Between graph theory and differential equations : the ★-product.

Manon Ryckebusch
ULCO LMPA
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Introduction.

$$(f \star g)(x,y) = \int_{-\infty}^{+\infty} f(x,\tau)g(\tau,y)d\tau$$

Introduction.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases}$$

$t' \geq t$

$$G = \frac{d}{dt'}U(t',t)H(t'-t) \iff G - 1_\star = MH \star G$$

$$\iff G = (1_\star - MH)^{\star^{-1}}$$

$$H(t'-t) = \begin{cases} 1 \text{ if } t' \geq t \\ 0 \text{ neither} \end{cases}$$

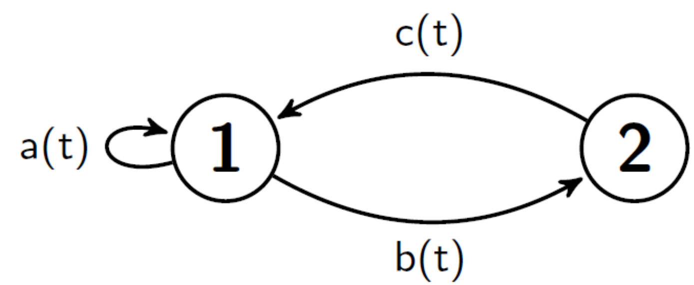
Introduction.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases}$$

$$(f \star g)(x,y) = \int_{-\infty}^{+\infty} f(x,\tau)g(\tau,y)d\tau$$

$$M(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & 0 \end{pmatrix}$$

Adjacency matrix



Replace $\times$ by $\star$

$$(Id_\star - zM)^{\star -1}$$

Summary

1 **Between physics and graph theory.**

- The method of path sum.
- Solving a differential equation with the method of path sum.

2 **Distributions.**

- Distribution reminders.
- Products between distributions.

3 **Compact ★-product on D.**

- A regular product.
- A remark on the ★-product on $\mathbb{R}$.
- Schwartz point of view : Action on $\mathcal{C}_c^\infty(\mathbb{R}^2)$.

4 **What's next ?**

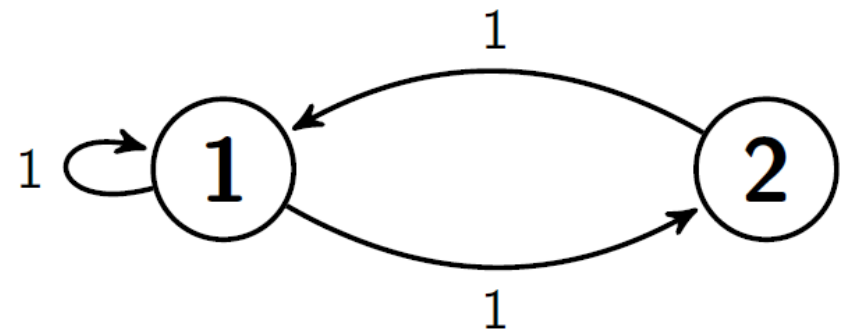
- ★-inverses and algebraic structure.

Between physics and graph theory.

The method of path sum.

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Adjacency matrix



Method of path sum

$$(Id - zM)^{-1}$$

Between physics and graph theory.

The method of path sum.

A Adjacency matrix

$\downarrow$

A^n The element (i, j) gives the number of walks of length n from vertex i to vertex j.

$\downarrow$

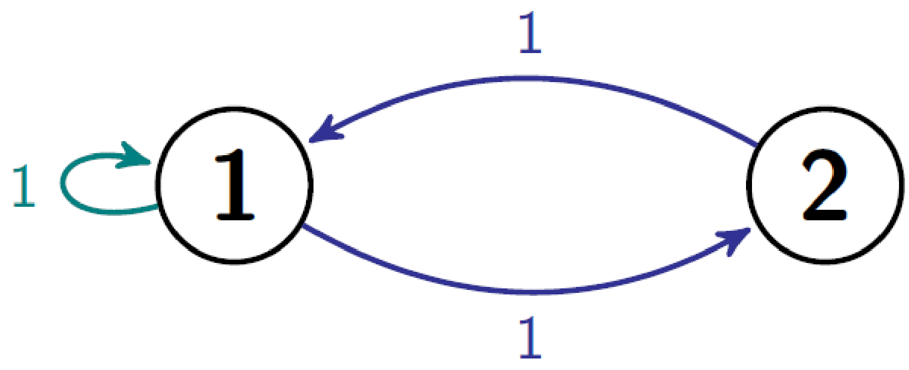
$\sum_n A^n$ The element (i, j) gives the number of walks from vertex i to vertex j.

$\nwarrow$

$$(Id - A)^{-1}$$

Between physics and graph theory.
The method of path sum.

