The $\star$-product formalism

Theoretical progresses in N-ODE analysis

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Table of Contents

The $\star$-product

From Volterra to Mikusiński

- Non-autonomous linear ODE (N-ODE)

$$\dot{\mathsf{U}}(t) = \mathsf{A}(t)\mathsf{U}(t)$$

Quantum dynamics $\mathsf{A} \equiv -i\mathsf{H}$

The $\star$-product

From Volterra to Mikusiński

- Non-autonomous linear ODE (N-ODE)

$$\dot{\mathsf{U}}(t) = \mathsf{A}(t)\mathsf{U}(t)$$

Quantum dynamics $\mathsf{A} \equiv -i\mathsf{H}$

- Solution by Picard iteration (Dyson series)

$$\mathsf{U}(t,s) - \mathsf{I} = \int_s^t \mathsf{A}(\tau)\mathsf{U}(\tau)d\tau$$

$$\vdots$$

$$\mathsf{U}(t,s) = \mathsf{I} + \int_s^t \mathsf{A}(\tau)d\tau + \int_s^t \int_s^\tau \mathsf{A}(\tau)\mathsf{A}(\tau')d\tau'd\tau + \int_s^t \int_s^\tau \int_s^{\tau'} \mathsf{A}(\tau)\mathsf{A}(\tau')\mathsf{A}(\tau'')d\tau'd\tau'' + \cdots$$

- Iterates an operation...

The precursor: Volterra composition

Origins of the $\star$-product

Dyson series iterates the *Volterra composition* $\star_v$

$$(\tilde{a} \star_v \tilde{b})(t,s) = \int_s^t \tilde{a}(t,\tau)\tilde{b}(\tau,s)d\tau$$

- $\tilde{a}(t,s)$ and $\tilde{b}(t,s)$ functions, smooth in both variables

Problems ! no unit & no inverse

V. Volterra (1860-1940)

J. Pérès (1890-1962)

Volterra, Pérès, *Leçons sur la composition et les fonctions permutables* (1924)

Constructing the $\star$-product

From Volterra to Mikusiński

Take $I \subset \mathbb{R}$ compact & define

$$(\tilde{f} \star_I \tilde{g})(t,s) := \int_I \tilde{f}(t,\tau)\tilde{g}(\tau,s)d\tau \qquad t,s, \in I$$

- Well defined on locally smooth functions $\mathcal{C}^\infty(I^2)$ and their weak limits $\overline{\mathcal{C}^\infty(I^2)}$

Ryckebusch, arXiv:2307.09037 (2024)
Mikusiński, Fund. Math. 35:235-239 (1948)

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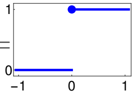
- Well defined on locally smooth functions $\mathcal{C}^\infty(I^2)$ and their weak limits $\overline{\mathcal{C}^\infty(I^2)}$
- Contains $\delta(t-s)$, $\Theta(t-s)$ and more...

$\Theta(x) =$ 

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The $\star$-product

From Volterra to Mikusiński

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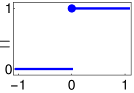
- Well defined on locally smooth functions $\mathcal{C}^\infty(I^2)$ and their weak limits $\overline{\mathcal{C}^\infty(I^2)}$

- Contains $\delta(t-s)$, $\Theta(t-s)$ and more... has a unit !

$\Theta(x) =$ 

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The $\star$-product

From Volterra to Mikusiński

- What is in $\overline{\mathcal{C}^\infty(I^2)}$?

The $\star$-product

From Volterra to Mikusiński

- What is in $\overline{\mathcal{C}^{\infty}(I^2)}$?

$$\mathcal{C}^{\infty}[I^2]$$

$$\mathcal{D} := \tilde{f}_{-1}(t,s)\,\Theta(t-s) + \sum_{i=0}^{\infty} \tilde{f}_i(t,s)\,\delta^{(i)}(t-s)$$

- And on $\mathcal{D}$...

The $\star$-product

From Volterra to Mikusiński

- What is in $\overline{\mathcal{C}^\infty(I^2)}$?

$$\mathcal{C}^\infty[I^2]$$

$$\mathcal{D} := \tilde{f}_{-1}(t,s)\,\Theta(t-s) + \sum_{i=0}^{\infty} \tilde{f}_i(t,s)\,\delta^{(i)}(t-s)$$

- And on $\mathcal{D}$... the $\star$-product is born !

$$(f \star g)(t,s) = \int_{-\infty}^{\infty} f(t,\tau)g(\tau,s)d\tau$$

$$\mathcal{D}$$

This operation is *really* nice

Properties of the $\star$-product

From Volterra to Mikusiński

$$(f \star g)(t,s) = \int_{\mathbb{R}} f(t,\tau) g(\tau,s) d\tau$$

- Induces $\star_v$

$$\tilde{f}(t,s)\Theta(t-s) \star \tilde{g}(t,s)\Theta(t-s)$$

Sm_Θ "Smooth-Θ"

Properties of the $\star$-product

$$\big(f \star g\big)(t,s) = \int_{\mathbb{R}} f(t,\tau)g(\tau,s)d\tau$$

- Induces $\star_v$

$$\tilde{f}(t,s)\Theta(t-s) \star \tilde{g}(t,s)\Theta(t-s) = \int_{\mathbb{R}} \tilde{f}(t,\tau)\Theta(t-\tau)\,\tilde{g}(\tau,s)\Theta(\tau-s)d\tau$$

Sm_Θ "Smooth-Θ"

