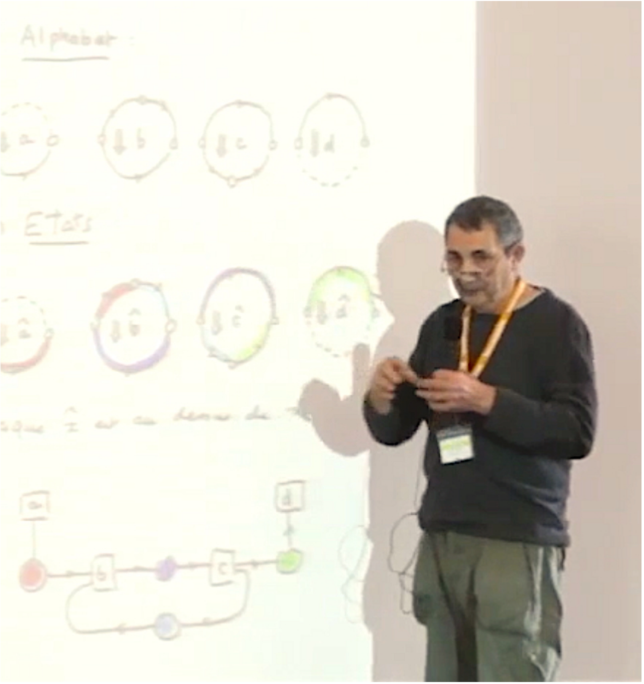
When is $\mathbf{Cat}(\mathcal{Q})$ Cartesian closed?

Isar Stubbe
joint work with Junche Yu

Université du Littoral, France

Lens, 6 June 2025

Mais d'abord…



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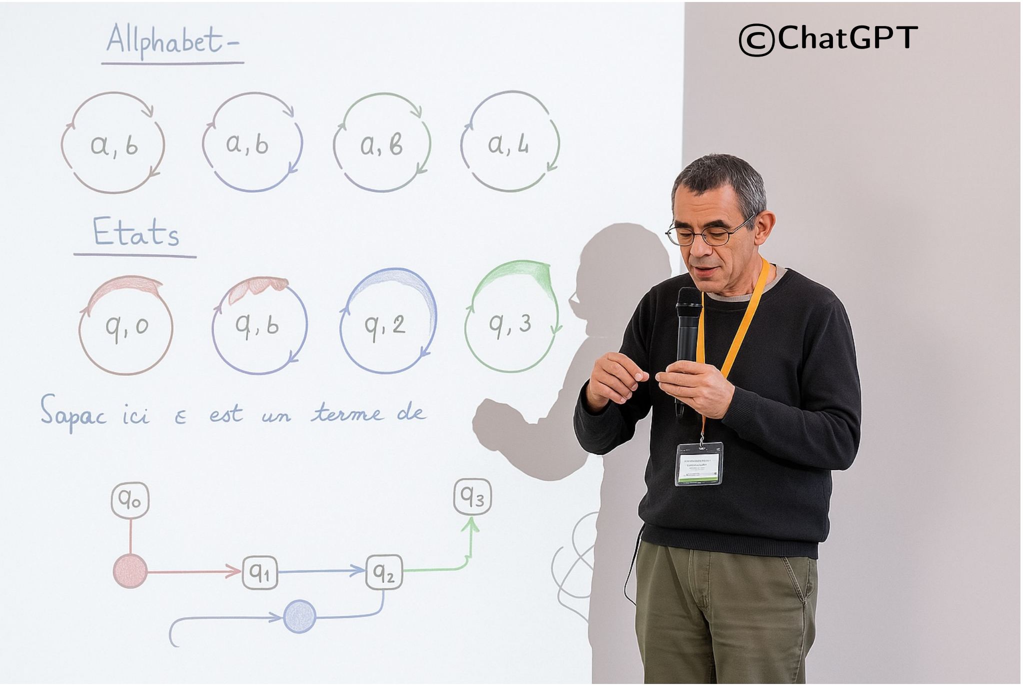
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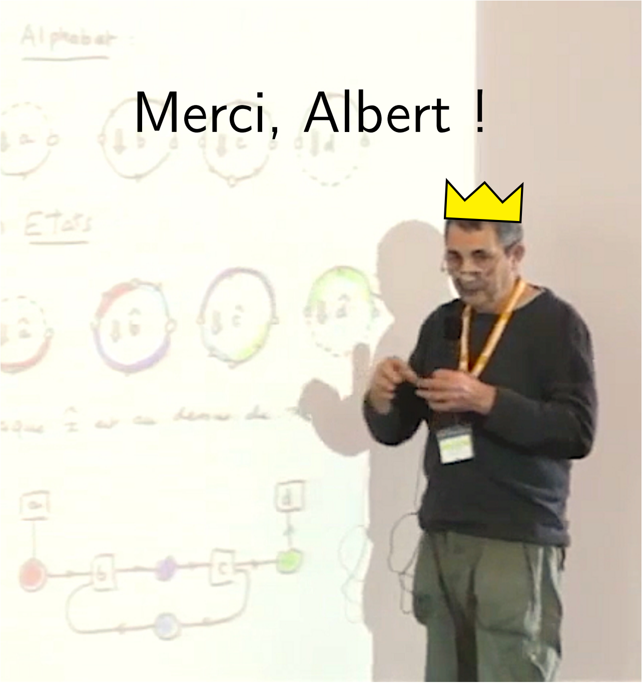

