Large audience introduction

Continued fractions = numeration system based on Euclid's algorithm.

They are known to give the best approximations of real numbers (e.g. π, φ, $\sqrt{2}$, ...) by ratios of integers.

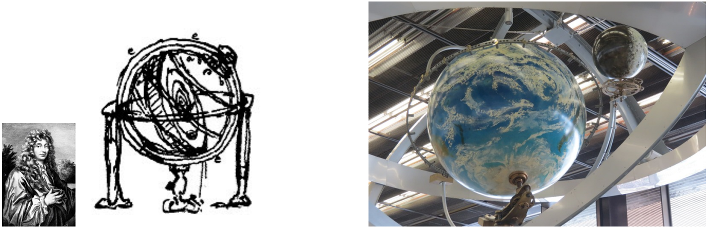

Figure: Planetary automaton imagined by Huygens (1686); a realization in Orly Airport.

They led to fruitful results:
- in mathematics: characterization of quadratic numbers (Lagrange, Galois), construction of transcendental numbers (Liouville),...
- in computer science: discretization of straight lines.

Large audience introduction

Since the 19th century, we want to extend continued fractions to higher dimension, so as to:

- *simultaneously* approximate pairs of real numbers
- discretize planes

Numerous algorithms have been proposed:
Jacobi-Perron (1868/1907), Poincaré, Brun (1957), Arnoux-Rauzy (1991), Cassaigne-Selmer (1961/2017)...

$\longrightarrow$ Are they satisfying? Which one is the best?

General research program: study these algorithms from the standpoint of the symbolic dynamical systems they generate.

My contribution: detect & study exceptional systems associated with a convergence anomaly: infinite imbalance.

Exceptional trajectories in the symbolic dynamics of multidimensional continued fraction algorithms

Mélodie Andrieu

PhD thesis defense

29th March 2021

I - Introduction

II - A semi-algorithm to explore the set of imbalances in a S-adic system

1. Construction of the tool
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III - Natural coding of minimal rotations of the torus, induction and exduction

1. A topological definition
2. Stability through induction & exduction
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I. Introduction

1. Motivations

Regular continued fraction algorithm & Sturmian words

Substractive continued fraction algorithm = iteration of the Farey map:

$$\begin{array}{rcll} (\mathbb{R}^+)^2 & \to & (\mathbb{R}^+)^2 & \\ (x,y) & \mapsto & (x-y,y) & \text{if } x \geq y, \\ & & (x,y-x) & \text{otherwise.} \end{array}$$

The symbolic trajectories under this dynamical system give rise to the class of Sturmian words.

Sturmian words enjoy multiple [combinatorial, geometrical, dynamical] characterizations.

Balance characterization :
Sturmian words are exactly the aperiodic binary words for which any two factors of same length contain, with +/- 1, the same number of 0s.

Ex
A word starting with $w = 0010001001000100010 01\ldots$ is possibly Sturmian.
A word starting with $w = 0\underline{11}011\underline{1}\underline{00}\ldots$ is not.

Regular continued fraction algorithm & Sturmian words

Consequences :
1.The letters 0 and 1 are uniformly distributed with respect to a probability measure ν on $\{0,1\}$.
2. Stronger : the difference between the observed frequency of 0s among the N first letters of w and its expected value $\nu(0)$ is bounded above by $1/N$.

Geometrically, the "broken line" made of the points $P_N := \sum_{n=0}^{N} e_{w[n]}$, where (e_0, e_1) is the usual basis of $\mathbb{R}^2$, remains at bounded distance from its average direction.

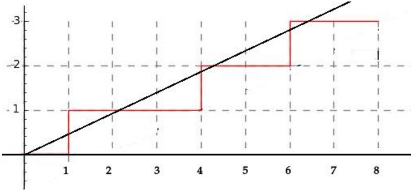


Figure: The broken line of 01000100100...

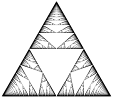
⟶ Sturmian words are used to approximate lines with irrational slopes.

MultiD continued fraction algorithms

Since Jacobi, several algorithms have been proposed to generalize continued fractions to triplets of nonnegative real numbers.