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Properties of the $\star$-product

$$(f \star g)(t,s) = \int_{\mathbb{R}} f(t,\tau) g(\tau,s) d\tau$$

- Induces $\star_v$

$$\tilde{f}(t,s)\Theta(t-s) \star \tilde{g}(t,s)\Theta(t-s) = \int_{\mathbb{R}} \tilde{f}(t,\tau)\Theta(t-\tau)\,\tilde{g}(\tau,s)\Theta(\tau-s)d\tau$$
$$= \int_{s}^{t} \tilde{f}(t,\tau)\,\tilde{g}(\tau,s)d\tau \;\; \Theta(t-s)$$

Sm_Θ "Smooth-Θ"

Properties of the $\star$-product

$$(f \star g)(t,s) = \int_{\mathbb{R}} f(t,\tau) g(\tau,s) d\tau$$

- Induces $\star_v$

$$\begin{aligned}
\tilde{f}(t,s)\Theta(t-s) \star \tilde{g}(t,s)\Theta(t-s) &= \int_{\mathbb{R}} \tilde{f}(t,\tau)\Theta(t-\tau)\, \tilde{g}(\tau,s)\Theta(\tau-s) d\tau \\
&= \int_s^t \tilde{f}(t,\tau)\, \tilde{g}(\tau,s) d\tau \;\; \Theta(t-s) \\
&= \big(\tilde{f} \star_v \tilde{g}\big)(t,s) \;\; \Theta(t-s)
\end{aligned}$$

Sm_Θ "Smooth-Θ": Volterra

Properties of the $\star$-product

$$(f \star g)(t,s) = \int_{\mathbb{R}} f(t,\tau) g(\tau,s) d\tau$$

- Induces $\star_v$

Theorem (Ryckebusch 2024)

Well defined on $\mathcal{D}$ & has a unit

Composition of endomorphisms of $\overline{\mathcal{C}^\infty(I^2)}$ *Think matrix-product*

Set $\mathcal{F}$ of invertible elements of $\mathcal{D}$ is dense in $\mathcal{D}$

$\mathcal{F}$ is a Fréchet-Lie group

Also for matrices ! $\mathcal{D}^{n\times n}$

- Induces Schwartz bracket, convolution $*$, Laplace transform etc.
- $\star$-transposition: reverses time ordering

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Back to N-ODEs

The $\star$-product

Back to N-ODEs

- Take $\dot{\mathsf{U}} = \mathsf{A}(t)\mathsf{U}$ in Sm_Θ
  - Write all the missing Θ: $\dot{\mathsf{U}}\Theta = \mathsf{A}(t)\Theta \,.\, \mathsf{U}\Theta$

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The $\star$-product
Back to N-ODEs

- Take $\dot{\mathsf{U}} = \mathsf{A}(t)\mathsf{U}$ in Sm_Θ
  - Write all the missing Θ: $\dot{\mathsf{U}}\Theta = \mathsf{A}(t)\Theta \,.\, \mathsf{U}\Theta$
  - A depends only on t so: $\mathsf{A}(t)\Theta \,.\, \mathsf{U}\Theta = (\mathsf{A}(t)\Theta) \star \delta' \star (\mathsf{U}\Theta)$

The $\star$-product

Back to N-ODEs

- Take $\dot{\mathsf{U}} = \mathsf{A}(t)\mathsf{U}$ in Sm_Θ
  - Write all the missing Θ: $\dot{\mathsf{U}}\Theta = \mathsf{A}(t)\Theta\,.\,\mathsf{U}\Theta$
  - A depends only on t so: $\mathsf{A}(t)\Theta\,.\,\mathsf{U}\Theta = (\mathsf{A}(t)\Theta) \star \delta' \star (\mathsf{U}\Theta)$
  - Define the Green's function: $\mathsf{G} := \delta' \star (\mathsf{U}\Theta) = \dot{\mathsf{U}}\Theta + \mathsf{I}_\star$
- Now the equation reads

$$\mathsf{G} - \mathsf{I}_\star = \mathsf{A}\Theta \star \mathsf{G}$$

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The $\star$-product

Back to N-ODEs

- Take $\dot{\mathsf{U}} = \mathsf{A}(t)\mathsf{U}$ in Sm_Θ
  - Write all the missing Θ: $\dot{\mathsf{U}}\Theta = \mathsf{A}(t)\Theta \,.\, \mathsf{U}\Theta$
  - A depends only on t so: $\mathsf{A}(t)\Theta \,.\, \mathsf{U}\Theta = (\mathsf{A}(t)\Theta) \star \delta' \star (\mathsf{U}\Theta)$
  - Define the Green's function: $\mathsf{G} := \delta' \star (\mathsf{U}\Theta) = \dot{\mathsf{U}}\Theta + \mathsf{I}_\star$
- Now the equation reads

$$\mathsf{G} - \mathsf{I}_\star = \mathsf{A}\Theta \star \mathsf{G}$$

That is

$$\mathsf{G} = (\mathsf{I}_\star - \mathsf{A}\Theta)^{\star-1} \qquad \mathsf{U} = \Theta \star (\mathsf{I}_\star - \mathsf{A}\Theta)^{\star-1}$$

The $\star$-product

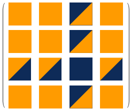
Back to N-ODEs

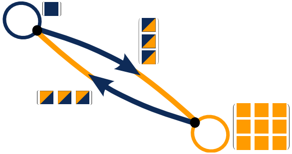
- Adjacency matrix A_G^ℓ counts $\#$ walks of length ℓ on G *What about other matrices ?*

The $\star$-product

Back to N-ODEs

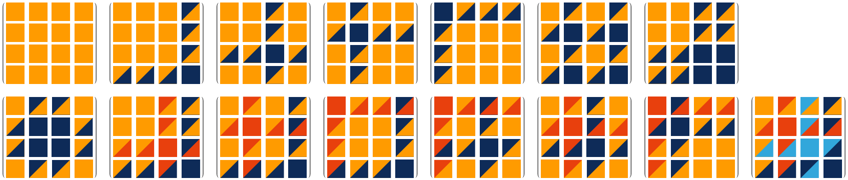
▶ Adjacency matrix A_G^ℓ counts # walks of length ℓ on G *What about other matrices ?*





