)}} \mathcal{D}_K(\phi_\nu, \xi_{d,k}) = \widehat{\phi}_\nu(0) - \frac12 \int_{-\nu}^{\nu} \left(\frac{1}{\nu} - \frac{|t|}{\nu^2} \right) \mathrm{d}t = \frac{1}{\nu} - \frac12,$$

and replacing in (7.1) with $\nu < 1$, we get

$$\lim_{K\to\infty} \frac{8}{K} \,\#\left\{1 \le k \le K,\, k \equiv \alpha \,(\mathrm{mod}\, 8) \;:\; L(\tfrac{1}{2}, \xi_{d,k}) \neq 0\right\} \ge 75\%.$$

Suppose now that α is odd and $\alpha \in S_+(d)$, i.e. for $k \equiv \alpha \,(\mathrm{mod}\, 8)$, we have $W(\xi_{d,k}) = 1$, which implies that $L(s, \xi_{d,k})$ vanishes with even order at $\frac{1}{2}$. Exactly as above, we have

$$\frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)\\ L(\frac{1}{2},\xi_{d,k})=0}} 1 \le \frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \frac{\mathrm{ord}_{s=\frac{1}{2}} L(s, \xi_{d,k})}{2} \le \frac{4}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \mathcal{D}_K(\phi_\nu, \xi_{d,k}) \tag{7.2}$$

but in this case, we have orthogonal symmetries from Theorem 1.1. We compute

$$\lim_{K\to\infty} \frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \mathcal{D}_K(\phi_\nu, \xi_{d,k}) = \widehat{\phi}_\nu(0) + \frac{1}{2}\int_{-\nu}^{\nu} \left(\frac{1}{\nu} - \frac{|t|}{\nu^2}\right) \mathrm{d}t = \frac{1}{\nu} + \frac{1}{2} \tag{7.3}$$

and replacing in (7.2) with $\nu < 1$, we get

$$\lim_{K\to\infty} \frac{8}{K} \,\#\left\{1 \le k \le K,\, k \equiv \alpha \,(\mathrm{mod}\, 8) \;:\; L(\tfrac{1}{2}, \xi_{d,k}) \neq 0\right\} \ge 25\%.$$

Finally, suppose that α is odd and $\alpha \in S_-(d)$, i.e. for $k \equiv \alpha \,(\mathrm{mod}\, 8)$, we have $W(\xi_{d,k}) = -1$, which implies that $L(s, \xi_{d,k})$ vanishes with odd order ≥ 1, and then

$$\frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)\\ \mathrm{ord}_{s=\frac{1}{2}} L(s,\xi_{d,k})>1}} 1 \le \frac{8}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \frac{\mathrm{ord}_{s=\frac{1}{2}} L(s, \xi_{d,k}) - 1}{2} \le \frac{4}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \mathcal{D}_K(\phi_\nu, \xi_{d,k}) - \frac{1}{2} + O(K^{-1}) \tag{7.4}$$

and we again have orthogonal symmetries. As in (7.3), we compute

$$\lim_{K\to\infty} \frac{4}{K} \sum_{\substack{1\le k\le K\\ k\equiv\alpha\,(\mathrm{mod}\,8)}} \mathcal{D}_K(\phi_\nu, \xi_d^k) - \frac{1}{2} = \frac{1}{2\nu} - \frac{1}{4}$$

and replacing in (7.4) with $\nu < 1$, we get

$$\lim_{K\to\infty} \frac{8}{K} \left\{1 \le k \le K,\, k \equiv \alpha \,(\mathrm{mod}\, 8) \;:\; \mathrm{ord}_{s=\frac{1}{2}} L(s, \xi_{d,k}) = 1\right\} \ge 75\%.$$

This completes the proof. □

Remark 7.1. For $\nu < 1$, the function $\phi_\nu(x) = \left(\frac{\sin(\pi\nu x)}{\pi\nu x}\right)^2$ is optimal among all functions $\phi \in S_1$ where

$$S_1 := \left\{\phi \in L^1(\mathbb{R}), \phi \ge 0, \phi(0) = 1, \text{ and } \mathrm{supp}(\widehat{\phi}) \subset (-1, 1)\right\}$$

for both symplectic and orthogonal symmetries, i.e.

$$\min_{\phi\in S_1} \int_{-\infty}^{\infty} \phi(x) W_G(x) \mathrm{d}x$$

is attained for ϕ_1 when $W_G(x) = 1 \pm \frac{1}{2}\frac{\sin(2\pi x)}{2\pi x}$. Indeed, for η as in (1.4), it follows by the criterion of [20, Corollary A.2], that ϕ_1 is optimal if and only if

$$\pm\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \eta(x-y)\mathrm{d}y$$

is independent of x, for all $0 \le x \le \frac{1}{2}$, which is the case.

