Algebraic walk theory

Habilitation à Diriger des Recherches

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LMPA
Laboratoire de mathématiques pures et appliquées
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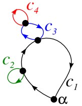
NAP-preLie^c bialgebra

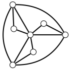
$(W_G, \Delta_{\mathrm{CP}}, \odot)$

$(W_G, \Delta_{\mathrm{CP}}, \odot)/\simeq$



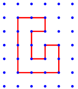
$$\frac{1}{1-c_1\frac{1}{1-c_2}\frac{1}{1-c_3\frac{1}{1-c_4}}}$$

Number theory $\hookrightarrow$ **Hike monoids** $\longleftrightarrow$ Realizability



Path-sum theorem $\dashrightarrow$ $\star$-algebra

Sieves on hikes



$$\sum_{d\in\mathcal{P}^{\text{s.a.}}} \mu(d)\,|\mathcal{M}_d|$$

Enumeration formulas

Walk sieves

Hopf $*$

Algorithms

Linear algebra

Matrix functions

Stat. inference

ODE(t)

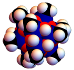
Special Functions

$\star$*-Lanczos*

Network analysis

Biology, social & computer science, econometry

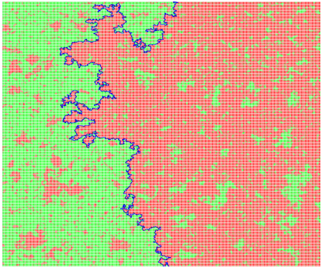
Quantum Dynamics

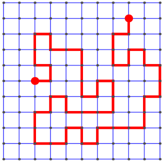


Theory

Applications

Loop erasing

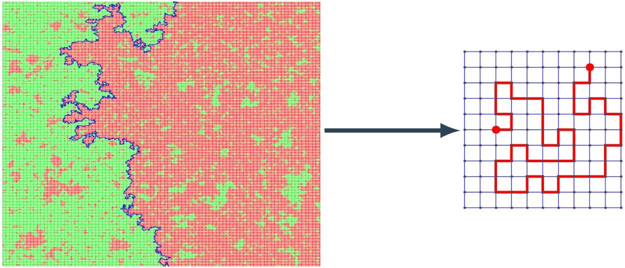




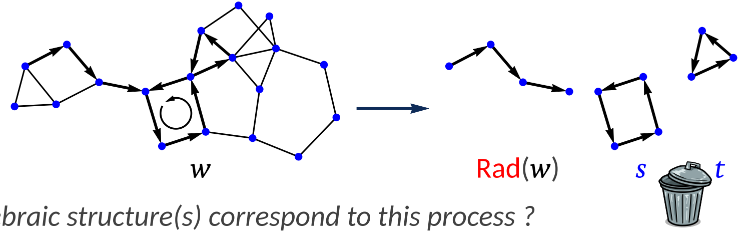
Counting ?

Random generation of Self-Avoiding Walks ?

Loop erasing



Random generation of Self-Avoiding Walks ? Gregory Lawler's loop erasing



▶ *What algebraic structure(s) correspond to this process ?*

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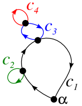


$$\frac{1}{1 - c_1 \frac{1}{1-c_2} \frac{1}{1-c_3 \frac{1}{1-c_4}}}$$

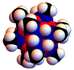
Path-sum theorem

$\star$-algebra

Linear algebra
Matrix functions
Stat. inference

ODE(t)
Special Functions
$\star$-Lanczos

Quantum Dynamics

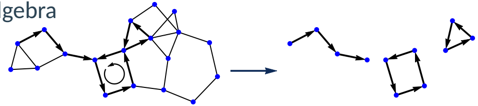


Theory

Applications

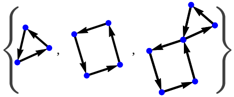
NAP-preLie^c bialgebra

Co-preLie coalgebra



Co-algebra ?

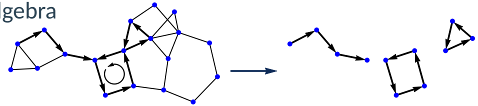
- Admissible cuts : [+ technical conditions on corollas...] $\mathrm{Adc}(w) =$



$$\Delta_{CP}(w) = \sum_{w^c \in \mathrm{AdC}(w)} w_c \otimes w^c$$

NAP-preLie^c bialgebra

Co-preLie coalgebra



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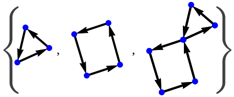
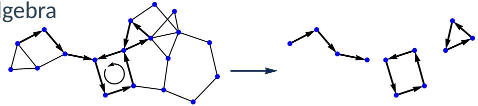
- Admissible cuts : [+ technical conditions on corollas...] $\text{Adc}(w) =$ 

$$\Delta_{CP}(w) = \sum_{w^c \in \text{AdC}(w)} w_c \otimes w^c$$

Admissible cuts are well defined. $\Delta_{CP}(w) = 0 \iff w$ self-avoids

NAP-preLie^c bialgebra

Co-preLie coalgebra



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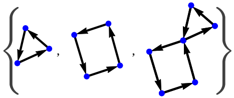
▶ Admissible cuts : [+ technical conditions on corollas...] $\mathrm{Adc}(w) =$ 

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Co-algebra

Lawler's loop erasure turns Δ_{CP} into a co-preLie coproduct on W_G

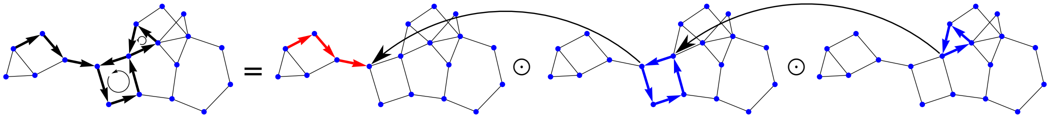
$$(\Delta_{CP} \otimes \mathrm{Id} - \mathrm{Id} \otimes \Delta_{CP}) \circ \Delta_{CP} = (\mathrm{Id} \otimes \tau) \circ (\Delta_{CP} \otimes \mathrm{Id} - \mathrm{Id} \otimes \Delta_{CP}) \circ \Delta_{CP}$$

(W_G, Δ_{CP}) is a co-preLie (not co-unital) co-algebra

NAP-preLie^c bialgebra

Non-associative permutative algebra

Ok we broke walks. Let's 'unbreak' them !