Matrix partitions

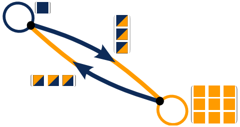
B_n distinct partitions into blocks & mappings onto graph



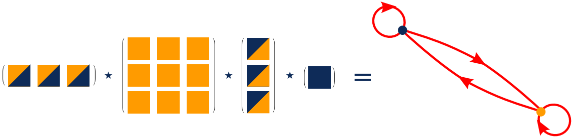
The $\star$-product

Back to N-ODEs

$M =$ 

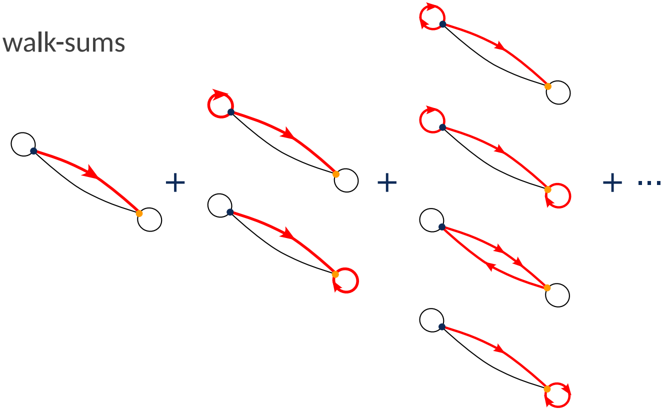
$G =$ 

$$M^{\star 4}_{\bullet\to\bullet} = M_{\bullet\to\bullet} \star M_{\bullet\to\bullet} \star M_{\bullet\to\bullet} \star M_{\bullet\to\bullet} + \dots$$



The ⋆-product
Back to N-ODEs

▶ Turning resolvent into walk-sums

$$\left(\sum_n M^{\star n}\right)_{\bullet\to\bullet} =$$ 

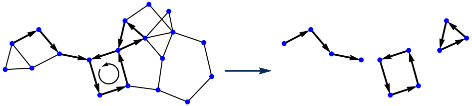
Evaluate via algebraic combinatorics of walks

Table of Contents

2 Bialgebra on walks

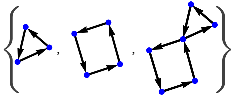
Bialgebra on walks

Co-product & product



Co-algebra ?

- Admissible cuts : [+ technical conditions on corollas...] $\mathrm{Adc}(w) =$

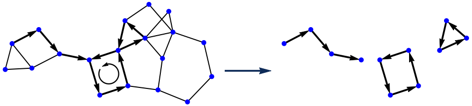


$$\Delta_{CP}(w) = \sum_{w^c \in \mathrm{AdC}(w)} w_c \otimes w^c$$

Foissy, Giscard, Mammez, Eur. J. Comb. 120: 103967 (2024)

Bialgebra on walks

Co-product & product



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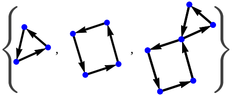
- Admissible cuts : [+ technical conditions on corollas...] $\mathrm{Adc}(w) =$ 

$$\Delta_{CP}(w) = \sum_{w^c \in \mathrm{AdC}(w)} w_c \otimes w^c$$

Theorem (Foissy, G., Mammez, 2023)

Lawler's loop erasure turns Δ_{CP} into a co-preLie coproduct on W_G

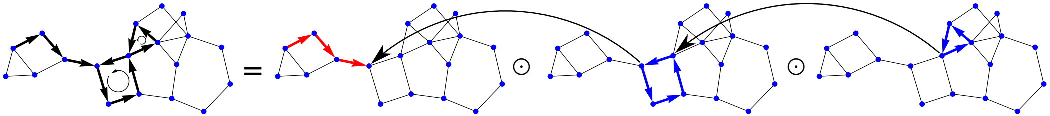
(W_G, Δ_{CP}) is a co-preLie (not co-unital) co-algebra. $\Delta_{CP}(w) = 0 \iff w$ self-avoids

Foissy, Giscard, Mammez, Eur. J. Comb. 120: 103967 (2024)

Bialgebra on walks

Co-product & product

Ok we broke walks. Let's 'unbreak' them !

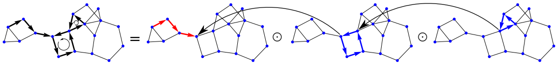


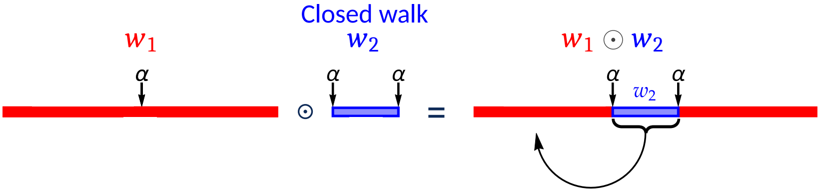
- Nesting re-inserts Lawler's cuts into walks

NAP-preLie^c bialgebra

Non-associative permutative algebra

Nesting re-inserts Lawler's cuts into walks





α appears once in w_1

$\mathrm{Rad}_{\text{Prior}}(w_1) \cap \mathcal{V}(w_2) = \emptyset$

- Otherwise $w_1 \odot w_2$ undefined

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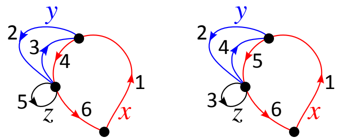
Bialgebra on walks