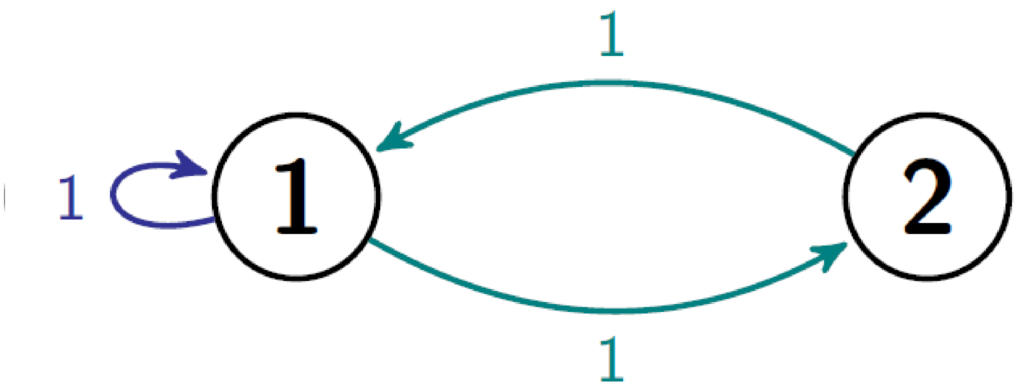
Going from 1 to 1



$$\left((Id - zM)^{-1}\right)_{11} = \frac{1}{1 - z - z^2}$$

Between physics and graph theory.
The method of path sum.

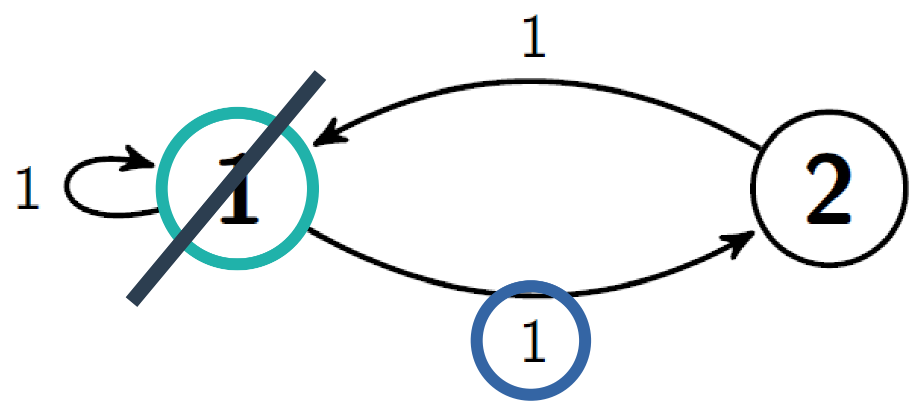
Going from 2 to 2



$$\left((Id - zM)^{-1}\right)_{22} = \frac{1}{1 - z^2 \times \frac{1}{1-z}}$$

Between physics and graph theory.
The method of path sum.

Going from 1 to 2

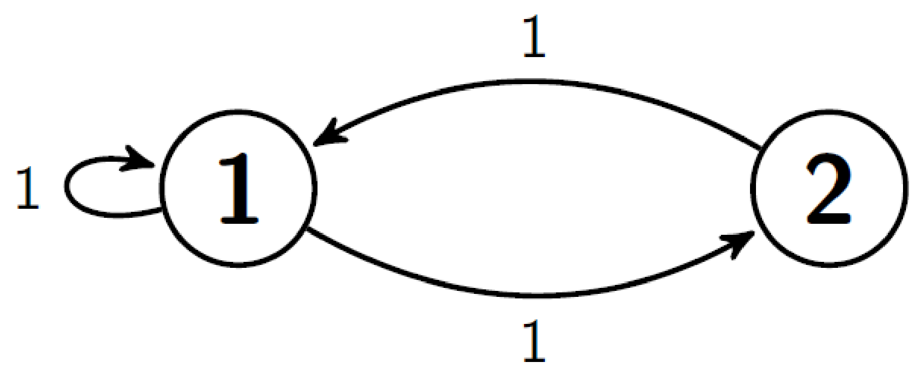


$$1 \quad \times \quad z \quad \times \quad \left((Id - zM)^{-1} \right)_{11}$$

Between physics and graph theory.

The method of path sum.

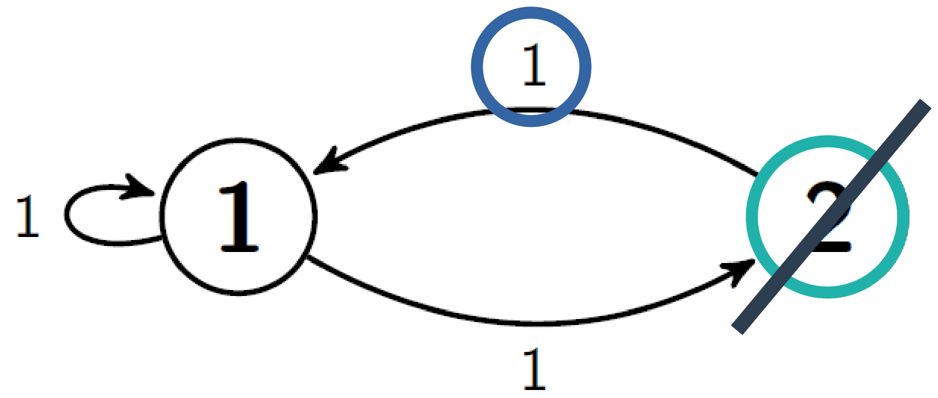
Going from 1 to 2



$$\left((Id - zM)^{-1}\right)_{12} = \frac{z}{1 - z - z^2}$$

Between physics and graph theory.
The method of path sum.

