Imbalances in Hypercubic Billiard Words

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Abstract

A hypercubic billiard word – or cubic billiard word in dimension d – is a word on the d-letter alphabet denoting the sequence of the faces successively hit by a billiard ball moving in the unit cube of $\mathbb{R}^d$, in which two parallel faces are encoded by the same letter. The imbalance of a word w is the maximal difference of the number of occurrences of a letter in two subwords of w of the same length. We know from the work of Morse and Hedlund (1940) that square billiard words generated by a momentum with rationally independent entries are exactly binary aperiodic words with imbalance equal to 1 (also known as Sturmian words). Vuillon (2003) showed that cubic billiard words whose momentum has rationally independent entries in dimension d have an imbalance lower than, or equal to, $d-1$. In this extended abstract, we completely describe the imbalances in this class of words.

1 Motivation

Sturmian words form a class of infinite words over the binary alphabet which sheds light on the remarkable interactions between combinatorics, dynamical systems, and number theory. These interactions are reflected in the various ways to define them (see the historic paper [MH40], or refer to [Lot97] for a recent general presentation). For instance, Sturmian words are equivalently

- words with complexity $n+1$, *i.e.*, admitting exactly $n+1$ factors of length n for all n (a *factor* of w of length n is a subword of w written with n consecutive letters);
- binary aperiodic words with an imbalance equal to 1: all factors of a given length contain, up to one, the same numbers of 1s (and thus, as well, the same numbers of 2s);
- the symbolic trajectories of a ball in a square billiard, launched with a momentum with rationally independent entries (see Remark 2 for a rigorous statement).

They give rise to several generalizations on the d-letter alphabet for $d \geq 3$, depending on the considered definition: strict episturmian words ([JP02], [GJ09]), words associated with d-dimensional continued fraction algorithms ([Sch00], [Ber11], [BD14]), polygonal ([Tab05], [CHT02] and references therein) or cubic billiard words ([AMST94]). It has been, and it is, a large program to determine which properties are still equivalent in higher dimension, and which are not.

In this work, we focus on hypercubic billiard words, which we will call cubic billiard words, regardless of the dimension. Cubic billiard words are defined in Section 2. Roughly speaking, a cubic billiard word in dimension d is a word on the d-letter alphabet denoting the sequence of the faces successively hit by a billiard ball moving in the unit cube of $\mathbb{R}^d$, in which two parallel faces are encoded by the same letter. This class of words (and, first of all, their complexity) was studied in [AMST94], [Bar95], [BH07] and [Bed09].

Hereafter, we focus on their imbalance, which is the maximal difference of occurrences of a letter in two factors of the same length:

$$\mathrm{Imb}(w) = \sup\{|u|_a - |v|_a \mid a \text{ letter and } u, v \text{ factors of } w \text{ with same length}\}, \tag{1}$$

where $|u|_a$ counts the numbers of a in u. The finiteness of this combinatorial quantity is associated with a fast convergence of the observed frequencies of letters in growing factors of w towards their limit value [Ada03].

In [Vui03], Vuillon provides a –somewhat optimal– upper bound for the imbalance of hypercubic billiard words.

Theorem 1 (Vuillon [Vui03]). *Let $d \in \mathbb{N}_{>0}$. In dimension d:*
- the imbalance of any cubic billiard word, whose momentum has rationally independent entries, is bounded above by $d-1$;
- there exists a cubic billiard word whose momentum has rationally independent entries and whose imbalance is exactly $d-1$.

This work answers the natural questions that follow: is the imbalance of such words always equal to $d-1$, as it is the case for $d=2$? If not, which values can be taken by the imbalances of cubic billiard words, and which sets of parameters (*i.e.*, initial position and momentum of the billiard ball) lead to these values?

Remark 1. *In our study, we focus on the case of initial momenta $\theta = (\theta_1, ..., \theta_d)$ with rationally independent entries,* i.e., $\dim(\mathrm{span}_{\mathbb{Q}}(\theta_1, \dots, \theta_d)) = d$. *Indeed:*
- the set of such momenta has full Lebesgue measure;
- the trajectory of a billiard ball with initial position x and initial momentum θ is dense in the unit cube of $\mathbb{R}^d$ if and only if θ has rationally independent entries (this follows from the minimality of the associated linear flow on $\mathbb{R}^d/\mathbb{Z}^d$, see for instance [KH95]);
- it is clear that for any $d \geq 2$, there exist cubic billiard words with rational momenta whose imbalance collapses to 1 (Example 1).