The Arnoux-Rauzy algorithm

$$F_{AR}: \begin{array}{rcll} (\mathbb{R}^+)^3 & \to & (\mathbb{R}^+)^3 & \\ (x,y,z) & \mapsto & (x-y-z,y,z) & \text{if } x \geq y+z, \\ & & (x,y-x-z,z) & \text{if } y \geq x+z, \\ & & (x,y,z-x-y) & \text{if } z \geq x+y. \end{array}$$



The Rauzy gasket

This algorithm gives rise to the class of Arnoux-Rauzy words.

The Cassaigne-Selmer algorithm

$$F_C: \begin{array}{rcll} (\mathbb{R}^+)^3 & \to & (\mathbb{R}^+)^3 & \\ (x,y,z) & \mapsto & (x-z,z,y) & \text{if } x \geq z, \\ & & (y,x,z-x) & \text{otherwise.} \end{array}$$

This algorithm gives rise to the class of C-adic words.

$\longrightarrow$ What can we say of their 3D broken lines?

Properties of their broken lines ?

Old belief : "the broken line of any Arnoux-Rauzy word remains at finite distance from its average direction"

$\longrightarrow$ disproved by the construction of an Arnoux-Rauzy word with infinite imbalance. [Cassaigne, Ferenczi, Zamboni 2000]

$\longrightarrow$ However, the set of these unbalanced Arnoux-Rauzy word has measure 0. [Delecroix, Hejda, Steiner 2013]

Questions: 1) What about Cassaigne-Selmer algorithm?
2) What can be said of these exceptional trajectories?

1. Introduction

2. Preliminaries

definitions & notations

Finite and infinite words

An alphabet $\mathfrak{A}$ is a finite set.
A finite word of length n is an element of $\mathfrak{A}^n$.
An infinite word is an element of $\mathfrak{A}^{\mathbb{N}}$.

Following `Python`, $u[k]$ denotes the $(k+1)$-th letter of u.

A finite word u of length n is a factor of a word w if there exists an index i such that:

$$\text{for all } k \in \{0, ..., n-1\},\ w[i+k] = u[k].$$

$\longrightarrow$If $i = 0$, u is a prefix of w.

Notations:
- $\mathfrak{A}^* =$ the set of all finite words over $\mathfrak{A}$
- $\epsilon =$ the empty word
- $\mathcal{F}_n(w) =$ the set of factors of w of length n of w
- $\mathcal{F}(w)$ the set of factors of all lengths.

The set $\mathfrak{A}^{\mathbb{N}}$ is endowed with the product topology.

definitions & notations

Substitutions and S-adic words

Let $\mathcal{A}$ an alphabet.

"Substitution": $\sigma \in End((\mathcal{A}^*, \cdot))$

ex : σ_{TM} : $1 \mapsto 12$
$2 \mapsto 21$

$\sigma_{TM}(122) = 122121$

Let S a *finite* set of substitutions over a *common alphabet* $\mathcal{A}$.

"S-adic word": an *infinite word* that can be written : $w = \lim_{n\to\infty} \sigma_0 \circ \ldots \circ \sigma_{n-1}(a)$

with : - $a \in \mathcal{A}$ $\longleftarrow$ "seed"
- $(\sigma_n) \in S^{\mathbb{N}}$ $\longleftarrow$ "directive sequence"

ex : $w_{TM} = \lim(\sigma_{TM})^n(1) = 1221211221121221211212211221211...$
- in fact, all substitutive words (but not only...)

"S-adic system": the set of all S-adic words (for a fixed S). Notation: X_S.

Central examples of S-adic systems

$C = \{c_1, c_2\}$ with:

$$c_1 : \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 2 \end{array} \qquad \text{and} \qquad c_2 : \begin{array}{ccc} 1 & \mapsto & 2 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 3. \end{array}$$

def: a C-adic word is a S-adic word with S = C.