Going from 2 to 1

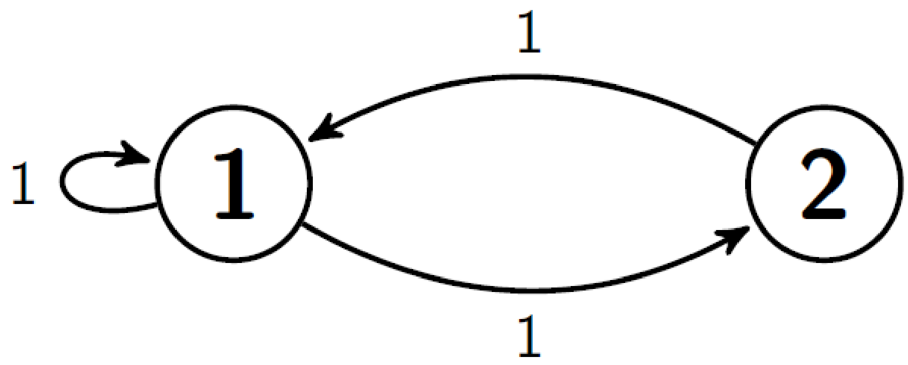


$$\frac{1}{1-z} \quad \times \quad z \quad \times \quad \left((Id - zM)^{-1}\right)_{22}$$

Between physics and graph theory.

The method of path sum.

Going from 2 to 1



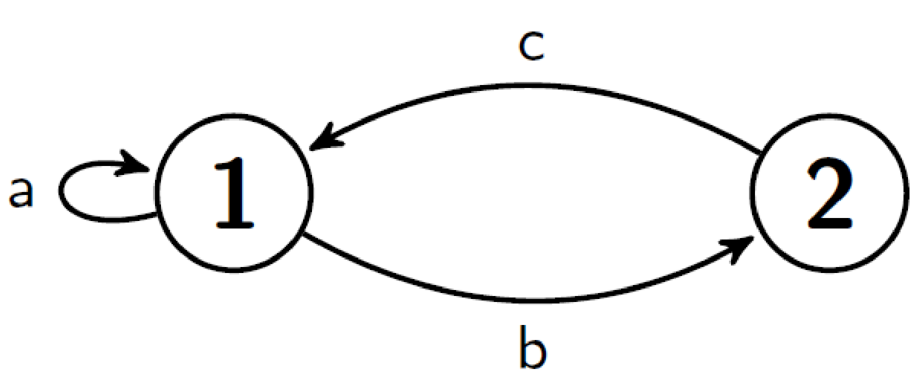
$$\left((Id - zM)^{-1}\right)_{21} = \frac{z}{1 - z - z^2}$$

Between physics and graph theory.

The method of path sum.

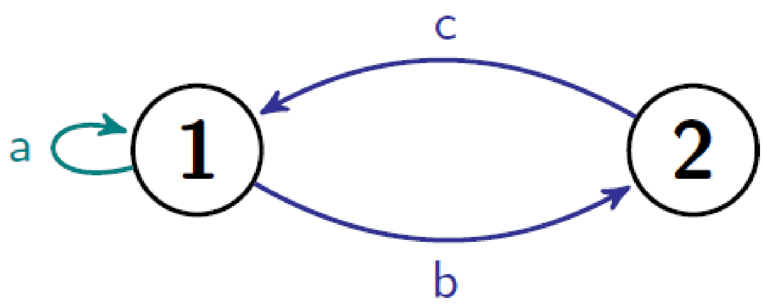
What if there are weights ?

$$M = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$$

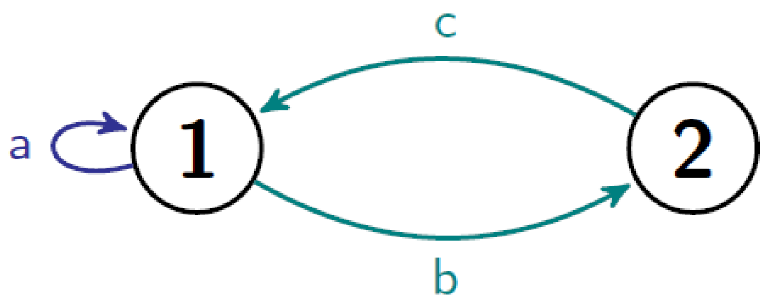


Between physics and graph theory.