Co-product & product

Nesting re-inserts Lawler's loops into walks



$$(x \odot y) \odot z \neq x \odot (y \odot z)$$

- Not always defined, non-associative & non-commutative

Lemma

Nesting $\odot$ is Non-Associative Permutative $(a \odot b) \odot c = (a \odot c) \odot b$

- Existence and uniqueness of walk factorisation into primitive elements

Giscard, Thwaite, Jaksch, arXiv:1202.5523 (2012)

Bialgebra on walks

Co-product & product

Theorem **(Foissy, G., Ronco, 2024+)**

Nesting and Δ_{CP} are Livernet compatible
$(W_G, \Delta_{CP}, \odot)$ is a NAP-copreLie bialgebra

- Idempotent E: $\mathcal{W}(G) \to \Pi(G) \cup \Gamma(G)$
- Identity map: $\mathcal{W}(G) \to \mathcal{W}(G)$ *resummations*

Foissy, Giscard, Ronco, upcoming (2024+)
Foissy, Giscard, Mammez, Eur. J. Comb. 120: 103967 (2024)
Livernet, J. Pure Appl. Algebra 207: 1-18 (2006)

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Path-sum theorem

Bialgebra on walks

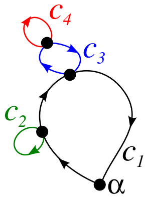
Path-sum theorem

Theorem (G., Thwaite, Jaksch 2012)

The formal series:

- $\sum_{w:\,\alpha\to\alpha} w$ is a ***finite*** continued fraction over simple cycles
- $\sum_{w:\,\alpha\to\omega} w$ is a ***finite*** sum over simple paths of ***finite*** continued fractions over simple cycles

▶ *Schematically*



$$\sum_{w:\,\alpha\to\alpha} w = \cfrac{1}{1 - c_1 \frac{1}{1-c_2} \cfrac{1}{1-c_3 \frac{1}{1-c_4}}}$$

▶ Universal formula for walk series, edge-orders respected

Giscard, Thwaite, Jaksch, arXiv:1202.5523 (2012)

Bialgebra on walks

Path-sum theorem

Product $\bullet$ with identity $\mathcal{I}_\bullet$

Operator $\mathcal{P}$ with discrete structure G

Bialgebra on walks

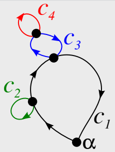
Path-sum theorem

Product $\bullet$ with identity $\mathcal{I}_\bullet$
Operator $\mathcal{P}$ with discrete structure G

Path-sum formulation of resolvents

The $\bullet$-resolvent

$$(\mathcal{I}_\bullet - \mathcal{P})^{\bullet -1}$$



is *formally* given by a path-sum continued fraction.

Each entry is a simple cycle or simple path on graph G representing the topology of $\mathcal{P}$ at some arbitrarily chosen scale.

- Quantum contexts, G = *quantum state space*

Table of Contents

3 Solutions to N-ODEs

Solutions to N-ODEs

Path-sum solution & Examples

Theorem (G. *et al.* 2015)

System of n-coupled linear non-autonomous ODEs with structure G

$$\dot{\mathsf{U}}(t) = \mathsf{H}(t)\mathsf{U}(t), \quad \mathsf{U}(0) = \mathsf{I}$$

Then any U_{ij} is a $\Pi(G)$-sum of finite $\star$-continued fractions over $\Gamma(G)$.

Giscard, Lui, Thwaite, Jaksch, J. Math. Phys. 56:053503 (2015)

Solutions to N-ODEs

Path-sum solution & Examples

Theorem (G. *et al.* 2015)

System of n-coupled linear non-autonomous ODEs with structure G

$$\dot{\mathsf{U}}(t) = \mathsf{H}(t)\mathsf{U}(t), \quad \mathsf{U}(0) = \mathsf{I}$$

Then any U_{ij} is a $\Pi(G)$-sum of finite $\star$-continued fractions over $\Gamma(G)$.

Proof:

We know that

$$\mathsf{G} = (\mathsf{I}_\star - \mathsf{H}\Theta)^{\star-1}$$

Invoke Path-Sum theorem with walk weights in $\left(\overline{\mathcal{C}^\infty(I^2)}^{m\times m}, \star\right)$. □

Giscard, Lui, Thwaite, Jaksch, J. Math. Phys. 56:053503 (2015)

Solutions to N-ODEs

Path-sum solution & Examples

- Time-ordered exp $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

$$\mathsf{A}(t) := \tilde{\mathsf{A}}(t)\Theta$$

$$\mathsf{A}(t) = \begin{pmatrix} \mathsf{A}_{11}(t) & \mathsf{A}_{12}(t) & 0 & 0 \\ 0 & 0 & \mathsf{A}_{23}(t) & 0 \\ \mathsf{A}_{31}(t) & \mathsf{A}_{32}(t) & 0 & \mathsf{A}_{34}(t) \\ 0 & 0 & \mathsf{A}_{43}(t) & \mathsf{A}_{44}(t) \end{pmatrix}$$

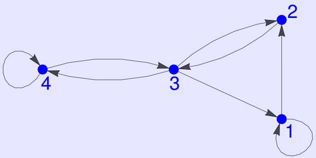
Solutions to N-ODEs

Path-sum solution & Examples

- Example: $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

$$\mathsf{A} := \tilde{\mathsf{A}}\Theta \Rightarrow \mathsf{U} = \Theta \star \mathsf{G} \;\text{ and }\; \mathsf{G} = (\mathsf{I} - \mathsf{A})^{\star -1}$$

$$\mathsf{A}(t) = \begin{pmatrix} \mathsf{A}_{11}(t) & \mathsf{A}_{12}(t) & 0 & 0 \\ 0 & 0 & \mathsf{A}_{23}(t) & 0 \\ \mathsf{A}_{31}(t) & \mathsf{A}_{32}(t) & 0 & \mathsf{A}_{34}(t) \\ 0 & 0 & \mathsf{A}_{43}(t) & \mathsf{A}_{44}(t) \end{pmatrix}$$