$S_{AR} = \{\sigma_1, \sigma_2, \sigma_3\}$ with:

$$\sigma_1 : \begin{array}{ccc} 1 & \rightarrow & 1 \\ 2 & \rightarrow & 12 \\ 3 & \rightarrow & 13 \end{array} \quad ; \quad \sigma_2 : \begin{array}{ccc} 1 & \rightarrow & 21 \\ 2 & \rightarrow & 2 \\ 3 & \rightarrow & 23 \end{array} \quad \text{and} \quad \sigma_3 : \begin{array}{ccc} 1 & \rightarrow & 31 \\ 2 & \rightarrow & 32 \\ 3 & \rightarrow & 3. \end{array}$$

def: w is an Arnoux-Rauzy word if there exists w_0 a S-adic word for $S = S_{AR}$ s.t.:

i. σ_1, σ_2 and σ_3 infinitively appear in its directive sequence

ii. $\mathcal{F}(w) = \mathcal{F}(w_0)$

Abelianization, broken line and average direction

Def: the abelianized of a finite word u is the vector $\mathrm{ab}(u) = (|u|_a)_{a\in A}$, where $|u|_a$ is number of occurrences of a in u.

Def: the broken line of w is $\mathcal{B}_w := \{\mathrm{ab}(\mathrm{pref}_k(w)) | k \in \mathbb{N}\}$.

Def: the frequency of a letter a in w is the limit, if it exists, of the proportion of a in the sequence of growing prefixes of w: $f_w(a) = \lim_{n\to\infty} \frac{|\mathrm{pref}_n(w)|_a}{n}$.

We denote by $f_w = (f_w(a))_{a\in A}$ the vector of letter frequencies of w, if it exists.

$\longrightarrow$ It gives the average direction of the broken line.

Fact : All Arnoux-Rauzy and C-adic words admit a vector of letter frequencies.

Link between words and CF algorithms Arnoux-Rauzy and [primitive] C-adic words admit a unique directive sequence which is driven by the symbolic trajectory of their frequency vector under the action of Arnoux-Rauzy and Cassaigne-Selmer continued fraction algorithms.

(A nice result)

Theorem (A. 20; Dynnikov, Hubert & Skripchenko 20)

The vector of letter frequencies of an Arnoux-Rauzy word has rationally independent entries.

$\longrightarrow$ This result was conjectured by Arnoux and Starosta in 2013.

(A nice result)

Theorem (A. 20; Dynnikov, Hubert & Skripchenko 20)

The vector of letter frequencies of an Arnoux-Rauzy word has rationally independent entries.

$\longrightarrow$ This result was conjectured by Arnoux and Starosta in 2013.

Stronger:

Theorem (A. 20)

Let $d \geq d' \geq 1$. Let w an episturmian word over A_d. Denote by $f = (f_1, \ldots, f_d)$ its vector of letter frequencies, and by $(s_n)_{n\in\mathbb{N}}$ one of its directive sequences. The following assertions are equivalent.

1. *Exactly d' substitutions appear infinitely many times in $(s_n)_{n\in\mathbb{N}}$.*
2. *The dimension of the linear space $f_1\mathbb{Q} + \ldots + f_d\mathbb{Q}$ is d'.*

$\longrightarrow$ The (generalized) Arnoux-Rauzy multidimensional continued fraction algorithm detects all kinds of rational dependencies.

definitions & notations

Discrepancy

A natural question is to study the difference between the predicted frequencies of letters and their observed occurrences, that is called discrepancy:

$$\begin{aligned} \mathrm{discr}: \quad & \mathbb{N} \to \mathbb{R} \\ & n \mapsto \max_{i \in A} \left| |\mathrm{pref}_n(w)|_i - n f_w(i) \right|. \end{aligned}$$

$\longrightarrow$ Geometrically, the discrepancy is linked to the distance between the broken line and its average direction.

A combinatorial counterpart: the imbalance

Let w a finite or infinite word over A.

imbalance of w : $\mathrm{imb}(w) := \sup_{n\in\mathbb{N}} \sup_{u,v\in F_n(w)} ||\mathrm{ab}(u) - \mathrm{ab}(v)||_\infty \quad \in \mathbb{N}$ or ∞

$\longrightarrow$ Equivalently, it is the smallest D such that:

$$\text{For all } u, v \in \mathcal{F}_n(w), \text{ for all } a \in A, \ |u|_a - |v|_a \leq D.$$

ex : $\mathrm{imb}(1221) = 1$

ex : - Thue-Morse: $w_{TM} = 1221211221121221\ldots$ $\quad \mathrm{imb}(w_{TM}) = 2$
- Any Sturmian word w: $\quad \mathrm{imb}(w) = 1$

Fact: $\mathrm{discr}(w) \leq \mathrm{imb}(w) \leq 4.\mathrm{discr}(w)$ [Ada03]

The broken line of w remains at bounded distance from its average direction if and only if the imbalance of w finite.