The method of path sum.



$$\left((Id - zM)^{-1}\right)_{11} = \frac{1}{1 - az - bcz^2}$$

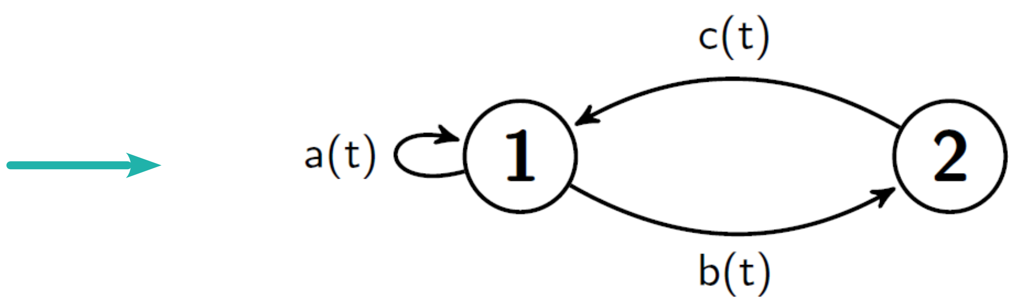


$$\left((Id - zM)^{-1}\right)_{22} = \frac{1}{1 - cbz^2 \times \frac{1}{1-az}}$$

Between physics and graph theory.
The method of path sum.

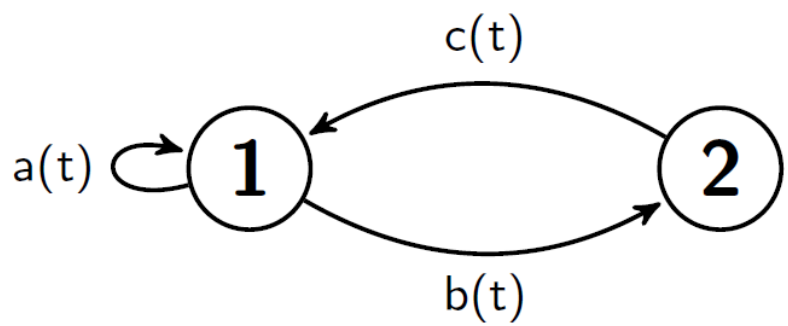
What if there are time depending weights ?

$$M(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & 0 \end{pmatrix}$$



Between physics and graph theory.

The method of path sum.



$1 \to 1:$

$$\left(Id - za(t) - z^2 b(t)c(t)\right)^{-1}$$

$2 \to 2:$

$$\left(Id - z^2 c(t)\left(Id - za(t)\right)^{-1} b(t)\right)^{-1}$$

$1 \to 2:$

$$z \times \left(Id - za(t) - z^2 b(t)c(t)\right)^{-1}$$

$2 \to 1:$

$$\left(Id - za(t)\right)^{-1} \times z \times \left(Id - z^2 c(t)\left(Id - za(t)\right)^{-1} b(t)\right)^{-1}$$

Between physics and graph theory.

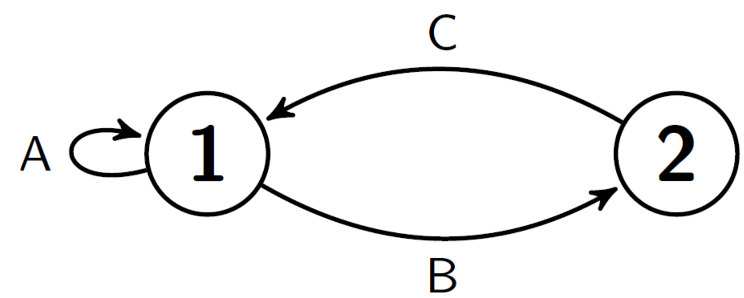
The method of path sum.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}\right)$$

Or

$$\left(\begin{array}{cc|c} 0 & -\omega & u(t) \\ \omega & 0 & -v(t) \\ \hline u(t) & v(t) & 0 \end{array}\right)$$

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$



Between physics and graph theory.

Solving a differential equation with the method of path sum.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases} \longleftarrow \; t' \geq t$$

$$G = \frac{d}{dt'}U(t',t)H(t'-t) \iff G = \frac{d}{dt'}U(t',t) \times H(t'-t) + U(t',t) \times \frac{d}{dt'}H(t'-t)$$

$$H(t) = \begin{cases} 1 \text{ if } t' \geq t \\ 0 \text{ neither} \end{cases}$$

$$\iff G = \frac{d}{dt'}U(t',t) \times H(t'-t) + U(0,0) \times \delta(t'-t)$$

$$\iff G - 1_\star = \frac{d}{dt'}U(t',t) \times H(t'-t)$$

Between physics and graph theory.

Solving a differential equation with the method of path sum.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases} \qquad (f \star g)(t',t) = \int_{-\infty}^{+\infty} f(t',\tau)g(\tau,t)d\tau$$

$$\begin{aligned} (MH \star G)(t',t) &= \int_{-\infty}^{+\infty} M(t')H(t'-\tau)G(\tau,t)d\tau \\ &= M(t') \times (H \star G)(t',t) \\ &= M(t') \times U(t',t) \times H(t'-t) \end{aligned}$$

Between physics and graph theory.