$$\mathsf{G}_{44} =$$

Solutions to N-ODEs

Path-sum solution & Examples

- Example: $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

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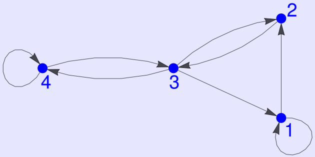


$$\mathsf{G}_{44} = \cfrac{1_\star}{1_\star - \mathsf{A}_{44} - \mathsf{A}_{43} \star \cfrac{1_\star}{1_\star - \mathsf{A}_{23}\star\mathsf{A}_{23} - \mathsf{A}_{32}\star\mathsf{A}_{21}\star\frac{1_\star}{1_\star - \mathsf{A}_{11}}\star\mathsf{A}_{13}} \star \mathsf{A}_{34}}$$

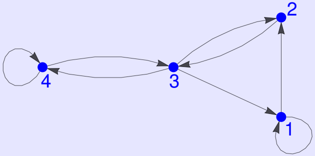
Solutions to N-ODEs

Path-sum solution & Examples

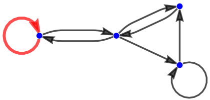
- Example: $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

$$\mathsf{A} := \tilde{\mathsf{A}}\Theta \Rightarrow \mathsf{U} = \Theta \star \mathsf{G} \text{ and } \mathsf{G} = (\mathsf{I} - \mathsf{A})^{\star-1}$$

$$\mathsf{A}(t) = \begin{pmatrix} \mathsf{A}_{11}(t) & \mathsf{A}_{12}(t) & 0 & 0 \\ 0 & 0 & \mathsf{A}_{23}(t) & 0 \\ \mathsf{A}_{31}(t) & \mathsf{A}_{32}(t) & 0 & \mathsf{A}_{34}(t) \\ 0 & 0 & \mathsf{A}_{43}(t) & \mathsf{A}_{44}(t) \end{pmatrix}$$



$$\mathsf{G}_{44} = \frac{1_\star}{1_\star - {\color{red}\mathsf{A}_{44}} - \mathsf{A}_{43} \star \frac{1_\star}{1_\star - \mathsf{A}_{23}\star\mathsf{A}_{23} - \mathsf{A}_{32}\star\mathsf{A}_{21}\star\frac{1_\star}{1_\star - \mathsf{A}_{11}}\star\mathsf{A}_{13}} \star \mathsf{A}_{34}}$$



Solutions to N-ODEs

Path-sum solution & Examples

- Example: $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

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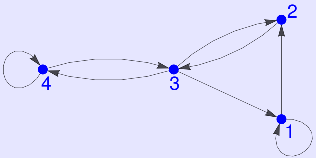
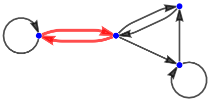


$$\mathsf{G}_{44} = \cfrac{1_\star}{1_\star - \mathsf{A}_{44} - \mathsf{A}_{43} \star \cfrac{1_\star}{1_\star - \mathsf{A}_{23}\star\mathsf{A}_{23} - \mathsf{A}_{32}\star\mathsf{A}_{21}\star\frac{1_\star}{1_\star-\mathsf{A}_{11}}\star\mathsf{A}_{13}} \star \mathsf{A}_{34}}$$



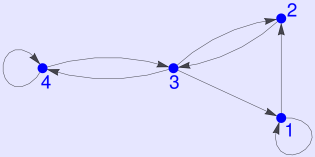
Solutions to N-ODEs

Path-sum solution & Examples

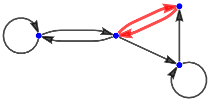
- Example: $U = \mathcal{T}\exp\left(\int \tilde{A}(\tau)d\tau\right)$

$$A := \tilde{A}\Theta \Rightarrow U = \Theta \star G \text{ and } G = (I - A)^{\star -1}$$

$$A(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) & 0 & 0 \\ 0 & 0 & A_{23}(t) & 0 \\ A_{31}(t) & A_{32}(t) & 0 & A_{34}(t) \\ 0 & 0 & A_{43}(t) & A_{44}(t) \end{pmatrix}$$



$$G_{44} = \frac{1_\star}{1_\star - A_{44} - A_{43} \star \frac{1_\star}{1_\star - A_{32}\star A_{23} - A_{32}\star A_{21}\star \frac{1_\star}{1_\star - A_{11}}\star A_{13}} \star A_{34}}$$



Solutions to N-ODEs

Path-sum solution & Examples

- Example: $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

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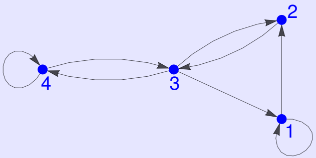
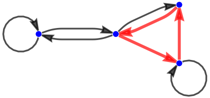


$$\mathsf{G}_{44} = \frac{1_\star}{1_\star - \mathsf{A}_{44} - \mathsf{A}_{43} \star \frac{1_\star}{1_\star - \mathsf{A}_{32}\star\mathsf{A}_{23} - \color{red}{\mathsf{A}_{32}\star\mathsf{A}_{21}}\star\frac{1_\star}{1_\star - \mathsf{A}_{11}}\star\color{red}{\mathsf{A}_{13}}} \star \mathsf{A}_{34}}$$



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Solutions to N-ODEs

Path-sum solution & Examples

- Example: $\mathsf{U} = \mathcal{T}\exp\left(\int \tilde{\mathsf{A}}(\tau)d\tau\right)$