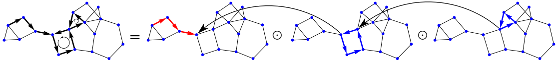


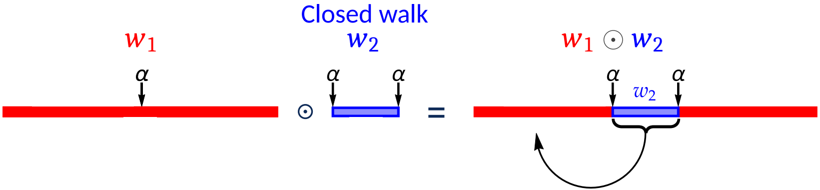
- Nesting re-inserts Lawler's cuts into walks

NAP-preLie^c bialgebra

Non-associative permutative algebra

Nesting re-inserts Lawler's cuts into walks





α appears once in w_1

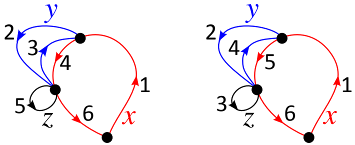
$\mathrm{Rad}_{\mathrm{Prior}}(w_1) \cap \mathcal{V}(w_2) = \emptyset$

- Otherwise $w_1 \odot w_2$ undefined

NAP-preLie^c bialgebra

Non-associative permutative algebra

Nesting re-inserts Lawler's loops into walks



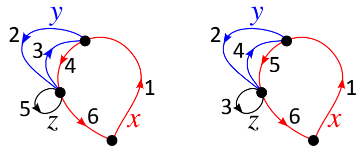
$$(x \odot y) \odot z \neq x \odot (y \odot z)$$

- Not always defined
- Nesting is non-associative & non-commutative

NAP-preLie^c bialgebra

Non-associative permutative algebra

Nesting re-inserts Lawler's loops into walks



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Lemma

Nesting $\odot$ is non-associative permutative $(a \odot b) \odot c = (a \odot c) \odot b$

- Existence and uniqueness of decomposition into simple cycles and one simple path

NAP-preLie^c bialgebra

Lemma

Nesting and Δ_{CP} are Livernet compatible: [Sweedler's notation]

$$\Delta_{CP}(a \odot b) = a \otimes b + a_{(1)} \odot b \otimes a_{(2)} + a \odot b_{(1)} \otimes b_{(2)}$$

NAP-preLie^c bialgebra

Lemma

Nesting and Δ_{CP} are Livernet compatible: [Sweedler's notation]

$$\Delta_{CP}(a \odot b) = a \otimes b + a_{(1)} \odot b \otimes a_{(2)} + a \odot b_{(1)} \otimes b_{(2)}$$

Bialgebraic structure on W_G

$(W_G, \Delta_{CP}, \odot)$ is a NAP-coprelie bialgebra

- Idempotent E: $\mathcal{W}(G) \to \Pi(G) \cup \Gamma(G)$ *prime counting ♯P-complete*
- Identity map: $\mathcal{W}(G) \to \mathcal{W}(G)$ *resummations*

Path-sum theorem

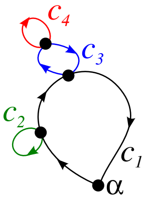
Turning walk series into continued fractions

Formal path-sums

The formal series:

- $\sum_{w:\,\alpha\to\alpha} w$ is a ***finite*** continued fraction over $\Gamma(G)$
- $\sum_{w:\,\alpha\to\omega} w$ is a ***finite*** $\Pi(G)$-sum of ***finite*** continued fractions over $\Gamma(G)$

▶ *Schematically*



$$\sum_{w:\alpha\to\alpha} w = \cfrac{1}{1 - c_1 \frac{1}{1-c_2} \frac{1}{1-c_3 \frac{1}{1-c_4}}}$$

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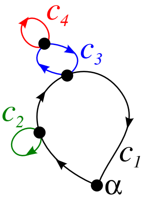


$$\sum_{w:\alpha\to\alpha} w = \cfrac{1}{1 - c_1 \cfrac{1}{1-c_2} \cfrac{1}{1-c_3 \cfrac{1}{1-c_4}}}$$

▶ Universal formula for walk series, edge-orders respected

Path-sum theorem

Turning walk series into continued fractions

Let $\mathcal{W}: \ W_G \to (\mathcal{A}, .)$ be a weight function, $\forall w \in W_G$, $\forall n \in \mathbb{N}: \ \mathcal{W}(w^n) = \mathcal{W}(w)^n$

Weighted path-sums

The weighted series:

- $\sum_{w:\,\alpha\to\alpha} \mathcal{W}(w)$ is a ***finite*** continued fraction over $\Gamma(G)$
- $\sum_{w:\,\alpha\to\omega} \mathcal{W}(w)$ is a ***finite*** $\Pi(G)$-sum of ***finite*** continued fractions over $\Gamma(G)$

Series may diverge, path-sums do not

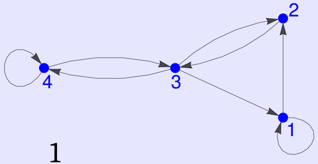
Path-sum theorem

Turning walk series into continued fractions

- Obvious example: $\mathsf{R}_{\mathsf{M}}(z) := (\mathsf{I} - z\,\mathsf{M})^{-1}$

$$\mathsf{R}_{\mathsf{M}}(z)_{ij} = \sum_{n} z^{n}(\mathsf{M}^{n})_{ij} = \sum_{w:i\to j} z^{\ell(w)}\mathcal{W}(w)$$

$$\mathsf{M} = \begin{pmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} & 0 & 0 \\ 0 & 0 & \mathsf{M}_{23} & 0 \\ \mathsf{M}_{31} & \mathsf{M}_{32} & 0 & \mathsf{M}_{34} \\ 0 & 0 & \mathsf{M}_{43} & \mathsf{M}_{44} \end{pmatrix}$$



$$\mathsf{R}_{\mathsf{M}}(z)_{44} = \cfrac{1}{1 - z\,\mathsf{M}_{44} - z^{2}\,\mathsf{M}_{43}\,\cfrac{1}{1 - z^{2}\,\mathsf{M}_{23}\mathsf{M}_{23} - z^{3}\,\mathsf{M}_{32}\mathsf{M}_{21}\,\cfrac{1}{1 - z\,\mathsf{M}_{11}}\,\mathsf{M}_{13}}\,\mathsf{M}_{34}}$$

Path-sum theorem

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- Obvious example: $\mathsf{R}_{\mathsf{M}}(z) := (\mathsf{I} - z\,\mathsf{M})^{-1}$

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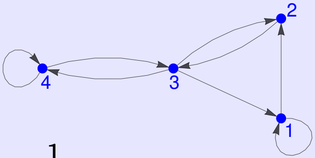

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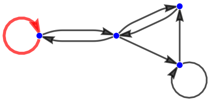

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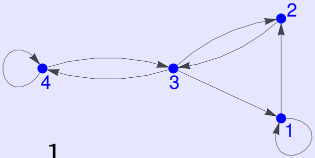

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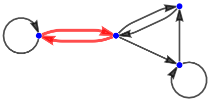

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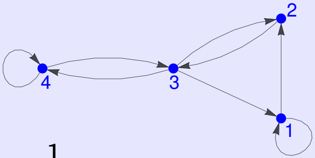