Merci, Albert !

Exponentials, Cartesian closedness

Let $\mathcal{C}$ be a category with finite products.

An object $A \in \mathcal{C}$ is **exponentiable** when

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This means: for any $B \in \mathcal{C}$ there exists an object B^A and a morphism $e_B \colon B^A \times A \to B$ in $\mathcal{C}$ inducing, for any $X \in \mathcal{C}$, a (natural) bijection

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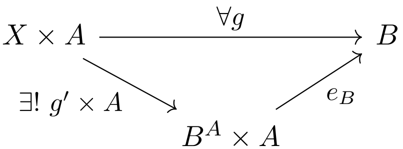

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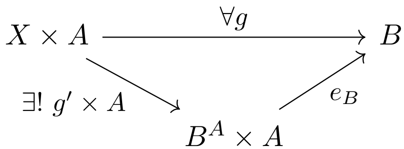

Putting $X = 1$ in the above, one has

$$\mathcal{C}(A, B) \cong \mathcal{C}(1, B^A),$$

so the "points" of the object B^A are precisely the morphisms from A to B; and similarly $e_B \colon B^A \times A \to B$ is the "pointwise" evaluation of such morphisms.

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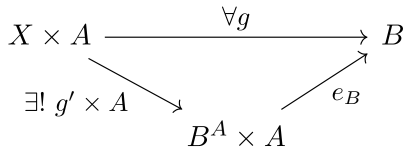

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The category $\mathcal{C}$ is **Cartesian closed** when every $A \in \mathcal{C}$ is exponentiable.

Example

Consider $(A, \leq)$ and $(B, \leq)$ in the category Ord of ordered sets and order-preserving maps.

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Because the "points" functor $\mathsf{Ord}(1, -)\colon \mathsf{Ord} \to \mathsf{Set}$ produces the underlying sets and maps of objects and morphisms in Ord, the underlying set and evaluation map of their powerobject must be

$$B^A = \mathsf{Ord}(A, B) \qquad \text{and} \qquad e_B\colon B^A \times A \to B\colon (f, a) \mapsto fa.$$

But is there an order on B^A that makes e_B order-preserving and universal?

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$$\text{if } f \leq g \text{ in } B^A \text{ and } a \leq b \text{ in } A \text{ then } fa \leq gb \text{ in } B \qquad \text{for all } f, g \in B^A \text{ and } a, b \in A,$$

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$$\begin{aligned} f \leq g \text{ in } B^A \quad &\overset{\text{def}}{\iff} \quad \text{for all } a, b \in A\text{: if } a \leq b \text{ then } fa \leq gb \\ &\iff \quad \text{for all } a \in A\text{: } fa \leq ga \end{aligned}$$

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The evaluation is a non-expansive map iff

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Theorem (Clementino and Hofmann, 2006). (A, d) *is exponentiable in* Met *if and only if, for every* $a, b \in A$*, whenever* $d(a, b) = r + s$ *in* $[0, \infty]$ *then, for every* $\varepsilon > 0$*, there exists* $m \in A$ *such that* $d(a, m) < r + \varepsilon$ *and* $d(m, b) < s + \varepsilon$*.*

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The category Met **is *not* Cartesian closed.**

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In fact, both Ord and Met are of the form "$\mathsf{Cat}(\mathcal{Q})$", the category of quantaloid-enriched categories and functors. One is Cartesian closed, the other is not. So Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$ is a significant property of the quantaloid $\mathcal{Q}$.