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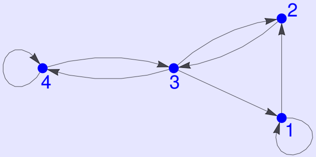
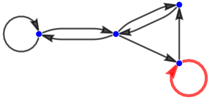


$$\mathsf{G}_{44} = \frac{1_\star}{1_\star - \mathsf{A}_{44} - \mathsf{A}_{43} \star \frac{1_\star}{1_\star - \mathsf{A}_{23}\star\mathsf{A}_{23} - \mathsf{A}_{32}\star\mathsf{A}_{21}\star\frac{1_\star}{1_\star - \color{red}{\mathsf{A}_{11}}}\star\mathsf{A}_{13}} \star \mathsf{A}_{34}}$$

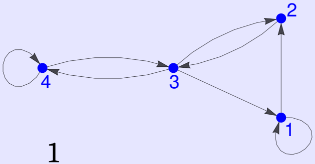


Solutions to N-ODEs

Path-sum solution & Examples

- Ordinary resolvent $R_M(z) := (I - z\,M)^{-1}$

$$M = \begin{pmatrix} M_{11} & M_{12} & 0 & 0 \\ 0 & 0 & M_{23} & 0 \\ M_{31} & M_{32} & 0 & M_{34} \\ 0 & 0 & M_{43} & M_{44} \end{pmatrix}$$



$$R_M(z)_{44} = \cfrac{1}{1 - z\,M_{44} - z^2\,M_{43}\cfrac{1}{1 - z^2\,M_{23}M_{23} - z^3\,M_{32}M_{21}\frac{1}{1 - z\,M_{11}}M_{13}}M_{34}}$$

Corollary: matrix functions

Let f be analytic over $\mathrm{Sp}(M)$: $f(M)$ has *the same* path-sum formulation.

Giscard, Thwaite, Jaksch, SIAM J. Matrix Anal. Appl. 34: 445-469 (2013)

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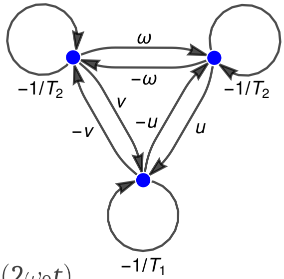
Solutions to N-ODEs

Path-sum solution & Examples

- Bloch equation, $\dot{\mathsf{U}}(t) = \tilde{\mathsf{A}}(t)\mathsf{U}(t)$ with

$$\tilde{\mathsf{A}}(t) = \begin{pmatrix} -\frac{1}{T_2} & -\omega & -v(t) \\ \omega & -\frac{1}{T_2} & -u(t) \\ v(t) & u(t) & -\frac{1}{T_1} \end{pmatrix}$$



with, as typical in NMR,

$$u(t) = \frac{1}{2}\big(- 2B_1\gamma - \gamma u_1 + \gamma v_2 - \gamma u_1 \cos(2\omega_0 t) + \gamma u_2 \sin(2\omega_0 t) - \gamma v_1 \sin(2\omega_0 t) - \gamma v_2 \cos(2\omega_0 t)\big),$$

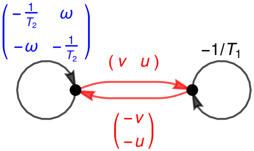
$$v(t) = \frac{1}{2}\big(\gamma u_2 + \gamma v_1 + \gamma u_1 \sin(2\omega_0 t) + \gamma u_2 \cos(2\omega_0 t) - \gamma v_1 \cos(2\omega_0 t) + \gamma v_2 \sin(2\omega_0 t)\big).$$

Solutions to N-ODEs

Path-sum solution & Examples

With a partition...

$$\tilde{\mathsf{A}} = \begin{pmatrix} -\frac{1}{T_2} & -\omega & -v(t) \\ \omega & -\frac{1}{T_2} & -u(t) \\ v(t) & u(t) & -\frac{1}{T_1} \end{pmatrix}$$



... yielding

$$\mathsf{G}_{33} = \left(1_\star + \frac{1}{T_1}\Theta - \begin{pmatrix} v & u \end{pmatrix}\Theta \star \left(\mathsf{I}_\star - \begin{pmatrix} -\frac{1}{T_2} & -\omega \\ \omega & -\frac{1}{T_2} \end{pmatrix}\Theta\right)^{\star-1} \star \begin{pmatrix} -v \\ -u \end{pmatrix}\Theta\right)^{\star-1}$$

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It comes $\mathsf{G}_{33} = (1_\star - \tilde{K}_{33}\Theta)^{\star -1}$

$$
\begin{aligned}
\tilde{K}_{33}(t,s) = &-\frac{1}{T_1} + c_0 + c_1\cos(2\omega_0 t) + c_2\sin(2\omega_0 t)\\
&- e^{-\frac{1}{T_2}(t-s)}\Big(c_3\sin\left(\omega\left(t-s\right)\right) + c_4\cos\left(\omega\left(t-s\right)\right)\\
&+ c_5\sin\left(\left(\omega-2\omega_0\right)\left(t-s\right)\right) + c_6\cos\left(\left(\omega-2\omega_0\right)(t-s)\right)\\
&+ c_7\sin\left(-2\omega_0 t+\omega(t-s)\right) + c_8\sin\left(2\omega_0 s+\omega(t-s)\right)\\
&+ c_9\cos\left(2\omega_0 t-\omega(t-s)\right) + c_{10}\cos\left(2\omega_0 s+\omega(t-s)\right)\Big)
\end{aligned}
$$

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$$
\begin{aligned}
\tilde{K}_{33}(t,s) = & -\frac{1}{T_1} + c_0 + c_1\cos(2\omega_0 t) + c_2\sin(2\omega_0 t)\\
& - e^{-\frac{1}{T_2}(t-s)}\Big(c_3\sin\left(\omega\left(t-s\right)\right) + c_4\cos\left(\omega\left(t-s\right)\right)\\
& + c_5\sin\left(\left(\omega-2\omega_0\right)\left(t-s\right)\right) + c_6\cos\left(\left(\omega-2\omega_0\right)(t-s)\right)\\
& + c_7\sin\left(-2\omega_0 t + \omega(t-s)\right) + c_8\sin\left(2\omega_0 s + \omega(t-s)\right)\\
& + c_9\cos\left(2\omega_0 t - \omega(t-s)\right) + c_{10}\cos\left(2\omega_0 s + \omega(t-s)\right)\Big)
\end{aligned}
$$

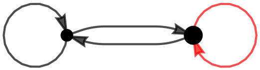
Virtual transitions !