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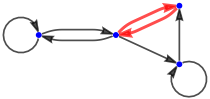

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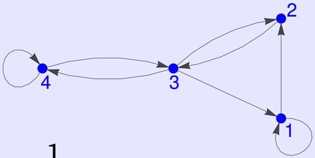

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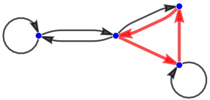

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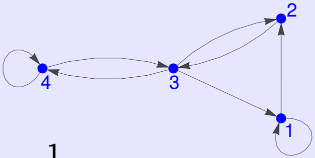

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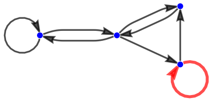

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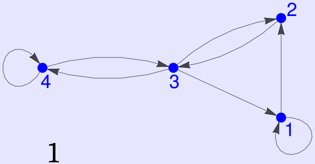

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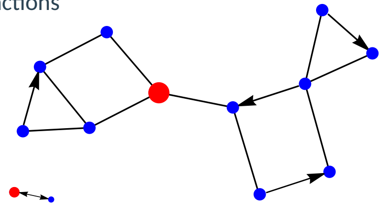
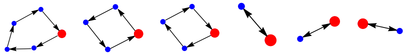
Corollary: matrix functions

Let f be analytic in A and M with $\mathrm{Sp}(\mathsf{M}) \subset A$. Then $f(M)$ has a path-sum formulation.

Path-sum theorem

Turning walk series into continued fractions



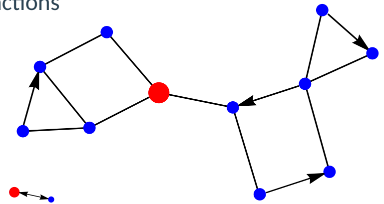


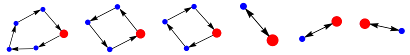
$$S(z) := \sum_{w:\bullet\to\bullet} z^{\Omega(w)} = 1 + 6z$$

- Not an algebraic function of adjacency matrix; but a walk-series...

Path-sum theorem

Turning walk series into continued fractions





$$S(z) := \sum_{w:\,{\color{red}\bullet}\to{\color{red}\bullet}} z^{\Omega(w)} = 1 + {\color{blue}6}z + 54z^2 + 499z^3 + 4628z^4 + 42957z^5 + 398845z^6 + \dots$$

- Not a function of adjacency matrix; but a walk-series... hence a path-sum!

$$S(z) = \frac{1}{1 - \frac{2z}{(1-z)\left(1-\frac{2z}{1-z}-z\right)} - \frac{z}{1-\frac{2z}{1-z}-z} - \frac{z}{1-\frac{z}{(1-3z)\left(1-\frac{z}{1-3z}\right)}-z} - \frac{z}{(1-z)\left(1-\frac{2z}{1-z}\right)\left(1-\frac{z}{1-\frac{2z}{1-z}}\right)} - \frac{z}{1-\frac{z}{1-\frac{2z}{1-z}}}}$$

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3 Differential calculus

‘Space-time’ path-sum

Major issue: $e \in E,\ \mathcal{W}(e)$ *not the same for all walks, e.g.* $\mathcal{W}(e) = f(t)$

'Space-time' path-sum

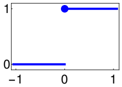
Major issue: $e \in E$, $\mathcal{W}(e)$ *not the same for all walks, e.g.* $\mathcal{W}(e) = f(t)$

- Solution: let them be different edges $G \mapsto \mathcal{G}$
  - Time-slice $\mathcal{G}|_t = G$
  - Space-slice $\mathcal{G}|_\alpha = f_\alpha(t)$

$$\sum_{w:\,(\alpha,t)\to(\omega,t')} \mathcal{W}(w) = \frac{d}{dt'}\mathcal{T}e^{\int_t^{t'} \mathsf{M}(\tau)d\tau} = \mathsf{G}$$

Green's function of $\dot{\mathsf{U}}(t) = \mathsf{M}(t)\mathsf{U}(t)$

- If $[\mathsf{M}(t'), \mathsf{M}(t)] \neq 0, \mathcal{G} \neq G \otimes \Theta(t) \longleftarrow$



"$\mathcal{G}$ does not decouple into space G and time $\Theta(t)$"

Non-autonomous ODEs

$\star$-product

Walk generation along the time-dimension

$$(\tilde{f} \star \tilde{g})(t', t) = \int_t^{t'} \tilde{f}(t', \tau)\, \tilde{g}(\tau, t) d\tau$$

- *Volterra composition* with problems [no unit!]

Non-autonomous ODEs

$\star$-product

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- *Volterra composition* with problems [no unit!]

Special instance of a more general product

Non-autonomous ODEs

$\star$-product

$$\mathcal{C}^\infty[I^2] \qquad I \subset \mathbb{R} \text{ compact}$$

$$\mathcal{D} := \boxed{\tilde{f}_{-1}(t',t)}\,\Theta(t'-t) + \sum_{i=0}^{\infty} \boxed{\tilde{f}_i(t',t)}\,\delta^{(i)}(t'-t)$$

$$\left(f \star_I g\right)(t',t) = \int_I f(t',\tau)g(\tau,t)d\tau$$

Fréchet-Lie algebra on distributions

The $\star_I$ and $\star_{\mathbb{R}}$ products well defined on $\mathcal{D}$

The set $\mathcal{F}$ of invertible elements of $\mathcal{D}$ is dense in $\mathcal{D}$

$\mathcal{F}$ is a Fréchet-Lie algebra

$\mathcal{F} \subset \text{Aut}\left(\overline{\mathcal{C}^\infty[I^2]}\right)$

Also for matrices !

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Non-autonomous ODEs

Path-sum solution

Path-sum solution of non-autonomous ODEs

System of n-coupled linear non-autonomous ODEs with structure G

$$\dot{\mathsf{U}}(t) = \mathsf{H}(t)\mathsf{U}(t), \quad \mathsf{U}(0) = \mathsf{Id}$$

Then any U_{ij} is a $\Pi(G)$-sum of finite $\star$-continued fractions over $\Gamma(G)$.

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Then any U_{ij} is a $\Pi(G)$-sum of finite $\star$-continued fractions over $\Gamma(G)$.

