Equ-saturating categories

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Outline

Normal subobjects

Syntactic equivalence relations

Generalized syntactic equivalence relations

Equ-saturating categories

Equ-saturating protomodular categories

The pointed protomodular and additive cases

The fibers $\mathrm{Cat}_X\mathbb{E}$

Outline

In a finitely complete category $\mathbb{E}$, a monomorphism $u : U \rightarrowtail X$ is normal to an internal equivalence relation R on X when we get $u^{-1}R = \nabla_U$ (the undiscrete equivalence relation) and when moreover, in the induced diagram:

$$\begin{array}{ccc} U \times U & \overset{\bar{u}}{\rightarrowtail} & R \\ p_0^U \downarrow \uparrow \downarrow p_1^U & & d_0^R \downarrow \uparrow \downarrow d_1^R \\ U & \underset{u}{\rightarrowtail} & X \end{array}$$

the square indexed by 0 (or equivalently by 1) is a pullback (= we have a discrete fibration).

- In any protomodular category, a monomorphism u is normal to at most one equivalence relation, as it is the case in the category Gp of groups.
- See also B+Metere (2021) for many other aspects of normal subobjects in any category $\mathbb{E}$.
- In any variety $\mathbb{V}$, U normal to R is equivalent to: $\forall (v,t) \in U \times X,\ [t \in U \iff tRv]$

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In this talk, normal will mean ”normal to some R”, and we shall be interested in their caracterizations without any mention of R.

- Well known: in Gp of groups, we have: $u : U \rightarrowtail X$ normal if and only if: $\forall (v,t) \in U \times X,\ tvt^{-1} \in U$.

- Less known: in Mon of monoids, we have: $u : U \rightarrowtail X$ normal if and only if: $\forall (v,x,y) \in U \times M \times M,\ xvy \in U \iff xy \in U$.

- **Proof.**
if U is normal to some R, we get $[v \in U \iff vR1]$, whence $\forall (v,x,y) \in U \times M \times M$:
$xvyRxy$, and $[xvy \in U \iff xy \in U]$. □

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Let M be a monoid and W any non-empty part of M.

Definition

Call syntactic relation associated with W the relation R_W defined by:
$mR_Wn \iff \forall(x,y) \in M \times M,\ [xmy \in W \iff xny \in W]$

- **Proposition**

  Given any non-empty W, the relation R_W is an internal equivalence relation in Mon *whose class of the unit* 1 *is the monoid $\overline{W}$ defined by:*
  $\overline{W} = \{m \in M / \forall(x,y) \in M \times M,\ xmy \in W \iff xy \in W\}$

  - When $1 \in W$, we get $\bar{W} \subset W$.
  - Then we get $\bar{W} = W$, when $\forall(v,x,y) \in W \times M \times M;\ xmy \in W \iff xy \in W$.
  - Accordingly, in Mon, any normal submonoid $u : U \rightarrowtail M$ (as described above) is normal to R_U.

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Corollary

Let G be a group and $u : U \rightarrowtail G$ a normal submonoid.
Then G is a subgroup.
In other words, Gp *is stable in* Mon *under normal submoids.*

- **Proof.**
  If $1 \in U$ and U normal, then
  $1 = v^{-1}.v.1 \in U \iff v^{-1}.1 = v^{-1} \in U.$ □

  - Now, if $u : U \rightarrowtail X$ is normal submonoid, then certainly the syntactic equivalence relation R_U must have an extremal position among the equivalence relations R to which U is normal:

- **Theorem**
  Let U be a normal submonoid of M. Then the syntactic equivalence relation R_U is the largest equivalence relation S on M in Mon *to which U is normal.*

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  Let U be a normal submonoid of M. Then the syntactic equivalence relation R_U is the largest equivalence relation S on M in Mon *to which U is normal.*

However we have not reached the strongest *universal property* of the syntactic equivalence relation R_U; for that we need the notion of saturated monomorphism between equivalence relations.

- **Definition**

  A monomorphism $u : U \rightarrowtail X$ is said to be saturated w.r. to an equivalence relation R on X when the cartesian map above u with codomain R:

  is fibrant i.e. a discrete fibration.