II. An automaton to explore the set of all imbalances in a S-adic system

1. Construction of the tool

Teaser

Given S a finite set of substitutions, we want to answer the questions:

1. Are the imbalances of S-adic words bounded?
2. If they are, give an upper bound.
3. If they are not, for an arbitrary $d \in \mathbb{N}$, exhibit a S-adic word whose imbalance is greater than d.

Our tool: the "automaton of imbalances" , an infinite directed graph such that:

imbalances	$\approx$ final states
directive sequences	$\approx$ paths labels

Result

Theorem (A. 21)

Let S denote a finite set of nonerasing substitutions over a common alphabet A, and assume that all letters in A appear in a S-adic word (not necessarily the same). If D_S denotes the quantity (possibly infinite):

$$D_S = \sup_{w \ S\text{-adic}} \operatorname{imb}(w),$$

*then a **Breadth First Search** in the **automaton of imbalances**, from its initial states, yields, for any $d \leq D_S$, a finite sequence of substitutions $(\sigma_i)_{i \in \{1,\ldots,n\}}$ in S such that the imbalance of S-adic word whose directive sequence starts with $(\sigma_i)_{i \in \{1,\ldots,n\}}$, is larger than d.*

$\longrightarrow$ The question "does there exists a word in X_S with imbalance greater than d?" is **semi-decidable**.

$\longrightarrow$ If there is a S-adic word with imbalance larger than d, the semi-algorithm will find it. Otherwise, it will run forever.

How does it work? (glimpse)

Its construction relies a much wider graph: **"automaton of pairs of factors"**:

* **states:** $\mathrm{V} := \{(u, v) \in \mathcal{F}(w) |\ w \text{ S-adic word}\}$
* **final states:** $\mathrm{F} := \{(u, v) \in \mathrm{V} \text{ s.t. } |u| = |v|\ \}$

Reminder: $\sup_{w \in X_S} \mathrm{imb}(w) = \sup_{u,v \in F} ||\mathrm{ab}(u) - \mathrm{ab}(v)||_\infty$

* **transitions?**

Lemma 1 ["finiteness of History"]:
For any $(u, v) \in \mathrm{V}$, there exists $n \in \mathbb{N}$, $d_0, ..., d_{n-1} \in S^n$ and $a \in A$ s.t. $u, v \in \mathcal{F}(d_0 \circ ... \circ d_{n-1}(a))$.

$\longrightarrow$ Transitions should lead from $(u_0, v_0) = (a, a)$ to (u, v) in *these* n steps.

How does it work? (glimpse)

The "Substitute and cut operation": $\sim$ *converse of desubsitution*

There is a transition from (u, v) to $(\tilde{u}, \tilde{v})$ labelled by $(\delta_1, \delta_2, \delta_3, \delta_4, \sigma) \in \mathbb{N}^4 \times S$ if:

$$\begin{cases} \mathrm{pref}_{\delta_1}(\sigma(u)) \cdot \tilde{u} \cdot \mathrm{suf}_{\delta_2}(\sigma(u)) = \sigma(u) \\ \mathrm{pref}_{\delta_3}(\sigma(v)) \cdot \tilde{v} \cdot \mathrm{suf}_{\delta_4}(\sigma(v)) = \sigma(v) \end{cases}$$

***Keypoints:**
- same substitution
- cuts may be different

***Additional good idea:**
wlog, we impose $\delta_1 < |\sigma(u[0])|$, $\delta_2 < |\sigma(u[-1])|$, $\delta_3 < |\sigma(v[0])|$ and $\delta_4 < |\sigma(v[-1])|$.

$\longrightarrow$ thus, the number of outgoing edges from any vertex is finite!