Appendix A. Evaluating the $c_j(1,0)$ constants numerically

In this appendix, we numerically approximate the values of $c_j(1,0)$ (see Proposition 5.1) for the cases $j=0$ and $j=2$, and explain the procedure for doing so for arbitrary $j \ge 0$. We write

$$c_j := c_j(1,0) = A_j + B_j - T_j,$$

where

$$A_j := \frac{(\log 2)^{j+1}}{j!}\mathrm{Li}_{-j}(\tfrac{1}{2}) - \frac{2}{j!}\left(\frac{\log 2}{2}\right)^{j+1}\mathrm{Li}_{-j}\left(\frac{1}{\sqrt{2}}\right) - \delta_0(j)\log 2\pi$$

$$B_j := \frac{(-1)^j}{j!}\left(\left(\frac{f'}{f}\right)^{(j)}(1) - \left(\frac{L'}{L}\right)^{(j)}(1,\chi_4)\right) \qquad f(s) := (s-1)\zeta(s) \tag{A.1}$$

and

$$T_j := \frac{2^{j+1}}{j!}\sum_{p\equiv 3 \,(\mathrm{mod}\, 4)} (\log p)^{j+1}\mathrm{Li}_{-j}\left(\frac{1}{p^2}\right).$$

As in [7], we find that for $n \ge 1$, the n^{th} Laurent–Stieltjes constant for $L(s,\chi)$ may be expressed as

$$\gamma_n(\chi) = \sum_{a=1}^{q}\chi(a)\gamma_n(a,q), \qquad \gamma_n(a,q) := \lim_{x\to\infty}\left(\sum_{\substack{0<m\le x\\ m\equiv a \,(\mathrm{mod}\, q)}}\frac{(\log m)^n}{m} - \frac{(\log x)^{n+1}}{q(n+1)}\right),$$

where $\gamma_n(a,q)$ are sometimes referred to as *generalized Euler constants* (for arithmetical progressions). In particular, we find that for $n \ge 1$,

$$\gamma_n(\chi_4) = \gamma_n(1,4) - \gamma_n(3,4), \tag{A.2}$$

and moreover that $\gamma_0(\chi_4) = L(1,\chi_4) = \frac{\pi}{4}$. For small values of n ($n \le 20$), $\gamma_n(1,4)$ and $\gamma_n(3,4)$ have been explicitly computed in [6]:

$\gamma_1(1,4)$	-0.154621845705		$\gamma_1(3,4)$	= 0.038279471092		$\gamma_1(\chi_4)$	-0.19290131679
$\gamma_2(1,4)$	-0.095836601153		$\gamma_2(3,4)$	= 0.058305123277		$\gamma_2(\chi_4)$	-0.15414172443
$\gamma_3(1,4)$	-0.049281458556		$\gamma_3(3,4)$	= 0.045601400650		$\gamma_3(\chi_4)$	-0.0948828592

Table 1. Values of $\gamma_n(1,4)$, $\gamma_n(3,4)$, and $\gamma_n(\chi_4)$

By (4.17) and (4.19) we may thus compute the B_j contribution by implementing the following code into Mathematica:

```
Bellzeta[j_,k_]:=BellY[j+1,k,
  Table[(-1)^i*(i+1)*StieltjesGamma[i], {i,0,j+1-k}]]
  (*Defining the Bell polynomial B_{j+1,k}(\gamma_0,...,(j+2-k)(-1)^{j+1-k}\gamma_{j+1-k})*);

Bzeta[j_]:=Sum[(-1)^{k+1}(k-1)! Bellzeta[j,k], {k,1,j+1}]
(*Summing over the relevant Bell polynomials*);

Lconstants:={-0.19290131679,-0.15414172443,-0.0948828592}
 (*Values of \gamma_j(\chi_4) extracted from previously published computations*)

BellL[j_,k_]:=BellY[j+1,k,Table[(-1)^{i}*Lconstants[[i]],{i,1,j+2-k}]]
(*Defining the Bell polynomial B_{j+1,k}(-\gamma_1(\chi_4),...,(-1)^{j+2-k}\gamma_{j+2-k}(\chi_4))*)

BL[j_]:=Sum[(-1)^{k+1}(k-1)!(Pi/4)^{-k} BellL[j,k],{k,1,j+1}]
(*Summing over the relevant Bell polynomials for the L-function factor*)

B[j_]:=(-1)^{j}/(j)!(Bzeta[j]-BL[j])
```