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So exactly when is Cat$(\mathcal{Q})$ **Cartesian closed?**

Quantale-enriched categories

A **quantale** $Q = (Q, \bigvee, \circ, 1)$ is a completely ordered monoid in which

$$a \circ (\bigvee_i b_i) = \bigvee_i (a \circ b_i) \quad \text{and} \quad (\bigvee_i a_i) \circ b = \bigvee_i (a_i \circ b)$$

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A Q-**category** $\mathbb{A}$ is

- a set $\mathbb{A}_0$ of "objects",
- a function $\mathbb{A}\colon \mathbb{A}_0 \times \mathbb{A}_0 \to Q\colon (x, y) \mapsto \mathbb{A}(x, y)$ assigning "homs",

such that, for any $x, y, z \in \mathbb{A}_0$,

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For $Q = (\{0, 1\}, \vee, \wedge, 1)$: $\mathbb{A}(x, y) = [\![x \leq y]\!]$ and $\mathsf{Cat}(Q) = \mathsf{Ord}$.

For $Q = ([0, \infty], \bigwedge, +, 0)$: $\mathbb{A}(x, y) = d(x, y)$ and $\mathsf{Cat}(Q) = \mathsf{Met}$.

But also t-norms and "fuzzy orders", probabilistic metric spaces, monoidal topology, ...

Quantaloid-enriched categories

A **quantaloid** $\mathcal{Q}$ is a (small) locally completely ordered bicategory in which

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such that, for any $x, y, z \in \mathbb{A}_0$,

$$\mathbb{A}(x, y) \in \mathcal{Q}(ty, tx) \qquad \text{and} \qquad \mathbb{A}(x, y) \circ \mathbb{A}(y, z) \leq \mathbb{A}(x, z) \qquad \text{and} \qquad 1 \leq \mathbb{A}(x, x).$$

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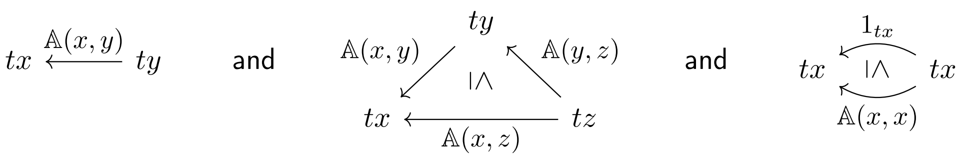

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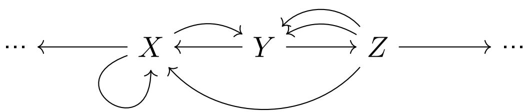

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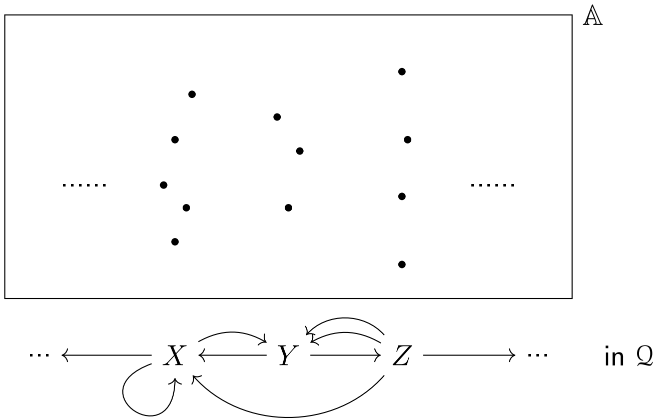

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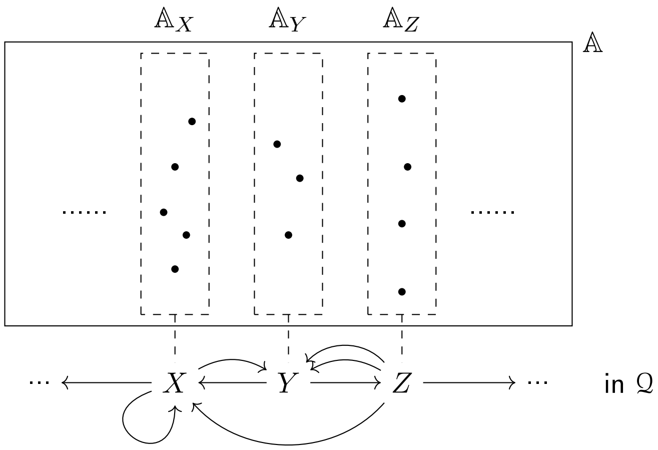