How does it work? (glimpse)

Properties of the automaton of pairs of factors:

1. All vertices can be accessed from a vertex of the form (a, a), (a, ϵ),(ϵ, a) or (ϵ, ϵ) with $a \in A$ $\longleftarrow$a finite set of "initial states"

2. If all letters appear in a S-adic word (not necessarily the same), the image of a pair of words in V by a S&C operation remains in V.

$\longrightarrow$ we want to traverse this graph!

3. Infinite number of vertices.

4. BUT: any vertex has a finite number of outgoing edges.

$\longrightarrow$ We can broadly traverse this graph.

How does it work? (glimpse)

The **automaton of imbalances** is the quotient graph of the automaton of pairs of factors, obtained after **semi-abelianization**:

def: the *semi-abelianization* of the pair (u, v) is:

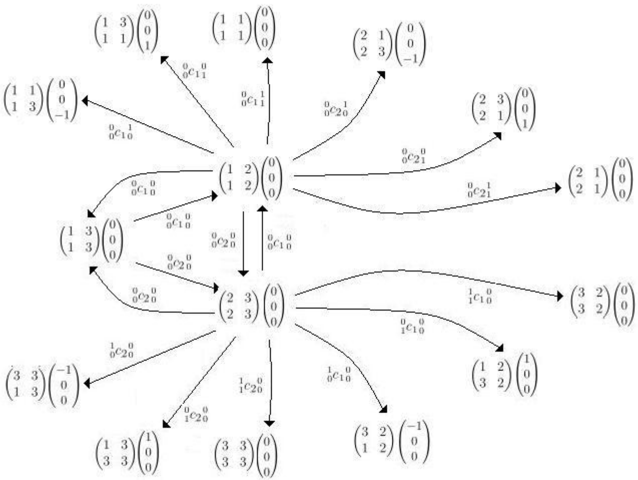
$$\begin{pmatrix} u[0] & u[-1] \\ v[0] & v[-1] \end{pmatrix}, (\mathrm{ab}(u) - \mathrm{ab}(v))$$

$\longrightarrow$ We keep the **minimal information** to perform the transitions & study imbalances.

The automaton of imbalances inherits from all the properties of accessibility of the automaton of pairs of factors.

$C = \{c_1, c_2\}$

$$c_1 : \begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 2 \end{array} \qquad c_2 : \begin{array}{ccc} 1 & \mapsto & 2 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 3 \end{array}$$



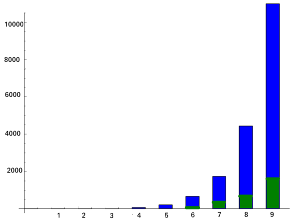
A portion of the automaton of imbalances for the Cassaigne-Selmer S-adic system.

II. An automaton to explore the set of all imbalances in a S-adic system

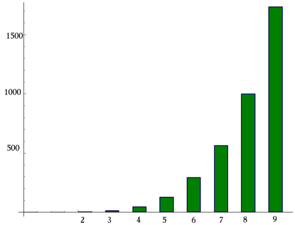
2. Application to Arnoux-Rauzy and Cassaigne-Selmer algorithms

Experimentation (sad) reality: example of Cassaigne-Selmer

Problem : the tree grows too fast!



Number of vertices in function of depth



Growth after cuttings

Solution : cut branches with no hope to reach new final states...

At depth 9, among 1 500 vertices, we found the first imbalance 3...
At depth 16, among 80 000 vertices, we found the first imbalance 4...

And yet, results!

1) I managed to find back the families of words constructed in [CFZ00]!
2) I spotted families of words with growing imbalances for C-adic words as well:

From any C-adic word w_0, construct:

$$\begin{cases} w_{n+1} = c_1^{2n+2} \circ c_2(w_n) & \text{if } n \text{ is odd} \\ w_{n+1} = c_2^{2n+2} \circ c_1(w_n) & \text{otherwise.} \end{cases}$$

Lemma 2: For all n, w_n is a C-adic word satisfying $\mathrm{imb}(w_n) \geq n$.

Theorem (A. 18)

There exists a C-adic word w_∞ with infinite imbalance.