It comes $G_{33} = (1_\star - \tilde{K}_{33}\Theta)^{\star -1}$

$$\tilde{K}_{33}(t,s) = -\frac{1}{T_1} + c_0 + c_1\cos(2\omega_0 t) + c_2\sin(2\omega_0 t)$$
$$-e^{-\frac{1}{T_2}(t-s)}\Big(c_3\sin\left(\omega\left(t-s\right)\right) + c_4\cos\left(\omega\left(t-s\right)\right)$$
$$+c_5\sin\left(\left(\omega - 2\omega_0\right)\left(t-s\right)\right) + c_6\cos\left(\left(\omega - 2\omega_0\right)(t-s)\right)$$
$$+c_7\sin\left(-2\omega_0 t + \omega(t-s)\right) + c_8\sin\left(2\omega_0 s + \omega(t-s)\right)$$
$$+c_9\cos\left(2\omega_0 t - \omega(t-s)\right) + c_{10}\cos\left(2\omega_0 s + \omega(t-s)\right)\Big)$$

'Actual' transitions !

It comes $G_{33} = (1_\star - \tilde{K}_{33}\Theta)^{\star-1}$



$$\tilde{K}_{33}(t,s) = -\frac{1}{T_1} + c_0 + c_1\cos(2\omega_0 t) + c_2\sin(2\omega_0 t)$$
$$- e^{-\frac{1}{T_2}(t-s)}\Big(c_3\sin\left(\omega\left(t-s\right)\right) + c_4\cos\left(\omega\left(t-s\right)\right)$$
$$+ c_5\sin\left(\left(\omega-2\omega_0\right)\left(t-s\right)\right) + c_6\cos\left(\left(\omega-2\omega_0\right)(t-s)\right)$$
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$$+ c_9\cos\left(2\omega_0 t - \omega(t-s)\right) + c_{10}\cos\left(2\omega_0 s + \omega(t-s)\right)\Big)$$

Generates *decay*

It comes $\mathsf{G}_{33} = (1_\star - \tilde{K}_{33}\Theta)^{\star-1}$

$$
\begin{aligned}
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& + c_5\sin\left(\left(\omega-2\omega_0\right)\left(t-s\right)\right) + c_6\cos\left((\omega-2\omega_0)(t-s)\right) \\
& + c_7\sin\left(-2\omega_0 t + \omega(t-s)\right) + c_8\sin\left(2\omega_0 s + \omega(t-s)\right) \\
& + c_9\cos\left(2\omega_0 t - \omega(t-s)\right) + c_{10}\cos\left(2\omega_0 s + \omega(t-s)\right)\Big)
\end{aligned}
$$

U_{33} available but... *not in closed form*

Giscard, Bonhomme, Faizy, upcoming (2024+)

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Further $\star$-fruits

Solution to N-ODEs

Further $\star$-fruits

Theorem (G., Pozza 2020-2022)

Consider $\dot{\mathsf{U}}(t) = \mathsf{A}(t)\mathsf{U}(t)$ with $\mathsf{A} \in \mathcal{C}^\infty[I]^{n\times n}$ and $\vec{u}$, $\vec{v}$ vectors

There exists $\mathsf{T} \in \mathcal{F}^{n\times n}$ tridiagonal with

$$\vec{u}^T.\mathsf{U}.\vec{v} = \Theta \star (\mathsf{I}_\star - \mathsf{T})^{\star-1}_{11}$$

$\hookrightarrow$ Constructive proof based non-Hermitian $\star$-Lanczos

Giscard, Pozza, Linear Algebra Appl. 624: 153-173 (2021)
Giscard, Pozza, Appl. Math. 1-21 (2020)
Giscard, Pozza, Boll. Unione Mat. Ital., 2198-2759 (2022)

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$\hookrightarrow$ Constructive proof based non-Hermitian $\star$-Lanczos

Path-sum corollary

There exists $\alpha_i,\ \beta_j \in \mathsf{Sm}_\Theta$ with

$$\vec{u}^T.\mathsf{U}.\vec{v} = \Theta \star \Big(1_\star - \alpha_0 - \big(1_\star - \alpha_1 - (1_\star - (\dots(1_\star - \alpha_{n-1})^{\star-1}\dots)^{\star-1} \star \beta_2\big)^{\star-1} \star \beta_1\Big)^{\star-1}$$

Giscard, Pozza, Linear Algebra Appl. 624: 153-173 (2021)
Giscard, Pozza, Appl. Math. 1-21 (2020)
Giscard, Pozza, Boll. Unione Mat. Ital., 2198-2759 (2022)

Solution to N-ODEs

Further $\star$-fruits

Theorem (Volterra, Pérés 1924)

Take $f, g \in \mathsf{Sm}_\Theta$, then

$$f{\star}g = g{\star}f \iff \exists m \in \mathcal{F} : \begin{cases} f = m(t,s) \star \lambda(t-s) \star m^{\star-1}(t,s) \\ g = m(t,s) \star \mu(t-s) \star m^{\star-1}(t,s) \end{cases}$$