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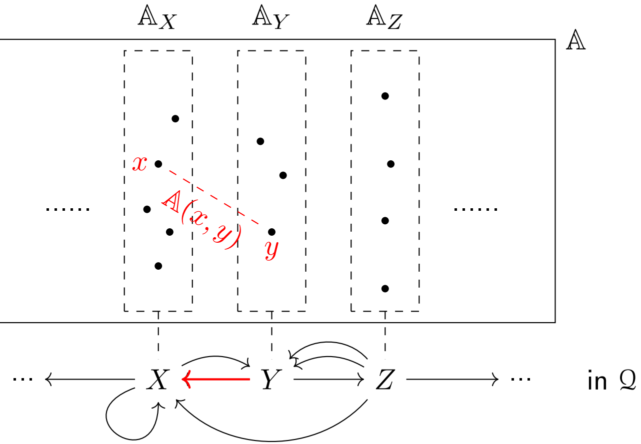

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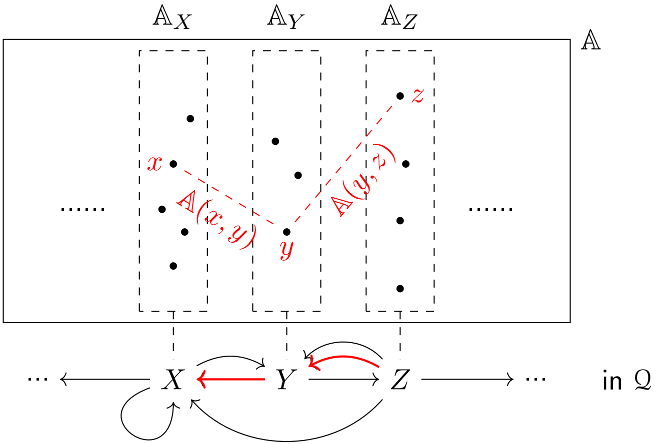

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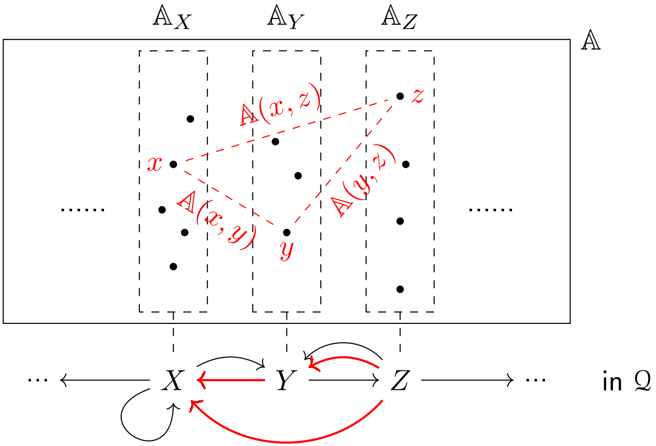

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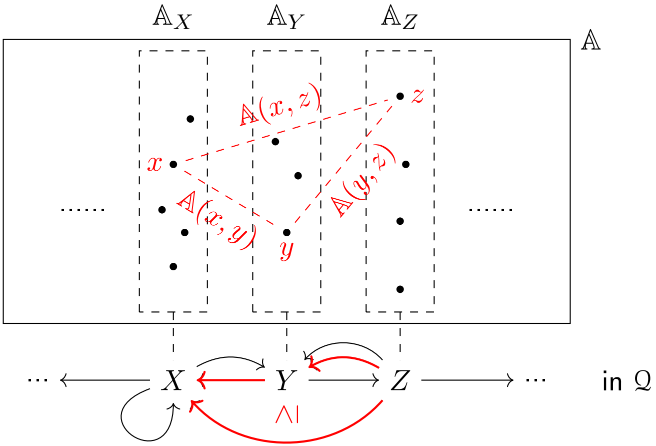

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A quantale is exactly a one-object quantaloid (think of "group" vs. "groupoid"), so we recover all previous examples.

Performing universal constructions on a quantale very often produce a quantaloid. These are used in many new examples: partial (probabilistic) metric spaces, sheaves, ...