$\longrightarrow$ w_∞ is contructed from $(w_n)_n$ by a pumping method which relies on:

Lemma 3: If w is a C-adic word s.t. $\mathrm{imb}(w) \geq 3n$, then $w' := c_1(w)$ (resp. $c_2(w)$) is a C-word satisfying $\mathrm{imb}(w') \geq n$.

And unexpected results!

Lemma 4: For any $(a, b, c) \in \mathbb{Z}^3$, there exists $s \in (S_{AR})^*$ and there exist $u, v \in \mathcal{F}(s(1))$ that satisfy $\mathrm{ab}(u) - \mathrm{ab}(v) = (a, b, c)$.

Rk: s can be explicitly described.

Theorem (A. 20)

There exists an Arnoux-Rauzy word $w_{\infty\infty}$ whose broken line is not trapped between two parallel planes - or, equivalently, whose Rauzy fractal is unbounded in all directions of the plane.

$\longrightarrow$ $w_{\infty\infty}$ is contructed from Lemma 4 by a pumping method which relies on the invertibility of incidences matrices of AR substitutions.

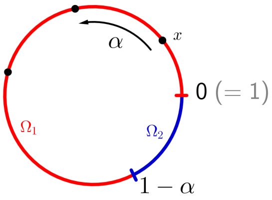
This result is surprising: its conflicts the intuition given by the Oseledets theorem on Lyapunov exponents.

III. Natural coding of minimal rotations of the torus, induction and exduction

Another remarkable property of Sturmian words

Consider:

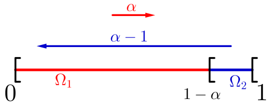
- ⋆ $\alpha \in \mathbb{R}\backslash\mathbb{Q}$
- ⋆ $R_\alpha : \begin{array}{ccc} \mathbb{R}/\mathbb{Z} & \to & \mathbb{R}/\mathbb{Z} \\ x & \mapsto & x+\alpha \end{array}$
- ⋆ the partition (Ω_1, Ω_2)



The partition (Ω_1, Ω_2) is remarkable:

1. The symbolic trajectory of any x under the iterations of R_α is a Sturmian word with frequency vector $(\frac{1}{1+\alpha}, \frac{\alpha}{1+\alpha})$.

2. Once lifted to $[0,1)$, Ω_1,Ω_2 are two intervals and R_α is the exchange of these two intervals.



⟶ Does this behaviour is preserved for multidimensional continued fractions words?

In higher dimension...

Roughly speaking

A word $w \in \{1, \ldots, d+1\}^{\mathbb{N}}$ is a natural coding of rotation with angle $\alpha \in \mathbb{R}^d$ on the d-dimensional torus if there exists a partition $\Omega_1, \ldots, \Omega_{d+1}$ of a fundamental domain such that:

- there exists a point whose symbolic trajectory is w
- the map induced by the rotation on the fondamental domain coincides with a **piecewise translation** (with pieces $\Omega_1, \ldots, \Omega_{d+1}$).

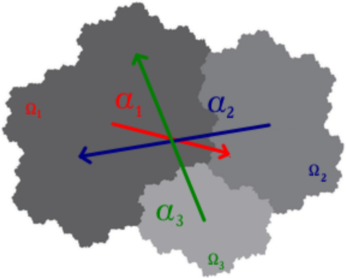
↖ this partition is **special!**

An encouraging example in dimension 2

The *Tribonacci word*:

$$w_{trib} := \lim_{n\to\infty} (\sigma_1 \circ \sigma_2 \circ \sigma_3)^n(1) = 1213121121312121\ldots$$

encodes an **exchange of** [fractal] **pieces**.



[Immediate consequence] It encodes the rotation by α_1 on the torus $\mathbb{R}^2/L$, with $L := (\alpha_2 - \alpha_1)\mathbb{Z} + (\alpha_3 - \alpha_1)\mathbb{Z}$.

Towards a generalization?

In the litterature:

- natural codings of rotations of the torus are often referred to, rarely defined
- big progress made for balanced words, with *ad hoc* assumptions.

In [CFZ00] appeared the idea that infinite imbalance might be incompatible with natural coding.

First (big) problem: give a **suitable definition**, which does not *a priori* prevent unbalanced words from being natural codings...