- *Floquet's theorem !*

$\tilde{\mathsf{A}}$ periodic: A $\star$-commutes with the T-translation operator

$$\mathsf{U}(t,s) = \mathsf{M}(t) e^{\mathsf{F}\times(t-s)} \mathsf{M}(s)^{-1}$$

Pinch of salt: proof not yet complete

V. Volterra (1860-1940)

J. Pérès (1890-1962)

Volterra, Pérès, *Leçons sur la composition et les fonctions permutables* (1924)

Solution to N-ODEs

Further $\star$-fruits

- Let $M = A + B$

$$\frac{1}{I - M} = \frac{1}{I - B}\frac{1}{I - A\frac{1}{I-B}}$$

- Taking $H(t) = H_0(t) + H_1(t)$ gives

$$\begin{aligned} G &= G_1 \star (I - H_0 \star G_1)^{\star -1} \\ &= G_1 + G_1 \star H_0 \star G_1 + G_1 \star H_0 \star G_1 \star H_0 \star G_1 + \cdots \end{aligned}$$

This is...

Solution to N-ODEs

Further $\star$-fruits

- Let $M = A + B$

$$\frac{1}{I - M} = \frac{1}{I - B}\frac{1}{I - A\frac{1}{I-B}}$$

- Taking $H(t) = H_0(t) + H_1(t)$ gives

$$\begin{aligned} G &= G_1 \star (I - H_0 \star G_1)^{\star -1} \\ &= G_1 + G_1 \star H_0 \star G_1 + G_1 \star H_0 \star G_1 \star H_0 \star G_1 + \cdots \end{aligned}$$

This is...just a frame change ! $U = U_1 . \mathcal{T} e^{\int U_1{}^\dagger(\tau) H_0(\tau) U_1(\tau) d\tau}$

Pay m integrals $\longrightarrow$ Get order m Picard

Solution to N-ODEs

Further $\star$-fruits

Let's improve this !

$$\frac{1}{\mathsf{I}-\mathsf{M}} = \frac{1}{\mathsf{I}-\mathsf{A}}\,\frac{1}{\mathsf{I}-\frac{\mathsf{B}}{\mathsf{I}-\mathsf{B}}\frac{\mathsf{A}}{\mathsf{I}-\mathsf{A}}}\,\frac{1}{\mathsf{I}-\mathsf{B}}$$

- Since $\mathsf{H}_i \star \mathsf{G}_i = \dot{\mathsf{U}}_i$

$$\mathsf{G} = \mathsf{G}_1 \star \left(\mathsf{I} - \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1\right)^{\star-1} \star \mathsf{G}_0$$

- "Bi-frame" change (nasty formula in exp. form), series form

$$\dot{\mathsf{U}} = \dot{\mathsf{U}}_0 + \dot{\mathsf{U}}_1 + \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1 + \dot{\mathsf{U}}_1 \star \dot{\mathsf{U}}_0 + \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1 \star \dot{\mathsf{U}}_0 + \dot{\mathsf{U}}_1 \star \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1 + \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1 \star \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1 + \dot{\mathsf{U}}_1 \star \dot{\mathsf{U}}_0 \star \dot{\mathsf{U}}_1 \star \dot{\mathsf{U}}_0 + \cdots$$

Solution to N-ODEs

Further $\star$-fruits

- "Bi-frame"

$$G = G_1 \star \left(I - \dot{U}_0 \star \dot{U}_1\right)^{\star -1} \star G_0$$

Pay $m + 2$ integrals $\longrightarrow$ Get order $2m + 1$ Picard

Solution to N-ODEs

Further $\star$-fruits

- “Bi-frame”

$$G = G_1 \star \left(I - \dot{U}_0 \star \dot{U}_1\right)^{\star -1} \star G_0$$

Pay $m + 2$ integrals $\longrightarrow$ Get order $2m + 1$ Picard

- “Quad-frame” etc.

Pay $m + 4$ integrals $\longrightarrow$ Get order $4m + 1$ Picard

Table of Contents

4 Beyond N-ODEs

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Beyond N-ODEs

Fractional & non-linear DEs

- Problems with path-sum formulations ? $\hookrightarrow$ Those with resolvents solutions
- How useful is $\mathcal{F}$ Fréchet-Lie structure ? $\hookrightarrow$ $\delta^{(\alpha)}$ and $\mathcal{T}e^{\int_s^t f(\tau,s)d\tau}$ exist in $\mathcal{F}$

Kosovtsov, arXiv:0910.3923 (2009)
Kosovtsov, arXiv:math-ph/0409035 (2004)
Kosovtsov, arXiv:math-ph/0202040 (2002)

Beyond N-ODEs

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- Problems with path-sum formulations ? $\hookrightarrow$ Those with resolvents solutions
- How useful is $\mathcal{F}$ Fréchet-Lie structure ? $\hookrightarrow$ $\delta^{(\alpha)}$ and $\mathcal{T}e^{\int_s^t f(\tau,s)d\tau}$ exist in $\mathcal{F}$

Fractional & Non-linear N-ODEs

Fractional linear N-ODEs have $\star$-resolvent solutions in $\mathcal{F}$

Corollary of Kosovtsov's (Kosovtsov 2002-2009):
Non-linear N-ODEs $\dot{\mathsf{U}} = f(t, \mathsf{U})$ have $\star$-resolvent solutions in $\mathcal{F}$

Kosovtsov's umbral linearization

$$\dot{y}(t) = a(t)\, y(t)^4 \Rightarrow U_y(t,s) := e^{y(t)s} \Rightarrow \partial_t U_y = a(t) s\, \partial_s^4\, U_y$$

Durastante, G., Pozza, upcoming (2024+)
Kosovtsov, arXiv:2406.08574 (2024)

Kosovtsov, arXiv:0910.3923 (2009)
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Thank you for listening!
Any questions?