Proof:

Let $\mathsf{G} := \delta' \star \mathsf{U}\Theta$ this yields $\mathsf{G} = \dot{\mathsf{U}}\Theta + \delta\mathsf{Id}$. Then the original equation reads

$$\mathsf{G} - \mathsf{Id}_\star = \mathsf{H}\Theta \star \mathsf{G} \Rightarrow \mathsf{G} = (\mathsf{Id}_\star - \mathsf{H}\Theta)^{\star-1}$$

Invoke Path-Sum theorem with walk weights in $(\mathcal{C}^\infty[I^2]^{m\times m}, \star)$. $\square$

Non-autonomous ODEs

$\star$-Lanczos

Tridiagonalisation of non-autonomous ODEs

Let $\dot{\mathsf{U}}(t) = \mathsf{H}(t)\mathsf{U}(t)$ a system of n non-autonomous ODEs with $\mathsf{H} \in \mathcal{C}^\infty[I]^{n\times n}$
u, v vectors
Then there exists $\mathsf{T} \in \overline{\mathcal{C}^\infty[I]}^{n\times n}$ tridiagonal with

$$u^T \mathsf{U} v = \Theta \star (\mathsf{Id}_\star - \mathsf{T})^{\star -1}_{11}$$

$\hookrightarrow$ Algorithmic & constructive proof

Non-autonomous ODEs

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$\hookrightarrow$ Algorithmic & constructive proof

Path-sum corollary

There exists $\alpha_i := \tilde{\alpha}_i \Theta,\ \beta_j := \tilde{\beta}_j \Theta$ with $\tilde{\alpha}_i, \tilde{\beta}_j \in \mathcal{C}^\infty[I]$ with

$$u^T \mathsf{U} v = \Theta \star \Big(1_\star - \alpha_0 - \big(1_\star - \alpha_1 - (1_\star - (\dots(1_\star - \alpha_{n-1})^{\star-1}\dots)^{\star-1} \star \beta_2\big)^{\star-1} \star \beta_1\Big)^{\star-1}$$

Some applications

Non-autonomous ODEs

Example 1: Heun functions

The **Heun function** solves

$$y''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right) y'(z) + \frac{\alpha\beta\, z - q}{z(z-1)(z-a)} y(z) = 0$$

Space-time ring-down of black hole $(z = 1)$ as seen from Earth $(z \to +\infty)$

Non-autonomous ODEs

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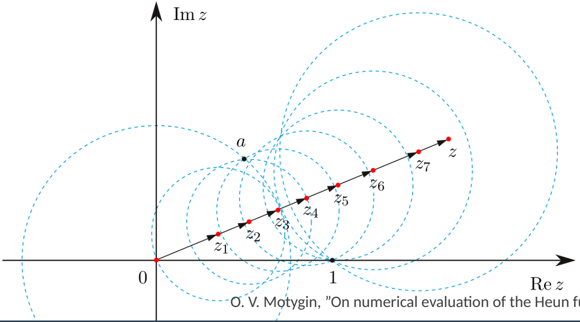
Space-time ring-down of black hole $(z = 1)$ as seen from Earth $(z \to +\infty)$



- Local series solutions known from 3-term recurrences

- Series near the event horizon $(z = 1)$ and Earth $(z \to +\infty)$ are of different nature

O. V. Motygin, "On numerical evaluation of the Heun functions" 2015 Days on Diffraction, pp. 1-6, doi: 10.1109/DD.2015.7354864.

Non-autonomous ODEs

Example 1: Heun functions

Representation of Heun functions with infinite convergence radius ?

Non-autonomous ODEs

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Representation of Heun functions with infinite convergence radius ?

- $y(z)$ solves degree 2 ODE, $y_0 = y(z_0)$, $y'_0 = y'(z_0)$

$$\text{There exists } M(z) \text{ such that } \dot{U}(z) = M(z)U(z), \quad y(z) = U(z)\psi_0$$

- Get $U(z)$ from path-sum theorem

Non-autonomous ODEs

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- Get $U(z)$ from path-sum theorem

$$y(z) = y_0 + y_0 \int_{z_0}^{z} G_1(\zeta, z_0)d\zeta + (y'_0 - y_0)\left(e^{z-z_0} - 1 + \int_{z_0}^{z}(e^{z-\zeta} - 1)G_2(\zeta, z_0)d\zeta\right),$$

where $G_i = (1_\star - \tilde{K}_i\Theta)^{\star-1}$ and

$$\tilde{K}_1(z) := 1 + e^{-z}\int_{z_0}^{z}\frac{\zeta_1^{\gamma}(\zeta_1 - 1)^{\delta}(a - \zeta_1)^{\epsilon}}{z^{\gamma}(z-1)^{\delta}(a-z)^{\epsilon}}e^{\zeta_1}\left(\frac{q - \alpha\beta\zeta_1}{(\zeta_1 - 1)\,\zeta_1\,(\zeta_1 - a)} - \frac{\epsilon}{a - \zeta_1} - \frac{\gamma}{\zeta_1} - \frac{\delta}{\zeta_1 - 1} - 1\right)d\zeta_1$$

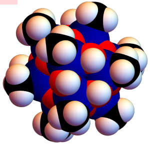
$$\tilde{K}_2(z) := \left(\frac{q - \alpha\beta z}{(z-1)z(z-a)} - \frac{\epsilon}{a - z} - \frac{\gamma}{z} - \frac{\delta}{z - 1} - 1\right)e^{z - z_0} - \frac{q - \alpha\beta z}{(z-1)z(z-a)}$$

Non-autonomous ODEs

Example 2: quantum dynamics

Schrödinger equation: $\dot{\mathsf{U}}(t) = -\frac{i}{\hbar}\mathsf{H}(t)\mathsf{U}(t)$

- Time-dependent
- For N particles, $\dim \mathsf{H} = 2^N$



42 protons $\Rightarrow \dim \mathsf{H} \simeq 4 \times 10^{12}$

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Non-autonomous ODEs

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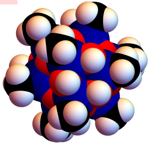

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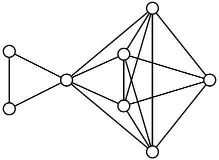

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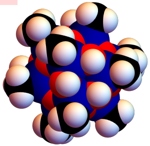

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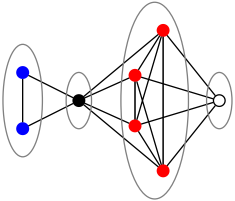

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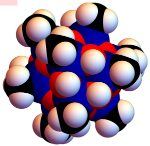

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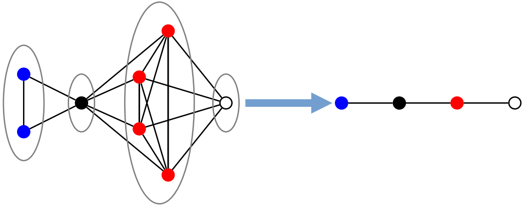

- Path-sums' scale-invariance [+ state-space reduction techniques]

