Action accessible and weakly action representable varieties of algebras

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- A category $\mathcal{C}$ is said to be *semi-abelian* if it is pointed, Barr-exact, Bourn-protomodular with finite coproducts.
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

in $\mathcal{C}$ such that $\alpha \circ \beta = \mathrm{id}_B$ and (X, k) is a kernel of α.

- For any object X in $\mathcal{C}$, we consider the functor

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  where, for every object B in $\mathcal{C}$, $\mathrm{SplExt}(B, X)$ is the set of isomorphism classes of split extensions of B by X.
- There is a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Act}(-, X).$$

Varieties of non-associative algebras

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- Any variety of non-associative algebras $\mathcal{V}$ can be seen as a full subcategory of **Alg**. Any such variety is a semi-abelian category.

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- **Lie** is the variety of *Lie algebras*, which is determined by anti-commutativity and the *Jacobi identity*: $xy + yx = 0$ and $x(yz) + y(zx) + z(xy) = 0$.

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- **Alt** is the variety of *alternative algebras*, which is determined by $(yx)x - y(xx) = 0$ and $x(xy) - (xx)y = 0$.

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- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.
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Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden, C. Vienne, 2021)

Let $\mathcal{V}$ be a variety of non-associative algebras over an infinite field $\mathbb{F}$ with $\mathrm{chat}(\mathbb{F}) \neq 2$. *If $\mathcal{V}$ is action representable, then $\mathcal{V} = \mathbf{AbAlg}$ or $\mathcal{V} = \mathbf{Lie}$.*

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- More recently, G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

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An object T as above is called *weak representing object* of X, the pair (T, τ) is called *weak representation* of $\mathrm{SplExt}(-, X)$ and $(\varphi \colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

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Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

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$$\mathrm{Bim}(X) = \{(f*-, -*f) \in \mathrm{End}(X)\times\mathrm{End}(X)^{\mathrm{op}} \mid f*(xy) = (f*x)y, \\ (xy)*f = x(y*f), x(f*y) = (x*f)y, \ \forall x, y \in X\}.$$

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- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable.

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- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable. A weak representing object of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y),$$
$$D(xy) = D(x)y - D(y)x, \ xd(y) = xD(y), \ \forall x, y \in X\}.$$

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The variety **Jord** of Jordan algebras is not action accessible.

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Conclusion: Jord is not weakly action representable.

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- We also suppose that $\mathcal{V}$ is *action accessible*, or equivalently that $\mathcal{V}$ is an *Orzech category of interest*.
- This is equivalent to saying that the λ/μ *rules* hold: there exist $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$\begin{aligned} x(yz) =& \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) + \\ &+ \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx), \end{aligned}$$

and

$$\begin{aligned} (yz)x =& \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) + \\ &+ \mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx) \end{aligned}$$

are identities in $\mathcal{V}$.

- We fix a set of multilinear identities

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  which *weakly represents* split extensions with kernel X.

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where

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Let $\mathcal{V}$ be a variety and let $U\colon \mathcal{V} \to \mathbf{PAlg}$ denote the forgetful functor.

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The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X.

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Then, the category $\mathcal{X}$ is not weakly action representable.

Remark (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

Any action representable category has the *amalgamation property*.

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Theorem (J. R. A. Gray, 2025)

The varieties $\mathbf{Sol}_t(\mathbf{Grp})$ *is not weakly action representable for any* $t \geq 3$.

- Consider the following groups

$$S = \langle a, b \mid [a, b] = a^5 = b^5 = 1 \rangle,$$
$$B = \langle x, a, b \mid [x, a] = b^{-1}, [x, b] = x^5 = a^5 = b^5 = 1 \rangle,$$
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Theorem (X. García Martínez, M. M., 2025)

The varieties $\mathbf{Sol}_2(\mathbf{Grp})$ *and* $\mathbf{Nil}_k(\mathbf{Grp})$ *(*$k \geq 3$*) are not weakly action representable.*

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Theorem (X. García Martínez, M. M., 2025)

The varieties $\mathbf{Sol}_t(\mathbf{Lie})$ *and* $\mathbf{Nil}_k(\mathbf{Lie})$ *are not weakly action representable for any* $t \geq 2$ *and* $k \geq 3$.

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The varieties $\mathbf{Sol}_t(\mathbf{Lie})$ *and* $\mathbf{Nil}_k(\mathbf{Lie})$ *are not weakly action representable for any* $t \geq 2$ *and* $k \geq 3$*.*

- **Open question:** is the variety $\mathbf{Nil}_2(\mathbf{Grp})$ weakly action representable?

Merci beaucoup!

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