j	$(f'/f)^{(j)}$	$(L'/L)^{(j)}$	B_j
$j=0$	0.57721566	0.24560958	0.33160608
$j=2$	0.10337726	0.29505047	−0.0958366

TABLE 2. Values of $(f'/f)^{(j)}$, $(L'/L)^{(j)}$, and B_j, for $j=0$ and $j=2$

To compute the inert prime contribution, we write

$$T_j = \frac{2^{j+1}}{j!} \sum_{p\equiv 3 \,(\mathrm{mod}\, 4)} (\log p)^{j+1} \mathrm{Li}_{-j}(p^{-2}) = T_j[x] + R_j[x],$$

where

$$T_j[x] := \frac{2^{j+1}}{j!} \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ p<x}} (\log p)^{j+1} \mathrm{Li}_{-j}(p^{-2})$$

and

$$R_j[x] := \frac{2^{j+1}}{j!} \sum_{\substack{p\equiv 3 \,(\mathrm{mod}\, 4)\\ p>x}} \sum_{n=1}^{\infty} \frac{(\log p)^{j+1} n^j}{p^{2n}} \leq \frac{2^{j+1}}{j!} \left(\frac{(\log x)^{j+1}}{x^2} + \sum_{k=0}^{j+1} \frac{(\log x)^k}{x} \frac{(j+1)!}{k!} \right),$$

so long as x is sufficiently large such that the map $t \mapsto (\log t)^{j+1}/t^2$ is decreasing in the range $[x,\infty)$. The resulting contribution may then be computed by implementing the following code:

```
X=1000000;  primes3 := Select[Range[3, X, 4],PrimeQ]
 (*List of primes 3 mod 4 up to X*);
```

```
T[j_]:=(2^{j+1}/j!)Sum[N[Log[primes3[[i]]]^{j+1}]
N[PolyLog[-j,(1/primes3[[i]]^2)]],{i,1,Length[primes3]}]
(*computes T_j[X]*)
```

Plugging in the values $x = 10^6$ yields the following values:

j	$T_j[10^6]$	Bound for $R_j[10^6]$	A_j	c_j
$j = 0$	0.45747	0.00003	-2.81814	-2.9440
$j = 2$	3.93	0.014	-1.00081	-5.0

Table 3. Numerical approximation of c_0 and c_2

References

[1] R. Chen, Y. Kim, J. Lichtman, S. J. Miller, A. Shubina, S. Sweitzer, E. Waxman, E. Winsor, J. Yang, A Refined Conjecture for the Variance of Gaussian Primes across Sectors, Exp. Math. (2020), 1–21.

[2] C. David and A. Güloğlu, One-level density and non-vanishing for cubic L-functions over the Eisenstein field, Int. Math. Res. Not. (2022), no. 23, 18833–18873.

[3] S. Drappeau, K. Pratt and M. Radziwiłł, One-level density estimates for Dirichlet L-functions with extended support, Algebra Number Theory 17 (2023), no. 4, 805–830.

[4] L. Devin, Discrepancies in the distribution of Gaussian primes, `arXiv:2105.02492`.

[5] L. Devin, D. Fiorilli, A. Södergren, Low-lying zeros in families of holomorphic cusp forms: the weight aspect, Quart. J. Math. 73 (2022), no. 4, 1403–1426.

[6] K. Dilcher, Supplement to Generalized Euler Constants for Arithmetical Progressions. Math. Comp. 59 (1992), no. 199, S21–S24.

[7] S. S. Eddin, On two problems concerning the Laurent-Stieltjes coefficients of Dirichlet L-series, Ph.D. thesis, University of Lille 1, France, 2013.

[8] F. Faà di Bruno, Note sur une nouvelle formule de calcul différentiel, The Quarterly Journal of Pure and Applied Mathematics 1 (1857), 359–360.