Exponentiability in $\mathsf{Cat}(\mathcal{Q})$

The product $\mathbb{A} \times \mathbb{B}$ of two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ exists in $\mathsf{Cat}(\mathcal{Q})$:

objects: $(\mathbb{A} \times \mathbb{B})_X = \mathbb{A}_X \times \mathbb{B}_X$ (for all $X \in \mathcal{Q}_0$),

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Theorem (Clementino, Hofmann and Stubbe, 2009). *A $\mathcal{Q}$-category $\mathbb{A}$ is exponentiable in* $\mathsf{Cat}(\mathcal{Q})$ *if and only if*

1. *for all* $tx \underset{(f_i)_i}{\overset{\mathbb{A}(x,y)}{\longleftarrow}} ty$: $\left(\bigvee_i f_i\right) \wedge \mathbb{A}(x,y) = \bigvee_i \left(f_i \wedge \mathbb{A}(x,y)\right)$,

2. *for all* $tx \xleftarrow{\mathbb{A}(x,y)} ty$, $tx \xleftarrow{g} Z \xleftarrow{f} ty$: $(g \circ f) \wedge \mathbb{A}(x,y) = \displaystyle\bigvee_{z \in \mathbb{A}_Z} \left(g \wedge \mathbb{A}(x,z)\right) \circ \left(f \wedge \mathbb{A}(z,y)\right)$.

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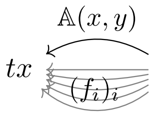
1. *for all* 

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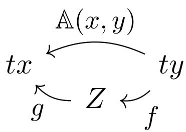
2. *for all* 

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The first condition says that "every hom of $\mathbb{A}$ is exponentiable"; the second condition says that "$\mathbb{A}$ has enough intermediate objects".

It is not very difficult to check that these conditions are always satisfied for ordered sets, and reduce to the "approximate intermediate points" for generalized metric spaces.

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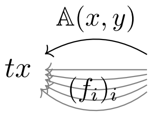
1. *for all* 

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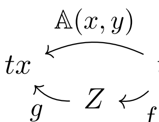
2. *for all* 

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Necessity of the conditions is checked by "probing" exponentibility of $\mathbb{A}$ on particular $\mathbb{B}$'s. $\square$

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By definition, $\mathsf{Cat}(\mathcal{Q})$ is Cartesian closed when **all** $\mathcal{Q}$-categories are exponentiable.

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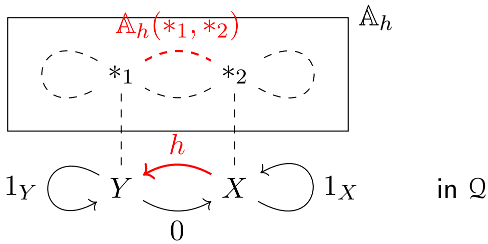
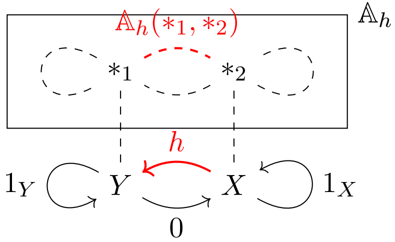


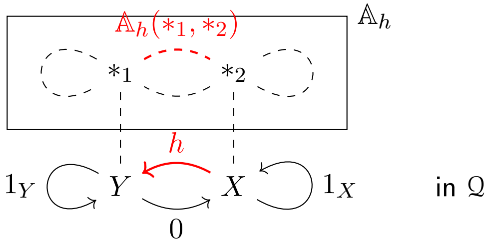
It is the *collage* (= universal cotabulation) of the image of h under the inclusion $\mathcal{Q} \to \mathsf{Dist}(\mathcal{Q})$.

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



in $\mathcal{Q}$

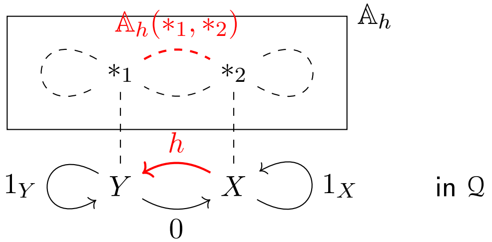
Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



Theorem. $\mathbb{A}_h$ *is exponentiable if and only if*

1. *for all* $tx \xleftarrow{\mathbb{A}_h(x,y)} ty$, $tx \xleftarrow{(f_i)_i} ty$: $\left(\bigvee_i f_i\right) \wedge \mathbb{A}_h(x,y) = \bigvee_i \left(f_i \wedge \mathbb{A}_h(x,y)\right)$,
2. *for all* $tx \xleftarrow{\mathbb{A}_h(x,y)} ty$, $tx \xleftarrow{g} Z \xleftarrow{f} ty$: $(g \circ f) \wedge \mathbb{A}_h(x,y) = \bigvee_{z \in (\mathbb{A}_h)_Z} \left(g \wedge \mathbb{A}_h(x,z)\right) \circ \left(f \wedge \mathbb{A}_h(z,y)\right)$.