  - In set-theoretical terms, this means that $u^{-1}R$, when it is non-empty, is a union of equivalence classes.
  - A normal monomorphism is then a special case of saturated monomorphism.

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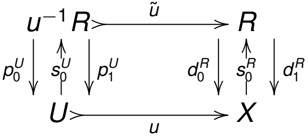

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Saturation has a very strong property:

Lemma

Given any category $\mathbb{E}$ and any monormorphism $u : U \rightarrowtail X$,
1) if u is saturated w.r. to R, then u is saturated w.r. to any equivalence relation $S \subset R$;
2) if u is normal to R, then u is saturated w.r. to any equivalence relation $S \subset R$.

- **Theorem**

  When U is a submonoid of M, the syntactic equivalence relation R_U on M is such that:
  1) the inclusion $u : U \rightarrowtail M$ is saturated w.r. to R_U;
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Actually the definition of a syntactic equivalence relation with its characteristic universal property can be extended to any variety $\mathbb{V}$, see Almeida (Finite semigroups and Universal Algebra, 1994).

- **Definition**
  Given any algebra $A \in \mathbb{V}$ and any non-empty subset L, call syntactic relation associated with L the relation R_L defined by mR_Ln when: for any $(a_1, \cdots, a_n) \in A^n$ and any term $\tau(x_0, x_1, \cdots, x_n)$ of $\mathbb{V}$, we get: $\tau(m, a_1, \cdots, a_n) \in L \iff \tau(n, a_1, \cdots, a_n) \in L$.

- **Theorem**
  When L is a subalgebra of A, the relation R_L is a congruence on A in $\mathbb{V}$ is such that:
  1) the inclusion $l : L \rightarrowtail A$ is saturated w.r. to R_L;
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Semi-rings

As for monoids, the definition of the synctatic relation can be reduced:

Proposition

Let $(A, +, .)$ *be a semi-ring. The* syntactic relation *associated with a subset* W *is the relation defined by:* mR_Wn *if and only if the following conditions hold:*

1) $\forall x \in A,\ [x + m \in W \iff x + n \in W]$;

2) $\forall (x, y) \in A \times A,\ [x + my \in W \iff x + ny \in W]$;

3) $\forall (x, y) \in A \times A,\ [x + ym \in W \iff x + yn \in W]$;

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- **Proposition**

  A subsemiring W *of* A *is normal if and only if for any* $v \in W$*, the following conditions hold:*

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Skew braces

Braces are algebraic structures which were introduced by Rump (2007) as producing set-theoretical solutions of the Yang-Baxter equation in response to a general incitement of Drinfeld to investigate this equation from a set-theoretical perspective.

- i.e. a pair (X, r) of a set X and an application $r : X \times X \to X \times X$, such that:

$$(r \times 1_X).(1_X \times r).(r \times 1_X) = (1_X \times r).(r \times 1_X).(1_X \times r)$$

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Later on, Guarnieri and Vendramin (2017) generalized this notion with the structure of left skew brace which again generates solutions of the Yang-Baxter equations.

- **Definition** *A left skew brace is a set X endowed with two group structures $(X, *, \circ)$ subject to a unique axiom:* $a \circ (b * c) = (a \circ b) * a^{-*} * (a \circ c)$ (1) *where a^{-*} denotes the inverse for the law $*$.*
- The simplest examples are the following ones: starting with any group $(G, *)$, then $(G, *, *)$ and $(G, *, *^{op})$ are left skew braces.
- We denote by SkB the category of left skew braces which is obviously a variety in the sense of Universal Algebra.