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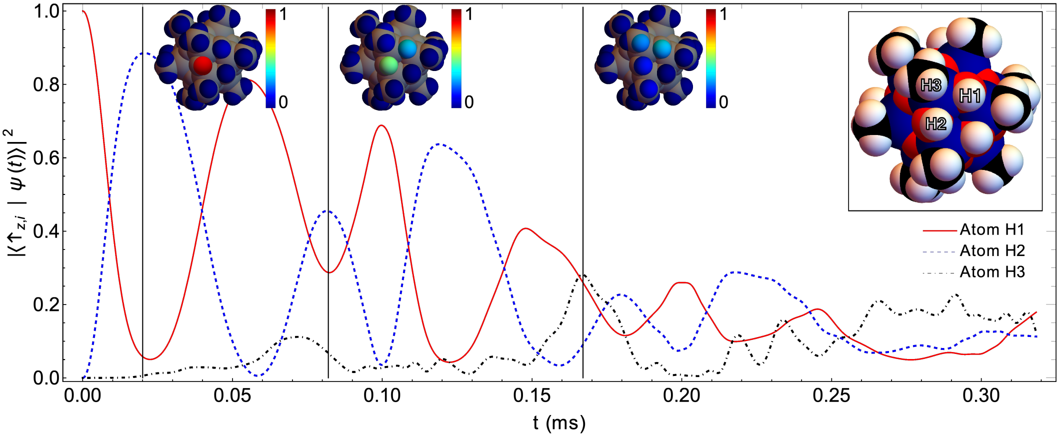
Non-autonomous ODEs

Example 2: quantum dynamics



It works ! This is **analytical !**

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4 Hike monoids

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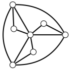
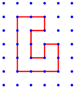
Number theory $\hookrightarrow$ **Hike monoids** $\longleftrightarrow$ Realizability



Sieves on hikes



$$\sum_{d \in \mathcal{P}^{\text{s.a.}}} \mu(d)|\mathcal{M}_d|$$

Enumeration formulas

Walk sieves

Hopf $*$

Algorithms

Network analysis

Biology, social & computer science, econometry

Theory

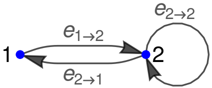
Applications

Hike monoids

Definition & origins

Combinatorial theory on NAP-preLie^c ? *impossible*

- Quotienting out the root



Hike monoids

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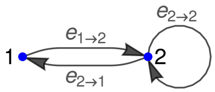

$$1 \to 2 \to 1 \equiv 2 \to 1 \to 2$$

$$e_{1\to 2}\, e_{2\to 1} = e_{2\to 1}\, e_{1\to 2}$$

Hike monoids

Definition & origins

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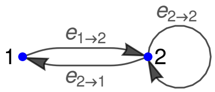
▶ Quotienting out the root



$$1 \to 2 \to 1 \equiv 2 \to 1 \to 2$$

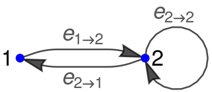
$$e_{1\to2}\, e_{2\to1} = e_{2\to1}\, e_{1\to2}$$

$$2 \to 2 \to 1 \to 2 \not\equiv 2 \to 1 \to 2 \to 2$$

$$e_{2\to2}\, e_{2\to1} \neq e_{2\to1}\, e_{2\to2}$$

Hike monoids

Definition & origins



Formally, $G = (E_G, V_G)$, consider the free monoid E_G^*

$$\mathcal{I} := \begin{cases} e_{i\to j}e_{i\to l} \neq e_{i\to l}e_{i\to j} \\ e_{i\to j}e_{k\to l} = e_{k\to l}e_{i\to j} \end{cases}$$

$$cl: \ \#e_{i\to\bullet} = \#e_{\bullet\to i}$$

Cartier-Foata monoid: $\mathcal{M}_{CF} := E_G^*|_{cl} \,/\, \mathcal{I}$

Hike monoids

Definition & origins

Formally, with $\mathcal{C}$ rootless simple cycles, the free monoid $\mathcal{C}^*$

$$\mathcal{I}' := \{cc' = c'c \iff \mathcal{V}(c) \cap \mathcal{V}(c') = \emptyset$$

Hike monoid: $\mathcal{H} := \mathcal{C}^* / \mathcal{I}'$

Hike monoids

Definition & origins

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Hike monoid: $\mathcal{H} := \mathcal{C}^* / \mathcal{I}'$

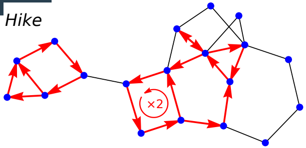
Isomorphism theorem

$$\mathcal{M}_{CF} \simeq \mathcal{H}$$

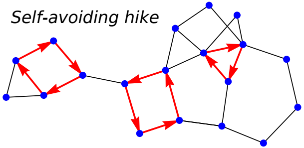
One can now cook up a combinatorial theory on $\mathcal{H}$

Hike monoids

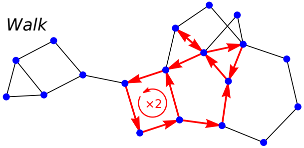
Hike



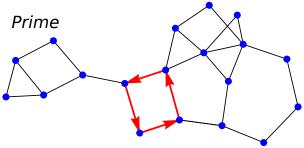
Self-avoiding hike



Walk



Prime



Hike monoids

Algebraic combinatorics on hike monoids

‘A combinatorial theory in three steps’ (A Recipe by Rota)

Hike monoids

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‘A combinatorial theory in three steps’ (A Recipe by Rota)

1. Left-divisibility poset on hikes $P_{\mathcal{H}} = (\mathcal{H}, \leq)$

$$h|h' : \;\; h' = h.s \Rightarrow h \leq h'$$

Hike monoids

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1. Left-divisibility poset on hikes $P_{\mathcal{H}} = (\mathcal{H}, \leq)$

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2. Incidence algebra $\mathcal{I}(P_{\mathcal{H}})$ of the poset, $\mathcal{H}$-indexed matrices

$$Q : \quad Q_{h,h} = 1, \quad \begin{cases} Q_{h,h'} = 0 & h \not\leq h' \\ Q_{h,h'} \in \mathbb{C} & \text{else} \end{cases}$$

Hike monoids

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3. Reduction $h'/h = k'/k \Rightarrow Q_{h,h'} = Q_{k,k'} = q$. Algebra of formal hike series $\mathcal{R}(P_{\mathcal{H}})$

$$Q \to \sum_{h \in \mathcal{H}} q(h)h$$

Hike monoids

Objects

If all hike commute

$\mathcal{R}(P_{\mathcal{H}}) \longrightarrow \mathfrak{D}$ Algebra of Dirichlet series

Number theory special case

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Hike monoids

Objects

If all hike commute

$\mathcal{R}(P_{\mathcal{H}}) \longrightarrow \mathfrak{D}$ Algebra of Dirichlet series

Number theory special case

Hikes	Number Theory
h hike	n integer
h, h' disjoint	n, m coprime
h self-avoiding	n square-free
p simple cycle	p prime
w walk	$n = p^k$