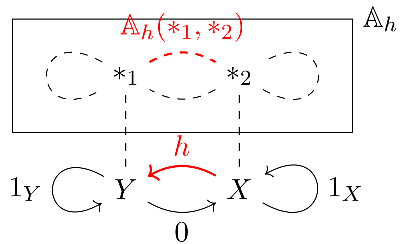
Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



Theorem. $\mathbb{A}_h$ *is exponentiable ~~if and~~ only if, for $x = *_1$ and $y = *_2$,*

1. *for all* $tx \xleftarrow{\mathbb{A}_h(x,y),\ (f_i)_i} ty$: $\left(\bigvee_i f_i\right) \wedge \mathbb{A}_h(x,y) = \bigvee_i \left(f_i \wedge \mathbb{A}_h(x,y)\right)$,
2. *for all* $tx \xleftarrow{\mathbb{A}_h(x,y)} ty$, $tx \xleftarrow{g} Z \xleftarrow{f} ty$: $(g \circ f) \wedge \mathbb{A}_h(x,y) = \bigvee\limits_{z \in (\mathbb{A}_h)_Z} \left(g \wedge \mathbb{A}_h(x,z)\right) \circ \left(f \wedge \mathbb{A}_h(z,y)\right)$.

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$

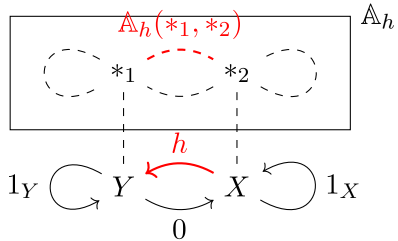


in $\mathcal{Q}$

Theorem. *$\mathbb{A}_h$ is exponentiable ~~if and~~ only if*

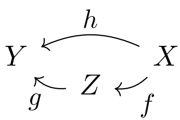
1. *for all* $Y \underset{(f_i)_i}{\overset{h}{\leftleftarrows}} X$: $\left(\bigvee_i f_i\right) \wedge h = \bigvee_i \left(f_i \wedge h\right)$,
2. *for all* $Y \xleftarrow{h} X$, $Y \xleftarrow{g} Z \xleftarrow{f} X$: $(g \circ f) \wedge h = \bigvee_{z \in (\mathbb{A}_h)_Z} \Big(g \wedge \mathbb{A}_h(*_1, z)\Big) \circ \Big(f \wedge \mathbb{A}_h(z, *_2)\Big)$.

Cartesian closedness of Cat($\mathcal{Q}$)



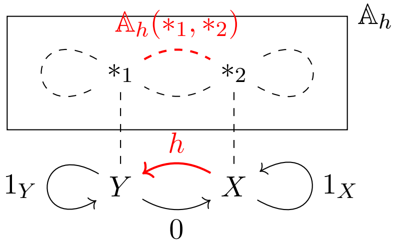
in $\mathcal{Q}$

Theorem. $\mathbb{A}_h$ *is exponentiable ~~if and~~ only if*

1. $- \wedge h \colon \mathcal{Q}(X,Y) \to \mathcal{Q}(X,Y)$ *preserves suprema,*
2. *for all* 

: $(g \circ f) \wedge h = \bigvee_{z \in (\mathbb{A}_h)_Z} \Big(g \wedge \mathbb{A}_h(*_1, z)\Big) \circ \Big(f \wedge \mathbb{A}_h(z, *_2)\Big)$.