2 Cubic billiard words in dimension d: definition and properties

We consider the trajectory of a billiard ball (formally, a point mass) in the unit cube in dimension $d \geq 1$, which moves with constant speed along straight lines until it hits a face, then bounces elastically, moving backwards along the direction given by the law of reflection. Given $x \in [0,1]^d$, $\theta \in \mathbb{R}^d \setminus \{0\}$ and $t \in \mathbb{R}_{>0} \cup \{+\infty\}$, we denote by $w(x, \theta, t) \in \{1, \dots, d\}^* \cup \{1, \dots, d\}^{\mathbb{N}}$ the [finite or infinite] sequence of the faces successively hit by the ball, with initial position and momentum (x, θ), during the time interval $(0, t]$: the *k-th* letter of the word w is i if and only if the *k-th* face hit by the ball is $x_i = 0$ or $x_i = 1$. In the sequel, we shorten the notation $w(x, \theta, +\infty)$ to $w(x, \theta)$.

Example 1. *Let $d \geq 2$. Consider*

$$\theta = \Big(1, \frac{1}{d-1}, ..., \frac{1}{d-1}\Big) \qquad \textit{and} \qquad x = \Big(\frac{1}{2}, \frac{d-2}{d-1}, \frac{d-3}{d-1}, ..., \frac{1}{d-1}, 0\Big).$$

One checks that $w(x,\theta)$ is the periodic word with period $1213...1d$. *An easy computation shows that the imbalance of this word is* 1. *For the initial position $y = (1-\varepsilon_1, \varepsilon_2, ..., \varepsilon_d)$, where ε_1 and $\varepsilon_2 > \varepsilon_3 > ... > \varepsilon_d$ are chosen positive, small enough, and rationally independent, $w(y,\theta)$ is the periodic word with period* $1^{d-1}234...d$, *which has imbalance $d-1$.*

Observe that the trajectory of the ball is well defined for any set of parameters (x, θ, t), while its symbolic coding $w(x, \theta, t)$ may encounter some problems:
1. when the ball has its entire trajectory on a face;
2. more generally, when the ball hits an *edge*, *i.e.*, the intersection of several faces.
For $\theta \in \mathbb{R}^d \backslash \{0\}$, we define the set $\mathrm{Init}(\theta) \subset [0,1]^d$ of all initial positions x such that $w(x,\theta)$ is unambiguously defined. For example, one checks that if θ has rationally independent entries, then $0 \in \mathrm{Init}(\theta)$.

Proposition 1. *Let $d \geq 1$ and $\theta \in \mathbb{R}^d \backslash \{0\}$. The set* $\mathrm{Init}(\theta)$ *is measurable and has Lebesgue measure* 1.

Yet, if θ has rationally independent entries, the set $\mathrm{Init}(\theta)$ has an empty interior – indeed, its complementary set contains the past trajectory of 0, which is dense in the cube.

Denote by $\mathcal{L}(w)$ the *language* of an infinite word w, which is the set of all its factors.

Proposition 2. *Let $d \geq 1$ and $\theta \in \mathbb{R}^d$ with rationally independent entries. For any $x, y \in \mathrm{Init}(\theta)$, we have $\mathcal{L}(w(x,\theta)) = \mathcal{L}(w(y,\theta))$.*

Proposition 2 does not always hold when the entries of θ have rational dependencies, as illustrated by Example 1.

Proof (ideas). Proposition 2 comes from the continuity of the coding function

$$\begin{array}{lll} \mathrm{Init}(\theta) & \longrightarrow & \{1, ..., d\}^{\mathbb{N}} \\ x & \longmapsto & w(x,\theta) \end{array}$$

together with the minimality of the linear flow of direction $(\theta_1, ..., \theta_d)$ on the torus $\mathbb{R}^d/(2\mathbb{Z})^d$. $\square$

Since the imbalance of a word only depends on its language, it will be sufficient to consider cubic billiard words [whose momenta have rationally independent entries] with initial position $x = 0$. These words will be called *standard* cubic billiard words. For $d = 2$, standard cubic (*i.e.*, square) billiard words are *standard Sturmian* words (as already defined in [MH40])– hence the name.

Remark 2. Link between square billiard words and Sturmian words. *In dimension* 2, *the complementary set of* $\mathrm{Init}(\theta)$ *is exactly the trajectories, back to the past, of the four corners of the unit square. For a momentum θ with rationally independent entries and an initial position $x \in [0,1]^2 \backslash \mathrm{Init}(\theta)$, exactly one corner is hit, and it is hit exactly once. We can consider two "twin" square billiard words, in which this event is encoded by the finite words* 12 *and* 21. *In this more general setting, square billiard words whose momentum has rationally independent entries are exactly Sturmian words.*

3 Main result and consequences

Below, $\mathcal{L}(\theta)$ denotes the language of all/any cubic billiard words with momentum θ (following Proposition 2), $\mathcal{L}_n(\theta)$ the subset of all of these factors with length n, and $\mathrm{pref}_n(w)$ the first n letters of a word w.