Solving a differential equation with the method of path sum.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases}$$

$$G = \frac{d}{dt'}U(t',t)H(t'-t) \iff G = \frac{d}{dt'}U(t',t) \times H(t'-t) + U(t',t) \times \frac{d}{dt'}H(t'-t)$$

$$\iff G = \frac{d}{dt'}U(t',t) \times H(t'-t) + U(0,0) \times \delta(t'-t)$$

$$\iff G - 1_\star = \frac{d}{dt'}U(t',t) \times H(t'-t)$$

$$\iff G - 1_\star = MH \star G$$

$$\iff G = (1_\star - MH)^{\star -1}$$

Summary

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- Distribution reminders.
- Products between distributions.

3 **Compact ★-product on D.**

- A regular product.
- A remark on the ★-product on $\mathbb{R}$.
- Schwartz point of view : Action on $\mathcal{C}_c^{\infty}(\mathbb{R}^2)$.

4 **What's next ?**

- ★-inverses and algebraic structure.

Definition :

A distribution on an open set Ω is a continuous linear form $T : \mathcal{C}_c^\infty(\Omega) \to \mathbb{C}$.

→ $\mathcal{D}'(\Omega)$, topological dual space of $\mathcal{C}_c^\infty(\Omega)$.

→ Duality bracket $\langle T_f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx$

Heaviside Theta function.

$$\langle H_a, \varphi \rangle = \int_{-\infty}^{+\infty} H(x-a)\varphi(x)dx = \int_{a}^{+\infty} \varphi(x)dx$$

Dirac Delta.

$$\langle \delta_a, \varphi \rangle = \int_{-\infty}^{+\infty} \delta(x-a)\varphi(x)dx = \varphi(a)$$

Dirac Delta derivatives.

$$\left\langle \delta_a^{(n)}, \varphi \right\rangle = (-1)^n \times \varphi^{(n)}(a)$$

Distributions.

Products between distributions.

Product with a smooth compactly supported function.

$$
\begin{aligned}
\langle a \times T_f, \varphi \rangle &= \int_{-\infty}^{+\infty} (a(x) f(x)) \varphi(x) dx \\
&= \int_{-\infty}^{+\infty} f(x) (a(x) \varphi(x)) dx \\
&= \langle T_f, a \times \varphi \rangle
\end{aligned}
$$

Generally : ~~Usual product between distributions.~~

Tensor product of two distributions.

$$
\begin{aligned}
\langle T_f \otimes T_g, \varphi \rangle &= \int_{-\infty}^{+\infty} f(x) \left(\int_{-\infty}^{+\infty} g(y)\varphi(x,y)dy \right) dx \\
&= \langle T_f, \langle T_g, \varphi(x,\cdot) \rangle \rangle \\
&= \int_{-\infty}^{+\infty} g(y) \left(\int_{-\infty}^{+\infty} f(x)\varphi(x,y)dx \right) dy \\
&= \langle T_g, \langle T_f, \varphi(\cdot, y) \rangle \rangle
\end{aligned}
$$

Convolution product of two distributions.

$$
\langle T_f * T_g, \varphi \rangle = \langle T_g \otimes T_f, \varphi(x+y) \rangle
$$

Distributions.

Products between distributions : convolution vs ★-product.

Convolution product.

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-\tau)g(\tau)d\tau$$

★-product.

$$(f \star g)(x,y) = \int_{-\infty}^{+\infty} f(x,\tau)g(\tau,y)d\tau$$

Dependance on the difference $x - y$ only.

$$(f \star g)(x,y) = \int_{-\infty}^{+\infty} f(x-\tau)g(\tau-y)d\tau$$

By substitution.

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3. **Compact ★-product on D.**
   - A regular product.
   - A remark on the ★-product on $\mathbb{R}$.
   - Schwartz point of view : Action on $\mathcal{C}_c^\infty(\mathbb{R}^2)$.
4. **What's next ?**
   - ★-inverses and algebraic structure.

Compact ★-product on D.

A regular product.

Distributions

Definition :

$$f \in \mathrm{D} \iff f(x,y) = g(x,y)H(x-y) + \sum_{n=0}^{+\infty} g_i(x,y)\delta^{(i)}(x-y)$$

Smooths coefficients

Compact ★-product on D.

A regular product.