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



in $\mathcal{Q}$

Theorem. $\mathbb{A}_h$ *is exponentiable ~~if and~~ only if*

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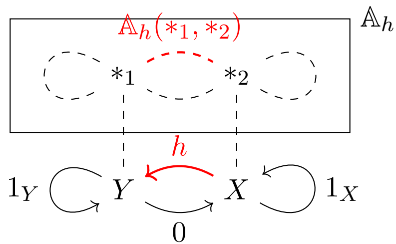
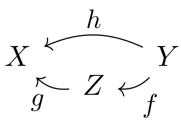
$$= \begin{cases} 0_{X,Y} & \text{if } X \neq Z \neq Y \text{ (so } (\mathbb{A}_h)_Z = \emptyset\text{)}, \\ (g \wedge h) \circ (f \wedge 1_X) & \text{if } X = Z \neq Y \text{ (so } (\mathbb{A}_h)_Z = \{*_2\}\text{)}, \\ (g \wedge 1_Y) \circ (f \wedge h) & \text{if } X \neq Z = Y \text{ (so } (\mathbb{A}_h)_Z = \{*_1\}\text{)}, \\ \Big((g \wedge h) \circ (f \wedge 1_X)\Big) \vee \Big((g \wedge 1_Y) \circ (f \wedge h)\Big) & \text{if } X = Z = Y \text{ (so } (\mathbb{A}_h)_Z = \{*_1, *_2\}\text{)}. \end{cases}$$

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



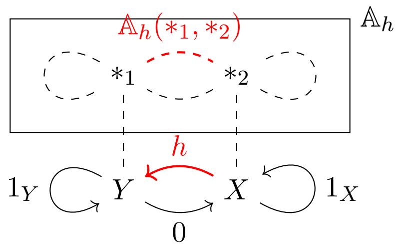
in $\mathcal{Q}$

Theorem. *Every $\mathbb{A}_h$ is exponentiable ~~if and~~ only if*

1. *every $- \wedge h\colon \mathcal{Q}(X,Y) \to \mathcal{Q}(X,Y)$ preserves suprema,*
2. *for all*  *:*

$$(g \circ f) \wedge h = \begin{cases} 0_{X,Y} & \textit{if } X \neq Z \neq Y, \\ (g \wedge h) \circ (f \wedge 1_X) & \textit{if } X = Z \neq Y, \\ (g \wedge 1_Y) \circ (f \wedge h) & \textit{if } X \neq Z = Y, \\ \Big((g \wedge h) \circ (f \wedge 1_X)\Big) \vee \Big((g \wedge 1_Y) \circ (f \wedge h)\Big) & \textit{if } X = Z = Y. \end{cases}$$

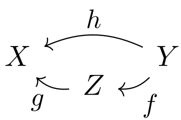
Cartesian closedness of Cat($\mathcal{Q}$)



in $\mathcal{Q}$

Theorem. *Every $\mathbb{A}_h$ is exponentiable ~~if and~~ only if*

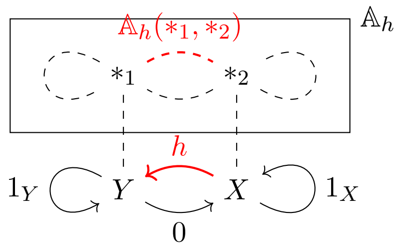
1. *every $\mathcal{Q}(X, Y)$ is localic,*
2. *for all*



:

$$(g \circ f) \wedge h = \begin{cases} 0_{X,Y} & \text{if } X \neq Z \neq Y, \\ (g \wedge h) \circ (f \wedge 1_X) & \text{if } X = Z \neq Y, \\ (g \wedge 1_Y) \circ (f \wedge h) & \text{if } X \neq Z = Y, \\ \Big((g \wedge h) \circ (f \wedge 1_X)\Big) \vee \Big((g \wedge 1_Y) \circ (f \wedge h)\Big) & \text{if } X = Z = Y. \end{cases}$$

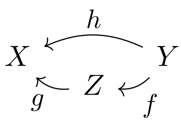
Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



in $\mathcal{Q}$

Theorem. $\mathsf{Cat}(\mathcal{Q})$ *is Cartesian closed ~~if and~~ only if*

1. *every* $\mathcal{Q}(X,Y)$ *is localic,*
2. *for all*



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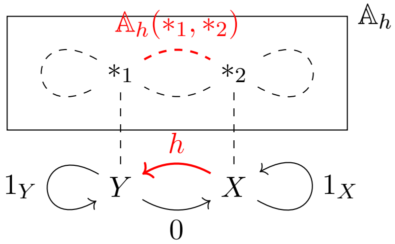
Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



in $\mathcal{Q}$

Theorem. $\mathsf{Cat}(\mathcal{Q})$ *is Cartesian closed if and only if*

1. *every* $\mathcal{Q}(X,Y)$ *is localic,*
2. *for all* $X \xleftarrow{h} Y$, $X \xleftarrow{g} Z \xleftarrow{f} Y$ *:*