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Hike monoids

Formal series

	Hikes	Number theory
Zeta	$\zeta = \mathcal{S}1 = \sum_{h\in\mathcal{H}} h$ $\zeta = \frac{1}{\det(\mathsf{I}-\mathsf{W})}$	$\zeta_R(s) = \sum_{n>0} \frac{1}{n^s}$
Möbius	$\mu(h) = \begin{cases} (-1)^{\Omega(h)}, & h \text{ self-avoiding} \\ 0, & \text{otherwise.} \end{cases}$	$\mu(n) = \begin{cases} (-1)^{\Omega(n)}, & n \text{ square-free} \\ 0, & \text{otherwise.} \end{cases}$
von Mangoldt	$\Lambda(h) = \begin{cases} \ell(p), & h \text{ walk, } p\|h \\ 0, & \text{otherwise.} \end{cases}$ $\mathcal{S}\Lambda = \mathrm{Tr}\left[(\mathsf{I}-\mathsf{W})^{-1}\right]$	$\Lambda(n) = \begin{cases} \log(p), & n = p^k \\ 0, & \text{otherwise.} \end{cases}$
Liouville	$\lambda(h) = (-1)^{\Omega(h)}$ $\mathcal{S}\lambda = \frac{1}{\mathrm{perm}(\mathsf{I}-\mathsf{W})}$ $\vdots$	$\lambda(h) = (-1)^{\Omega(n)}$ $\vdots$

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Hike monoids

Combinatorial relations

	Hikes	Number Theory
Number of divisors	ζ^2	$\zeta_R(s)^2$
Log Zeta	$\log\zeta = \sum_h \frac{\Lambda(h)}{\ell(h)} h$	$\log\zeta_R(s) = \sum_n \frac{\Lambda(n)}{\log(n)}\frac{1}{n^s}$
Log-Mangoldt	$\ell(h) = \sum_{h'\vert h}\Lambda(h')$	$\log(n) = \sum_{d\vert n}\Lambda(n)$
Totally multiplicative functions	$f^{-1} = \sum_h \mu(h) f(h) h$ $f'/f = -\sum_h \Lambda(h) f(h) h$	$f^{-1} = \sum_n \mu(n) f(n)\frac{1}{n^s}$ $f'/f = -\sum_n \Lambda(n) f(n)\frac{1}{n^s}$
Totally additive functions	$(f*\mu)(h) = \begin{cases} f(p), & h \text{ walk}, p\vert h \\ 0, & \text{otherwise.}\end{cases}$	$(f*\mu)(n) = \begin{cases} f(p), & n = p^k \\ 0, & \text{otherwise.}\end{cases}$
ζ'/ζ from the primes	$-\sum_{\gamma:\text{ simple cycle}} \ell(\gamma)\frac{\det(\mathsf{I}-\mathsf{W}_{\setminus\gamma})}{\det(\mathsf{I}-\mathsf{W})}$	$-\sum_p \log p \frac{p^{-s}}{1-p^{-s}}$
Number Ω of prime factors	$\sum_{w:\text{ walk}} w = \det(\mathsf{I}-\mathsf{W})\sum_{h\in\mathcal{H}}\Omega(h)\, h$ $\vdots$	$\sum_{p,n}\frac{1}{p^{-ns}} = \zeta_R(s)^{-1}\sum_n \frac{\Omega(n)}{n^s}$ $\vdots$

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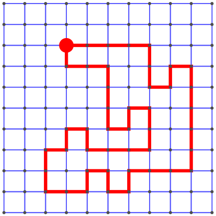
Hike monoids

Counting primes



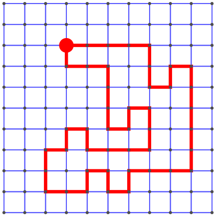
Number $\pi(\ell)$ of self-avoiding polygons of length ℓ ?

"A widely open problem" [Flajolet & Sedgewick]

$$\pi(\ell) \sim \mu^{\ell}\, \ell^{-5/2} \quad \text{as } \ell \to +\infty$$

Hike monoids

Counting primes



Number $\pi(\ell)$ of self-avoiding polygons of length ℓ ?

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Semi-commutative generalization of
Prime Number Theorem

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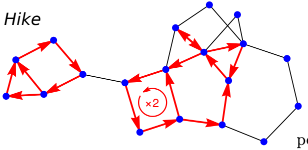
First strategy: exact counts

Enumeration: formulas & algorithms

Hike

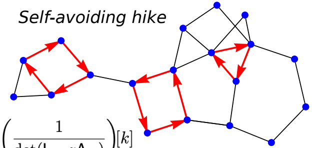


Hopf $(\mathcal{H}, *)$

Euler

$$\mathrm{perm}(\mathsf{I} + z\mathsf{A}) = \sum_{H \prec G} \sum_{k} \binom{N - |H|}{k - |H|} (-1)^{|H|+k} z^k \left(\frac{1}{\det(\mathsf{I} - z\mathsf{A}_H)}\right)[k]$$

Self-avoiding hike



Euler
$D \log \zeta$

Dynkin
Λ

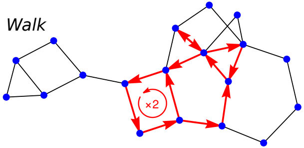
Hopf $(\mathcal{H}, .)$

Euler

$$\sum_{H \prec G} \det(-z\mathsf{A}_H) \frac{d}{dz} \mathrm{perm}(\mathsf{I} + z\mathsf{A}_{G-H})$$

Hopf $(\mathcal{H}^{s.a.}, *)$

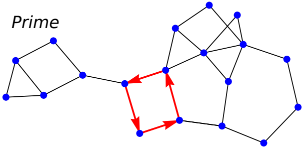
Walk



$$\sum_{\substack{H \prec G \\ H \text{ connecté}}} \mathrm{Tr}\left((z\mathsf{A}_H)^{|H|}(\mathsf{I} - z\mathsf{A}_H)^{|N(H)|}\right)$$

Structure ?

Prime



Enumeration: formulas & algorithms

Algorithmic considerations

$$\frac{d}{dz}\Gamma(z) = \sum_{\substack{H \prec G \\ H \text{ conn.}}} \mathrm{Tr}\Big((z\mathsf{A}_H)^{|H|}(\mathsf{I} - z\mathsf{A}_H)^{|N(H)|}\Big)$$

Counting simple cycles/paths is $\sharp$P-complete, cost up to length ℓ

$$O\Big(|V| + |E| + \big(\underbrace{\ell^{\omega}}_{\omega \text{ matrix multiply exponent}} + \ell\,\underbrace{\Delta}_{\text{max degree}}\big)\,\overbrace{|S_\ell|}^{\#\text{ of connected subgraphs on } \ell \text{ vertices}}\Big) \sim O\big(|V|\,\ell^{-1}\Delta^{\ell}\big)$$

- Best general purpose algorithm for certain graphs
  [*less connected induced subgraphs than cycles*]