Definition :

Let I be a real compact interval. The compact ★-product is defined as follows :

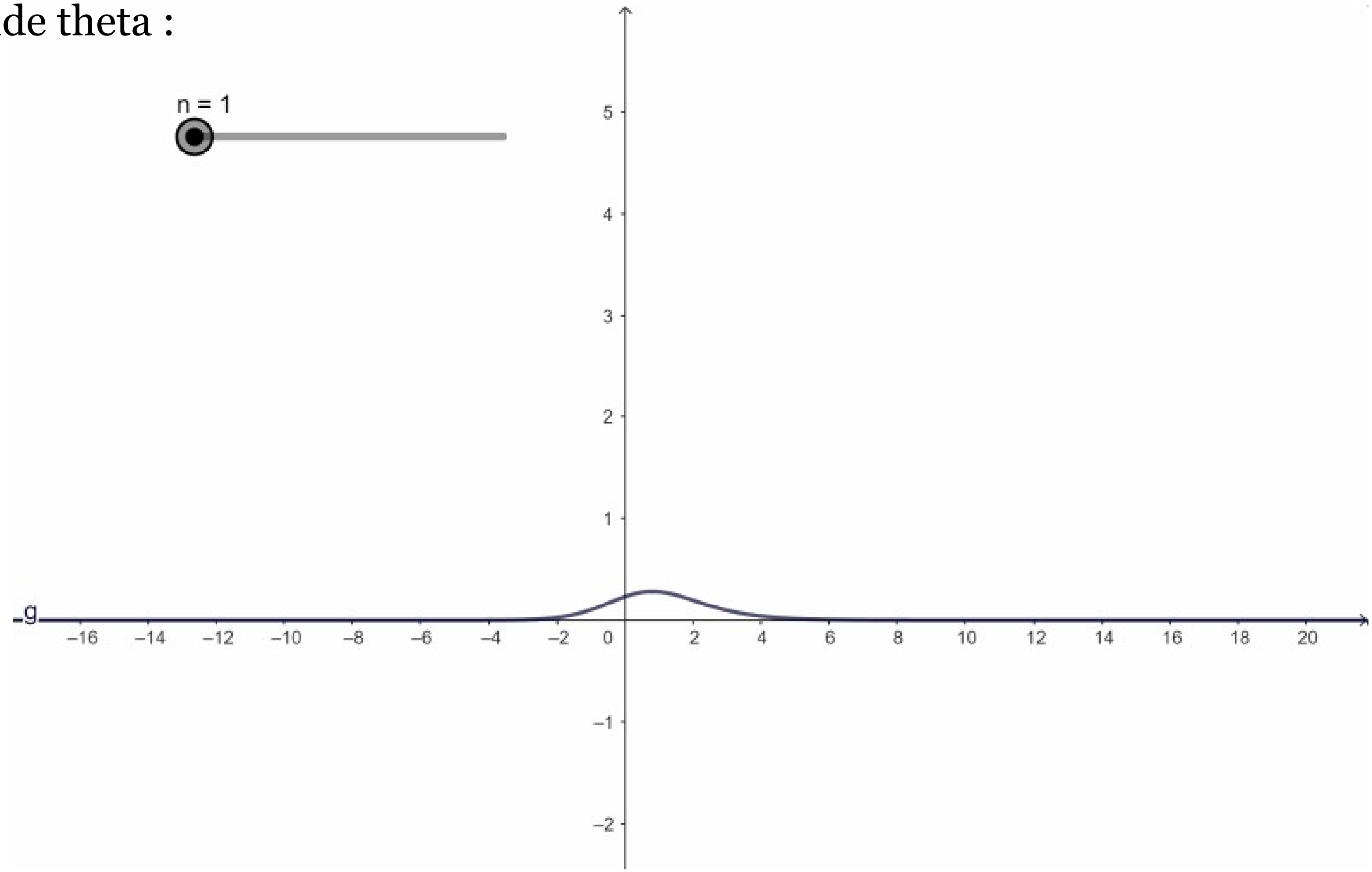
$$\begin{cases} \mathcal{C}^{\infty}(I^2) \times \mathcal{C}^{\infty}(I^2) \to \mathcal{C}^{\infty}(I^2), \\ \qquad\qquad (f, \varphi) \mapsto \int_I f(x,\tau)\varphi(\tau,y)d\tau. \end{cases}$$

Main idea : Extend this product thanks to sequences to a larger set, which must contain D, to be able to make this product between distributions.

Compact ★-product on D.

A regular product.