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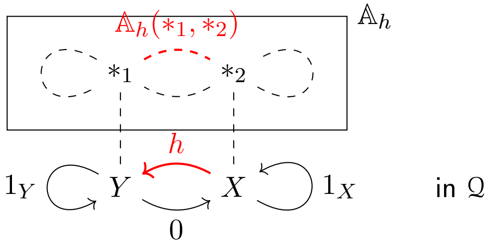

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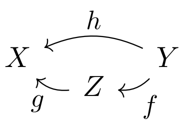
2. *for all*  *:*

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Proof: "case analysis" for the exponentiability of any $\mathcal{Q}$-category. □

More examples

For a quantale $Q = (Q, \circ, 1)$, $\mathsf{Cat}(Q)$ is Cartesian closed if and only if the underlying suplattice of Q is a locale and

$$\text{for all } a, b, c \in Q: \quad (a \circ b) \wedge c = \Big((a \wedge c) \circ (b \wedge 1)\Big) \vee \Big((1 \wedge a) \circ (b \wedge c)\Big),$$

because the other conditions in the Theorem are void.

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because the other conditions in the Theorem are void.

If Q is an integral quantale (meaning that $1 = \top$), this further simplifies to

$$\text{for all } a, b, c \in Q\text{:} \quad (a \circ b) \wedge c = ((a \wedge c) \circ b) \vee (a \circ (b \wedge c)).$$

This condition was first observed in [Lai and Luo, 2025], but only for so-called complete subquantales of continuous t-norms, and with a different proof.

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Some quantale examples:

1. whenever $a \circ b = a \wedge b$ in Q (i.e. $\mathcal{Q}$ is a locale),
2. for $Q = \{0, \frac{1}{2}, 1\}$ with natural order and multiplication $x \circ y = \max\{x + y - 1, 0\}$ (in [Lai and Zhang, 2016] with an *ad hoc* proof),
3. for $\mathcal{Q} = [0, 1]$ with natural order and multiplication $x \circ y = \begin{cases} 0 & \text{if } x, y \leq \frac{1}{2} \\ x \wedge y & \text{otherwise} \end{cases}$ (this is a *left-continuous* t-norm),
4. the only *continuous* t-norm satisfying the above condition is the Gödel t-norm, i.e. $\mathcal{Q} = [0, 1]$ with natural order and $x \circ y = x \wedge y$ (see [Lai and Zhang, 2016]).

More examples

Some quantaloidal examples:

1. The conditions in the Theorem are stable under coproducts (but not under splitting of idempotents nor the construction of diagonals), so any coproduct of quantales satisfying the conditions is a quantaloid satisfying the conditions

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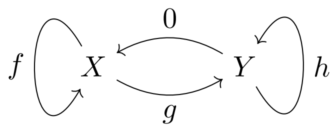
2. There are quantaloids that satisfy the conditions that are *not* a coproduct of quantales. For example, let $L = (L, \bigvee, \wedge, \top)$ be a locale, take $u, v \in L$ and define $\mathcal{Q}$ as follows:



all $f \leq u$
all $g \leq u \wedge v$
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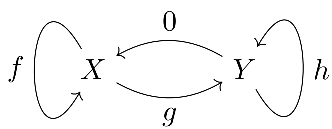

all $f \leq u$
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3. The free quantaloid $\mathcal{PC}$ on a (small) category $\mathcal{C}$ is given by:

$(\mathcal{PC})_0 = \mathcal{C}_0$,
$\mathcal{PC}(X, Y) = \mathcal{P}(\mathcal{C}(X, Y))$ with $\bigcup$ as suprema,
$1_X = \{1_X\}$ and $G \circ F = \{g \circ f \mid g \in G, f \in F\}$.

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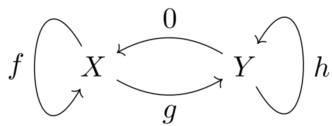

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   $\mathcal{PC}$ is always locally localic, and satisfies the conditions in the Theorem if and only if

   when two morphisms compose in $\mathcal{C}$, then at least one of them is an identity.

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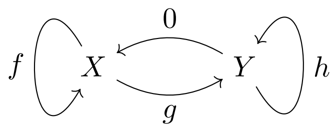

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   when two morphisms compose in $\mathcal{C}$, then at least one of them is an identity.

   In particular, the only free quantale $\mathcal{P}M$ (on a monoid $(M, \circ, 1)$) satisfying this condition is when $M = \{*\}$ (and so $\mathcal{P}M = (\{0, 1\}, \vee, \wedge, 1)$).

   (This corrects a mistake in an example in [Clementino, Hofmann and Stubbe, 2009].)

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