Enumeration: formulas & algorithms

Algorithmic considerations

$$\frac{d}{dz}\Gamma(z) = \sum_{\substack{H \prec G \\ H \text{ conn.}}} \mathrm{Tr}\Big(\big(z\mathsf{A}_H\big)^{|H|}\big(\mathsf{I} - z\mathsf{A}_H\big)^{|N(H)|}\Big)$$

Counting simple cycles/paths is $\sharp$P-complete, cost up to length ℓ

$\#$ of connected subgraphs on ℓ vertices

$$O\Big(|V| + |E| + \big(\,\ell^{\omega} \;+ \ell\,\Delta\big)\;|S_\ell|\;\Big) \sim O\Big(|V|\,\ell^{-1}\Delta^{\ell}\Big)$$

ω matrix multiply exponent

max degree

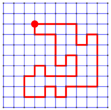
- ▶ Best general purpose algorithm for certain graphs
  [*less connected induced subgraphs than cycles*]
- ▶ Social networks "*Checking Heider's conjecture is NP-hard*"

$+++ = + \qquad ++- = -$

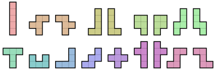
Enumeration: formulas & algorithms

Exact results

- *What about infinite graphs ?*



$$\sum_{\text{Self-avoiding polygons}} \ell(\gamma) z^{\ell(\gamma)} = \sum_{\text{Polyominos}} \mathrm{Tr}\left((z\mathsf{A}_p)^{\mathsf{Area}(p)} (\mathsf{I} - z\mathsf{A}_p)^{\mathsf{Perimeter}(p)} \right)$$



Asymptotics unknown on both sides [conjectures!] $\rightarrow$ Change strategy

Second strategy: asymptotic analysis

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Sieving on hikes

Finite sieve theorem

▶ **Step 1:** estimate $W_\gamma(\ell)$ number of walks of length ℓ multiples of a prime γ

Erathostenes-Legendre sieve ...

$$W_\gamma(\ell) = \sum_{d \in \mathcal{P}^{\text{s.a.}}_{G\backslash\gamma}} \mu(d) |\mathcal{M}_d(\ell)|$$

... skipping lots of details ...

$$W_\gamma(\ell) = \Bigg(\underbrace{\frac{1}{\lambda^{\ell(\gamma)}} \det\left(\mathsf{I} - \frac{1}{\lambda}\mathsf{A}_{G\backslash\gamma}\right)}_{\text{Good term}} + \underbrace{\sum_{k\geq 1} \frac{\nabla^k[f](\ell)}{\lambda^k\, k!} \det{}^{(k)}\left(\mathsf{I} - \frac{1}{\lambda}\mathsf{A}_{G\backslash\gamma}\right)}_{\text{Bad term}} \Bigg) |\mathcal{H}_\ell|$$

Number of hikes of length ℓ

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Sieving on hikes

Finite sieve theorem

Finite sieve theorem

Good term is dominant.

$$\lim_{\ell\to\infty}\frac{W_\gamma(\ell)}{|\mathcal{H}_\ell|}=\frac{1}{\lambda^{\ell(\gamma)}}\det\left(\mathsf{I}-\frac{1}{\lambda}\mathsf{A}_{G\backslash\gamma}\right):=\boldsymbol{c}(\gamma)$$

- **Fraction** of hikes which are walks with unique right prime factor γ, $0\le \boldsymbol{c}(\gamma)\le 1$

$$\frac{1}{\alpha}\sum_{\gamma\in\Gamma(G)}\boldsymbol{c}(\gamma)=1$$

Ratio of walks to hikes

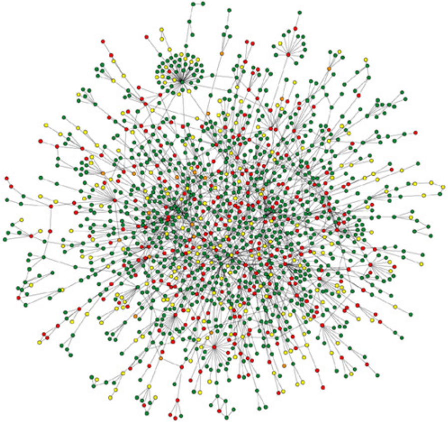
$\boldsymbol{c}(\gamma)$ measures the "*importance*" of γ for all walks... ...and of terms in path-sums!

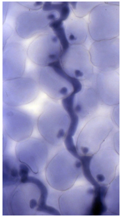
Sieving on hikes

Finite sieve theorem

- ... even in biology

Arabidopsis





Hyaloperonospora arabidopsidis

PPI cycles *maximizing* $c(\gamma)$: immune + **target** + special protein

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Sieve on hikes

Infinite sieve theorem

- **Step 2:** Go to infinite graphs...

Sieve on hikes

Infinite sieve theorem

▶ **Step 2:** Go to infinite graphs...

Good term   Bad term

$$W_\gamma(\ell) = \left(\mathrm{Frac}_\gamma + \sum_{k\geq 1}^{\infty} \frac{\nabla^k[f_w](\ell-\ell(p))}{f_w(\ell)\lambda^k\, k!}\, \mathrm{Frac}_\gamma^{(k)}(1/\lambda) \right) W(\ell)$$

Infinite sieve theorem

Good term is still dominant.

$$W_\gamma(\ell) \sim \mathrm{Frac}_\gamma\, W(\ell) + O\big((\ell - \ell(\gamma))^{-1}\big)$$

Furthermore: $0 < \mathrm{Frac}_\gamma < 1$ and $\sum_{\gamma\in\Gamma(G)} \mathrm{Frac}_\gamma = 1$

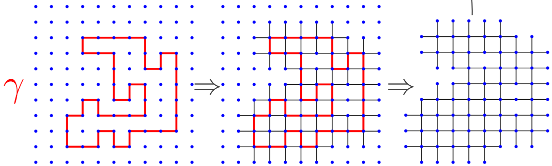
Sieve on hikes

Infinite sieve theorem

Good term: Fraction of walks with last erased loop γ

$$\mathsf{Frac}_\gamma = \frac{1}{\lambda^{\ell(\gamma)}}\,\mathbf{deg}_\gamma^T\,.\,\mathrm{adj}\left(\mathsf{I} + \frac{1}{\lambda}\,\mathsf{C}\,.\,\boxed{\mathsf{B}_\gamma}\right).\,\mathbf{1}$$

- B_γ adjacency matrix



- C resistor lattice matrix $(\mathsf{C})_{ij} = -\frac{1}{\pi}\int_0^\infty \frac{1}{\tau}\left(1 - \left(\frac{\tau-\mathrm{i}}{\tau+\mathrm{i}}\right)^{x_{ij}-y_{ij}}\left(\frac{\tau-1}{\tau+1}\right)^{x_{ij}+y_{ij}}\right)d\tau$

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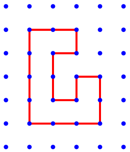
Sieve on hikes