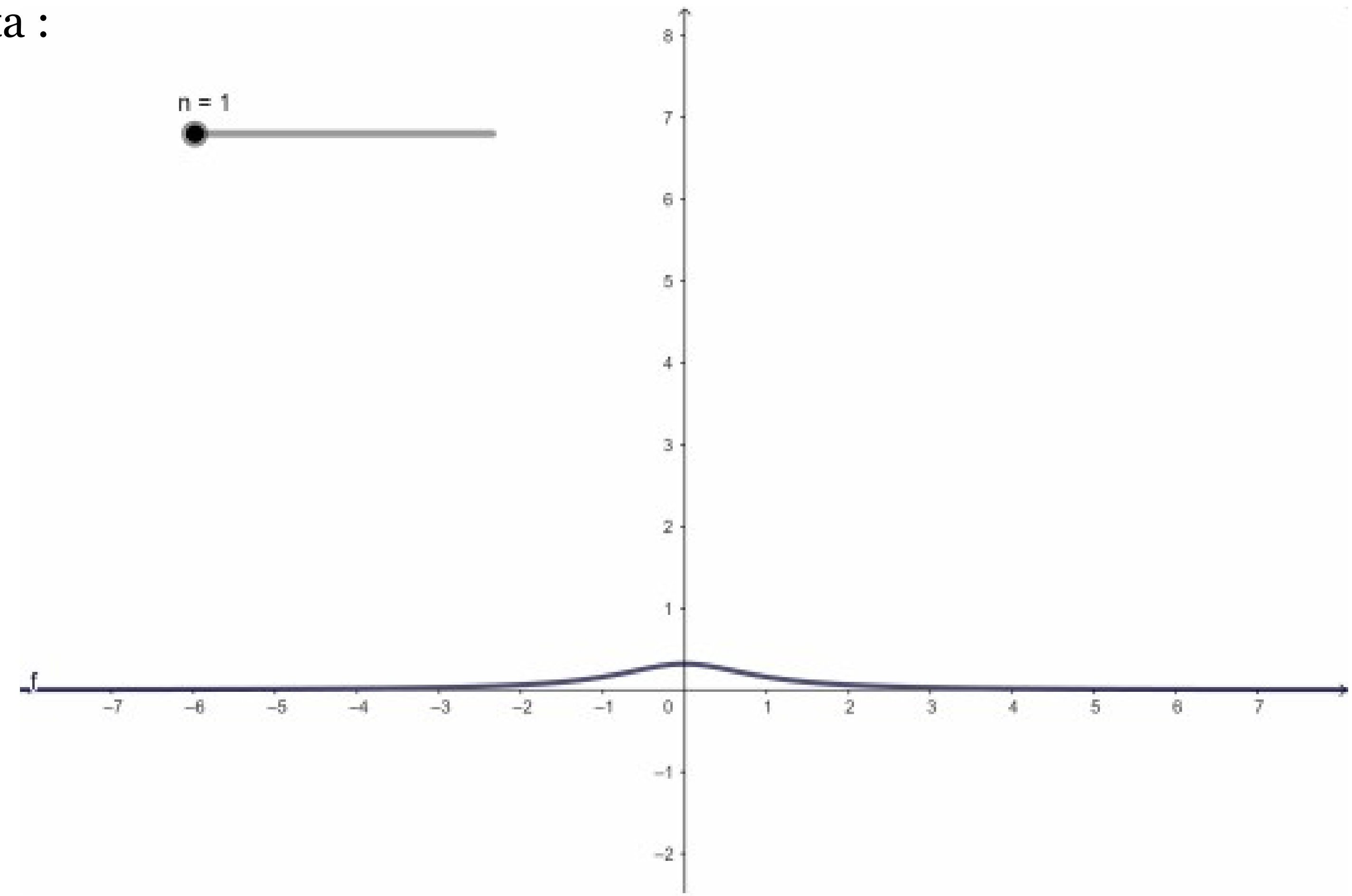
Heaviside theta :



Compact ★-product on D.

A regular product.

Dirac delta :



Compact ★-product on D.

A regular product.

Theorem :

The compact ★-product can be extended to the set D.

Proof :

- ✔ Use the sequential approach and show that the compact star-product respects the convergence, ie is a regular product.
- ✔ Show that the obtained limit is unique.
- ✔ Obtain elements of D as a limit of smooth sequences.

Compact ★-product on D.

A remark on the ★-product on $\mathbb{R}$.

Corollary :

The ★-product on $\mathbb{R}$ is well defined for elements of D.

Proof :

- ✔ Use the notion of compactly supported distributions.
- ✔ Adapt it to the ★-product on $\mathbb{R}$.
- ✔ Define the ★-product on $\mathbb{R}$ for elements of D.

Compact ★-product on D.

A remark on the ★-product on $\mathbb{R}$.

$$(f \star g)(x,y) = \int_{-\infty}^{+\infty} f(x,\tau)g(\tau,y)d\tau$$

Properties.

- ✔ Continuous matrix multiplication ⟶ $\sum_{k=1}^{n} a_{i,k}b_{k,j}$
- ✔ Generalizes convolution ⟶ $(f * g)(x) = \int_{-\infty}^{+\infty} f(x-\tau)g(\tau)d\tau$
- ✔ Generalizes Volterra's compositions ⟶ $\int_{x}^{y} f(x,\tau)g(\tau,y)d\tau$ and $\int_{a}^{b} f(x,\tau)g(\tau,y)d\tau$
- ✔ Induces Schwartz bracket ⟶ $\langle T_f, \varphi\rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)dx$

Compact ★-product on D.

Schwartz point of view : Action on $\mathcal{C}_c^{\infty}(\mathbb{R}^2)$.

Definition :

$$f \in \mathrm{D} \iff f(x,y) = g(x,y)H(x-y) + \sum_{n=0}^{+\infty} g_i(x,y)\delta^{(i)}(x-y)$$

⟶ Subset of $\mathcal{D}'(\mathbb{R}^2)$.