A topological definition

Let $d \geq 1$.
Let $L \subset \mathbb{R}^d$ a lattice and $\alpha \in \mathbb{R}^d$ such that $R_{\alpha,L}$ (the rotation with angle α on the torus $\mathbb{R}^d/L$) is minimal.

Definition (A. 21)

The word $w_0 \in \{1, \ldots, d\}^{\mathbb{N}}$ is a natural coding of $R_{\alpha,L}$ if:

- [partition of a pseudo-fundamental domain] There exist $\Omega_1,\ldots,\Omega_{d+1}$ **nonempty**, **open** sets of $\mathbb{R}^d$ such that:
  - the sets $\Omega_1,\ldots,\Omega_{d+1}$ are **pairwise disjoint**;
  - the **projection** $p_L : \Omega \to \mathbb{T}_L$, with $\Omega := \cup\Omega_i$, is **one-to-one**;
  - the **image set** $p_L(\Omega)$ is **dense** in the torus $\mathbb{R}^d/L$.
- [exchange of pieces] There exist $\alpha_1, \ldots, \alpha_{d+1} \in \mathbb{R}^d$ such that for all indices $i \in \{1, \ldots, d+1\}$ and for all point $\tilde{x} \in p_L(\Omega_i) \cap R_\alpha^{-1}(p_L(\Omega))$, $r_{\Omega,L}(R_\alpha(\tilde{x})) = r_{\Omega,L}(\tilde{x}) + \alpha_i$, with $r_{\Omega,L} : p_L(\Omega) \mapsto \Omega$ the lift map.
- [a coding trajectory] There exists $\tilde{x}_0$ in $p_L(\Omega)$ such that, for all $n \in \mathbb{N}$, $R_\alpha^n(\tilde{x}_0) \in p_L(\Omega_{w_0[n]})$, where $w_0[n]$ denotes the (n+1)-th letter of w_0.

Borders assignments

***Keypoint:** under the axiom of choice, we can **wisely assign borders**, i.e., complete each piece Ω_i so as to obtain the partition of a true fundamental domain $\Omega' = \Omega'_1 \sqcup ... \sqcup \Omega'_d$, while preserving:

- the exchange of pieces property
- the "continuity" of the coding function $f : \Omega' \to \{1, ..., d\}^{\mathbb{N}}$:

Lemma 5 [weak sequencial continuity]:
For all $x \in \Omega'$, there exists a sequence $(y_n)_n \in \Omega^{\mathbb{N}}$ such that $y_n \to x$ and $f(y_n) \to f(x)$.

Strengh of the definition is to fully know what happens on borders

Good and expected properties

Theorem (stability by induction,(A. 21))

If w is a natural coding of a minimal rotation of a d-dimensional torus and admits $d+1$ return words to a letter i, then the derivated word to the letter i, $D_i(w)$ is also a natural coding of a minimal rotation on a d-dimensional torus.

And "conversely":

Theorem (stability by *exduction*,(A. 21))

Let w a natural coding of a minimal rotation of a d-torus and i a letter. If $\sigma : \{1, ..., d+1\}^* \to \{1, ..., d+1\}^*$ is a substitution such that:
- all images of letters start with i and contain no other occurrences of i
- the incidence [integer] matrix of σ is invertible,
then $\sigma(w)$ is a natural coding of a minimal rotation of a d-torus.

In both cases:
- the lattice, the angle, the fundamental domain and its partition are explicitely given;
- borders assignment are inherited.

Consequences for AR and C-adic words with infinite imbalance

1. A theorem of Rauzy for bounded remainder sets gives:

Corollary (A. 21; Thuswalder 20)

No Arnoux-Rauzy / C-adic word with infinite imbalance is a natural coding of a minimal rotation of the 2-torus with a bounded pseudo-fundamental domain.

$\longrightarrow$ True question: does this still hold **without the assumption of boundedness??**

2. By studying the S-adic expression of their return words, we obtain:

Corollary (A. 21)

For Arnoux-Rauzy and C-adic words, the property of being a natural coding of a minimal rotation of the 2-torus does not depend on any prefix of the directive sequence.

$\longrightarrow$ neither does the infinite imbalance property...

Thank you!