Infinite sieve theorem



$$
\begin{aligned}
\mathrm{Frac}_\gamma = \frac{8388608}{8303765625\pi^{12}}\Big(&172151036713123194478159462 4 - 673302912063441661102915584 0\pi \\
&+ 1200172504512664753714652774 4\pi^2 - 1289567574563800792193984102 4\pi^3 \\
&+ 930398263935998467457522073 6\pi^4 - 474890311567953703602015436 8\pi^5 \\
&+ 175841856045601919604464025 6\pi^6 - 475910723284488375970037760\pi^7 \\
&+ 93430267561362281294131200\pi^8 - 12973459941155225172708000\pi^9 \\
&+ 1209211981439562793530000\pi^{10} - 67906363349663583525000\pi^{11} \\
&+ 1736896666805181140625\pi^{12}\Big)
\end{aligned}
$$

- Analytically on 144 738 980 SAPs Numerically on 220 167 804 196 SAPs

Hike monoids

Sieving on hikes

- **Step 3:** Deduce the asymptotics for $\pi(\ell)$

$$\pi(\ell) \sim_{\ell \gg 1} \left(1 - \sum_{\gamma:\ell(\gamma)<\ell-2} \mathsf{Frac}_\gamma\right) |W_\ell| \sim 4^\ell\, \ell^{-13/5} \neq (2.638\cdots)^\ell\, \ell^{-5/2}$$

Accumulation of errors when $\ell(\gamma)$ is near ℓ

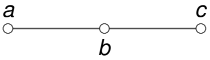
Sieving gap $\ell - \max \ell(\gamma)$ too small, can be fixed (probably)

Realizability: understanding hike monoids

Realizability: why we need a theory of walks

Which trace monoids are hike monoids ?

$$\mathcal{T} = \langle a, b, c | ac = ca \rangle$$



Is it a hike monoid ? *Can you draw a graph G with this cycle dependency ?*

Realizability: why we need a theory of walks

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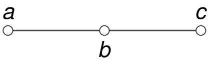

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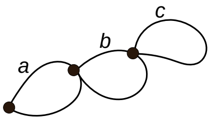

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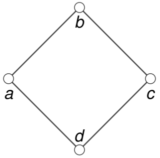
Realizability: why we need a theory of walks

Which trace monoids are hike monoids ?

$$\mathcal{T} = \langle a, b, c, d | ac = ca, bd = db \rangle$$



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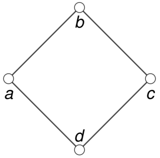
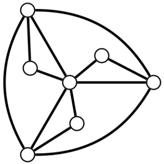


Is it a hike monoid ? *Can you draw a graph G with this cycle dependency ?*

NO !

Hike monoids 'can be found'

No discernable criterion in general.
But the problem is decidable



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Realizability: why we need a theory of walks

Which trace monoids are hike monoids ?

Invariant of hike monoids

Let G, G' two graphs with identical hike monoid $\mathcal{H}$. Are G and G' alike ?

Realizability: why we need a theory of walks

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Then G and G' may **not** share:

- Vertex-transitivity

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Realizability: why we need a theory of walks

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- Regularity

Realizability: why we need a theory of walks

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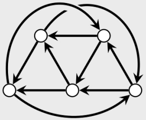
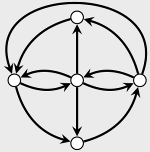
- Regularity
- Planarity





Realizability: why we need a theory of walks

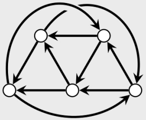
Which trace monoids are hike monoids ?

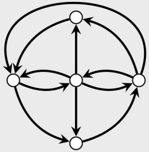
Invariant of hike monoids

Let G, G' two graphs with identical hike monoid $\mathcal{H}$. Are G and G' alike ?

Then G and G' may **not** share:

- Vertex-transitivity
- Regularity
- Planarity
- Bipartiteness, (bi)directedness, Hamiltonicity, being Eulerian, being chordal, being triangle-free, chromatic number, graph spectra, vertex numbers, edge numbers, in- and out-degree distributions, a majority of algebraic quantities computable from adjacency matrices...





Invariant of hike monoids do exist ! *path-sum topologies*, $\text{perm}(\mathsf{I} + \mathsf{A}_G)$ ...

$$a, b \in \text{Inv}(\mathcal{H}) \Rightarrow a * b \in \text{Inv}(\mathcal{H}) \text{ in Hopf } (\mathcal{H}, *)$$

Table of Contents

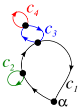
5 Summary

The two branches

NAP-preLie^c bialgebra

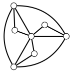
$(W_G, \Delta_{\text{CP}}, \odot)/\simeq$

$(W_G, \Delta_{\text{CP}}, \odot)$



$$\frac{1}{1-c_1\frac{1}{1-c_2}\frac{1}{1-c_3\frac{1}{1-c_4}}}$$

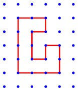
Number theory $\hookrightarrow$ Hike monoids $\longleftrightarrow$ Realizability



Path-sum theorem

$\star$-algebra

Sieves on hikes



$\sum_{d\in\mathcal{P}^{\text{s.a.}}} \mu(d)|\mathcal{M}_d|$

Enumeration formulas

Walk sieves

Hopf $*$

Algorithms

Linear algebra

Matrix functions

Stat. inference

ODE(t)

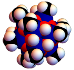
Special Functions

$\star$-Lanczos

Network analysis

Biology, social & computer science, econometry

Quantum Dynamics



Theory

Applications

The two branches

Further leaves

- Sieves on hikes

Two-sided sieves widen the sieving gap to $\ell - \max \ell(\gamma) = \ell/2$ eq. to $\sqrt{x}$ in number theory

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The two branches

Further leaves

- What equations have path-sum solutions ?
  $\hookrightarrow$ Resolvents $R_{\mathsf{M}} := (1 - \mathsf{M})^{-1}$ solve **linear equations** $R_{\mathsf{M}} - 1 = \mathsf{M} \,.\, R$
- Walk-based techniques only for linear equations? What is linear ?

The two branches

Further leaves

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  $\hookrightarrow$ Resolvents $R_{\mathsf{M}} := (1 - \mathsf{M})^{-1}$ solve **linear equations** $R_{\mathsf{M}} - 1 = \mathsf{M} \,.\, R$
- Walk-based techniques only for linear equations? What is linear ?

$$\dot{y}(t) = a(t)\, y(t)^4$$

Introduce the umbral operator $U_y(\omega, t) := e^{\omega\, y(t)}$

$$\boxed{\partial_t U_y = a(t)\omega\, \partial_\omega^4\, U_y}$$

- Systems of smooth non-autonomous NLODEs/FODEs (should) have a $\star$-path-sum solution in their umbral algebra
- (Likely) extends to PDEs

Thank you for listening!
Any questions?