[9] D. Fiorilli, J. Parks, A. Södergren, Low-lying zeros of elliptic curve L-functions: Beyond the Ratios Conjecture, Math. Proc. Cambridge Philos. Soc. 160 (2016), no. 2, 315–351.

[10] D. Fiorilli, J. Parks, A. Södergren, Low-lying zeros of quadratic Dirichlet L-functions: Lower order terms for extended support, Compos. Math. 153 (2017), no. 6, 1196–1216.

[11] P. Gao, L. Zhao, One-level density of low-lying zeros of quadratic Hecke L-functions of imaginary quadratic number fields, J. Aust. Math. Soc. 112 (2022), no. 2, 170–192.

[12] D. R. Heath-Brown, The average analytic rank of elliptic curves, Duke Math. J.122 (2004), no. 3, 591–623.

[13] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen. I, Mathematische Zeitschrift 1 (1918), 357–376; II, Mathematische Zeitschrift 6 (1920), 11–51.

[14] G. Harman, P. Lewis, Gaussian primes in narrow sectors. Mathematika 48 (2001), no. 1-2, 119-135.

[15] K. Holm, The 1-Level Density For Zeros of Hecke L-Functions of Imaginary Quadratic Number Fields of Class Number 1, `arXiv:2309.10018`

[16] C. P. Hughes, Z. Rudnick, Linear statistics of low-lying zeros of L-functions, Q. J. Math. **54** (2003), no. 3, 309–333.

[17] K. Ireland, M. Rosen, *A classical introduction to modern number theory*, Second edition, Graduate Texts in Mathematics, 84, Springer-Verlag, New York, 1990.

[18] H. Iwaniec, *Topics in classical automorphic forms*, American Mathematical Society Graduate Texts in Mathematics, 17, American Mathematical Society, Providence, RI, 1997.

[19] H. Iwaniec, E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, 53, American Mathematical Society, Providence, RI, 2004.

[20] H. Iwaniec, W. Luo, P. Sarnak, Low lying zeros of families of L-functions, Inst. Hautes Études Sci. Publ. Math. (2000), no. 91, 55–131.
[21] N. Katz, P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloq. Publ. 45 (1999).
[22] N. Katz, P. Sarnak, Zeroes of zeta functions and symmetry, Bull. Amer. Math. Soc. (N.S.) 36 (1999), no. 1, 1–26.
[23] F. Lemmermeyer, *Reciprocity laws : From Euler to Eisenstein*, Springer Monogr. Math. Springer-Verlag, Berlin, (2000).
[24] S. J. Miller, One- and two-level densities for rational families of elliptic curves: evidence for the underlying group symmetries, Compos. Math. 140 (2004), no. 4, 952–992.
[25] B. Maxwell, S. J. Miller, C. Rapti, C. Turnage-Butternaugh, Some results in the theory of low-lying zeros of families of L-functions, in Families of automorphic forms and the trace formula, Simons Symp., Springer (2016), 435–476.
[26] J. Neukirch, *Algebraic number theory*, Translated from the 1992 German original and with a note by Norbert Schappacher. With a foreword by G. Harder Grundlehren Math. Wiss., 322 Springer-Verlag, Berlin, (1999).
[27] A. E. Özlük, C. Snyder, Small zeros of quadratic L-functions, Bull. Austral. Math. Soc. 47 (1993), no. 2, 307–319.
[28] G. Ricotta, E. Royer, Lower order terms for the one-level densities of symmetric power L-functions in the level aspect, Acta Arith. 141 (2010), no. 2, 153–170.
[29] G. Ricotta, E. Royer, Statistics for low-lying zeros of symmetric power L-functions in the level aspect, Forum Math. 23 (2011), no. 5, 969–1028.
[30] Z. Rudnick and E. Waxman, Angles of Gaussian Primes, Israel Journal of Mathematics (2019) 1–41.
[31] P. Sarnak, S. W. Shin, N. Templier, Families of L-functions and their symmetry, Proceedings of Simons Symposia, *Families of Automorphic Forms and the Trace Formula*, Springer-Verlag (2016), 531–578.
[32] E. Waxman, Lower Order Terms for the One Level Density of a Symplectic Family of Hecke L-Functions. J. Number Theory 221 (2021), 447–483.
[33] M. P. Young, Low-lying zeros of families of elliptic curves, J. Amer. Math. Soc. 19 (2006), no. 1, 205–250.