Theorem 2. *Let $d \geq 3$ and $\theta = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$ with rationally independent entries. With no loss of generality, assume that $|\theta_1| > |\theta_i|$ for all $i \neq 1$. Then*

$$\mathrm{Imb}(\mathcal{L}(\theta)) = 1 + \max_{f \in \mathcal{L}_{d-2}(\theta)} |f|_1 = 2 + \left|\mathrm{pref}_{d-3}(w(0, \theta))\right|_1.$$

Proof (ideas). For $i \in \{1, ..., d\}$, we compute how many times the faces i are hit at least/at most during a time interval of length t, from which we obtain restrictions on the form of the vectors $(|u|_i - |v|_i)_{1 \leq i \leq d}$ for $u, v \in \mathcal{L}(\theta)$. $\square$

This theorem immediately covers the results of Vuillon (Theorem 1): we trivially have $|\mathrm{pref}_{d-3}(w(0, \theta))|_1 \leq d - 3$, and this inequality becomes an equality when θ_1 is chosen large enough. It also answers the questions of Section 1:

Corollary 1. *For $d \in \{1, ..., 4\}$, the imbalance of any cubic billiard word generated by a momentum with rationally independent entries is exactly $d - 1$.*

Corollary 2. *Let $d \geq 5$.*
- The imbalance of any cubic billiard word generated by a momentum with rationally independent entries belongs to $\{3, ..., d - 1\}$.
- For every $k \in \{3, ..., d - 1\}$, there exists a cubic billiard word generated by a momentum with rationally independent entries whose imbalance is equal to k.

We conclude this work with geometric and algorithmic remarks.

Denote $\Delta := \{(\theta_1, ..., \theta_d) \in (\mathbb{R}_{\geq 0})^d | \sum_{i=1..d} \theta_i = 1\}$. Due to the symmetries of our billiard table, and because the trajectory of the ball does not depend on the norm of the momentum, we assume without loss of generality that the momentum of a cubic billiard word belongs to Δ. Observe that, for all k, Theorem 2 gives an explicit system of inequalities on the entries of θ which generates cubic billiard words with an imbalance equal to k. In other words, the simplex Δ is partitioned into polytopes, on each of which the imbalance is constant for all momenta *with rationally independent entries.* For instance for $d = 5$, the imbalance takes the value 4 in the region $Z_4 := \cup_i \cap_{j \neq i} \theta_i > 2\theta_j$, and the value 3 in the rest of the simplex.

Finally, observe that it is in general hard to determine the imbalance of a language: the definition/formula (1) only enables us to compute larger and larger lower bounds. In the case of cubic billiard words with rationally independent entries, Theorem 2 gives an algorithm to compute it:
- from the knowledge of θ (compute the $d - 3$ first faces hit by a ball initially in 0);
- from the knowledge of a large enough portion of the word (or, equivalently, of its language). Indeed, we are able to detect when all factors of length $d - 2$ have been visited, since we know the cardinality of this set ([Bar95]):

$$\mathrm{card}\, \mathcal{L}_{d-2}(\theta) \;=\; \sum_{i=0}^{d-2} \frac{(d-1)!(d-2)!}{(d-1-i)!(d-2-i)!i!}. \tag{2}$$

However, the formula holds under a little additional condition: the inverses of any three coordinates of θ have to be rationally independent [Bed09]. Furthermore, no upper bound on the computation time can be guaranteed: the portion of the word/language required to compute the imbalance can be arbitrarily long, as illustrated by Example 2.

Example 2. *For $d = 5$, it is possible to choose a momentum θ, with rationally independent entries, such that $|\theta_1| \approx 2|\theta_2| \gg |\theta_i|$ for all $i \in \{3, 4, 5\}$, that would give:*
- $w_1 := w(0, \theta) = 112112...1123...$ *if* $|\theta_1| > 2|\theta_2|$;
- $w_2 := w(0, \theta) = 12112112...1123...$ *otherwise.*

Observe that w_1 and w_2 share a factor $112112...112$ *that can be made arbitrarily long. Yet, by Theorem 2, we have* $\mathrm{Imb}(w_1) = 4 \neq 3 = \mathrm{Imb}(w_2)$.

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