Compact ★-product on D.

Schwartz point of view : Action on $\mathcal{C}_c^\infty(\mathbb{R}^2)$.

Calculate the ★-product between two elements of D.

$$\begin{aligned}\langle H \star \delta, \varphi\rangle &= \int_{\mathbb{R}^2} \left(\int_{-\infty}^{+\infty} H(x-\tau)\delta(\tau-y)d\tau\right)\varphi(x,y)d(x,y) \\ &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} H(x-\tau)\varphi(x,y)dx\right)\delta(\tau-y)dy\right)d\tau \\ &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} H(x-\tau)\left(\int_{-\infty}^{+\infty}\delta(\tau-y)\varphi(x,y)dy\right)dx\right)d\tau \\ &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} H(x-\tau)\varphi(x,\tau)dx\right)d\tau \\ &= \langle H, \varphi\rangle\end{aligned}$$

⟶ $H \star \delta = H$

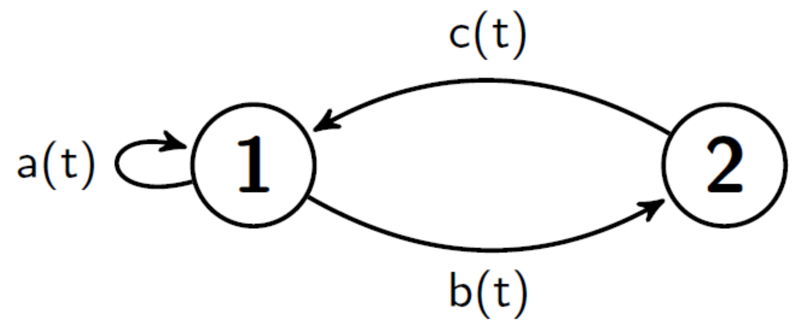
Compact ★-product on D.

Schwartz point of view : Action on $\mathcal{C}_c^\infty(\mathbb{R}^2)$.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases} \qquad (f \star g)(x,y) = \int_{-\infty}^{+\infty} f(x,\tau)g(\tau,y)d\tau$$

$$M(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & 0 \end{pmatrix}$$

Adjacency matrix

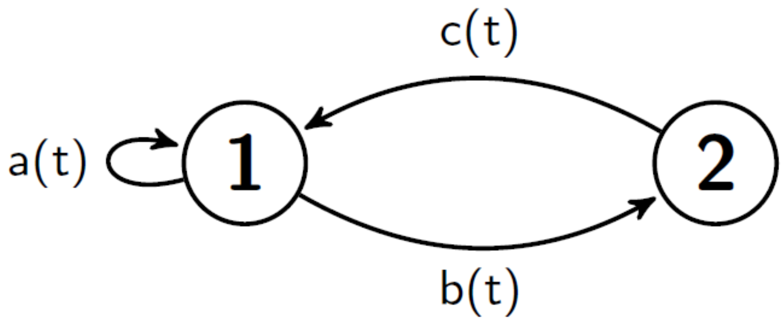


Replace $\times$ by $\star$

$$(Id_\star - MH)^{\star -1}$$

Compact ★-product on D.

Schwartz point of view : Action on $\mathcal{C}_c^\infty(\mathbb{R}^2)$.



1 → 1 :

$$\left(Id_\star - a(t) - b(t) \star c(t)\right)^{\star-1}$$

2 → 2 :

$$\left(Id\star - c(t) \star (Id_\star - a(t))^{\star-1} \star b(t)\right)^{\star-1}$$

1 → 2 : $$\left(Id_\star - a(t) - b(t) \star c(t)\right)^{\star-1}$$

2 → 1 : $$(Id_\star - a(t))^{\star-1} \star \left(Id_\star - c(t) \star (Id_\star - a(t))^{\star-1} \star b(t)\right)^{\star-1}$$

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- ★-inverses and algebraic structure.

What's next ?

★-inverses and algebraic structures.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases} \longrightarrow (Id_\star - MH)^{\star-1} \quad ?$$

What we know :

- ✔ The inverse does not exist everytime, but the resolvent does.
- ✔ How to calculate ★-inverses for elements in a particular subset of D.
- ✔ Calculate some ★-inverses with numerical methods.

What's next ?

★-inverses and algebraic structures.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases} \longrightarrow (Id_\star - MH)^{\star -1} \quad ?$$

What we know :

✔ There is a dense set inside of D ★-invertible.

✔ This set is a group of distribution with the ★-product.

What's next ?

★-inverses and algebraic structures.

$$\begin{cases} \frac{d}{dt'}U(t',t) = M(t')U(t',t) \\ U(0,0) = Id \end{cases} \longrightarrow \quad (Id_\star - MH)^{\star -1} \; ?$$

What we know :

✔ $(\mathcal{D}, \star)$ is a subgroup of the infinite-dimensional Frechet Lie group

$(Aut(\mathcal{C}^\infty(I^2, \mathbb{C})), \star)$.

Thank you !