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- **Definition** *A left skew brace is a set X endowed with two group structures $(X, *, \circ)$ subject to a unique axiom:*
  $a \circ (b * c) = (a \circ b) * a^{-*} * (a \circ c)$ (1)
  where a^{-*} denotes the inverse for the law $*$.
- The simplest examples are the following ones: starting with any group $(G, *)$, then $(G, *, *)$ and $(G, *, *^{op})$ are left skew braces.
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SKB is a protomodular category, since it is a full subcategory of DiGp *(digroups).*

- As any variety, SkB allows the definition of the syntactic equivalence relation and the characterization of normal monomorphisms, even though, in this case, the list of its axioms defining it does not seem to be reductible to a finite one.
- However, we have a finite list to characterize normal monomorphism:

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A subobject $u : (U, *, \circ) \rightarrowtail (X, *, \circ)$ *is normal in the category* SkB *if and only if the three following conditions hold:*
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Outline

The idea here will be to refined the definition of the syntactic relation:

Definition

Given any algebra A in a variety $\mathbb{V}$, any subalgebra L and any congruence S on L, call generalized syntactic relation associated with the pair (L, S) the relation R_L^S on A defined by $mR_L^S n$ when:

for any $(a_1, \cdots, a_n) \in A^n$ and any term $\tau(x_0, x_1, \cdots, x_n)$ of $\mathbb{V}$, we get:

$[\tau(m, a_1, \cdots, a_n) \in L \iff \tau(n, a_1, \cdots, a_n) \in L]$

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If U is a subalgebra of A in a variety $\mathbb{V}$ and S a congruence on U, the relation R_U^S is a congruence on A in $\mathbb{V}$ such that:
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Definition

A category $\mathbb{E}$ is said to be Equ-*saturating when, given any pair (u, S) of a monomorphism $u : U \rightarrowtail X$ and an internal equivalence relation S on U, the set of saturated monomorphisms above u with a domain smaller than S has a supremum. We shall denote the codomain of this supremum by $\forall_u S$ and call it the saturating equivalence relation associated with S.*

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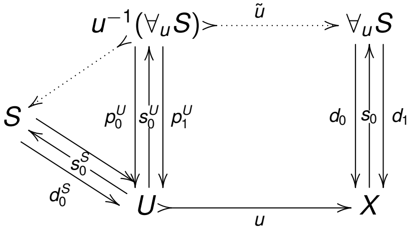

where the horizontal monomorphism of equivalence relations is saturated.

Outline

Proposition

Let $\mathbb{E}$ be a Equ-*saturating category and $u : U \rightarrowtail X$ any mono. Then an equivalence R on U is the domain of a saturated subobject above u if and only if $R = u^{-1}(\forall_u R)$. Then $\forall_u R$ is the largest saturated equivalence relation on X above u with domain R.*

- **Corollary**

  Let $\mathbb{E}$ be a Equ-*saturating category and $u : U \rightarrowtail X$ any mono. If u is normal in $\mathbb{E}$, the set of equivalence relations S on X w.r. to which u is normal has a supremum.*

Proof.
Apply the previous proposition to $R = \nabla_U$ □

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  Let $\mathbb{E}$ be a Equ-*saturating category and $u : R \rightarrowtail S$ any monomorphism in* Equ$\mathbb{E}$ *above $u : U \rightarrowtail X$. Then $\overline{S} = S \cap (\forall_u R)$ is the largest of the equivalence relations T on X such that u is saturated w.r. to T, such that $u^{-1}(T) \subset R$ and $T \subset S$.*

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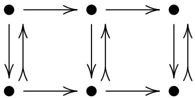
The Equ-saturating axiom has very good stability properties:

Proposition

Let $\mathbb{E}$ *be a* Equ-*saturating category.*
Then any slice category $\mathbb{E}/Y$, *any coslice category* $Y/\mathbb{E}$ *is* Equ-*saturating as well.*
Accordingly any fiber $\mathrm{Pt}_Y\mathbb{E}$ *of split epimorphisms above* Y *and any fiber* $RGh_Y\mathbb{E}$ *of reflexive graphs on the object* Y *is* Equ-*saturating.*

Outline

Recall that a (finitely complete category) $\mathbb{E}$ is protomodular when, given any pair of commutative squares of vertical split epimorphisms:



the right hand side square is a pullback as soon as so are the left hand side one and the whole rectangle.

- Any protomodular category is a Mal'tsev one; so there is a notion of centralization of pairs (R, S) of equivalence relations ($[R, S] = 0$) on an object X.
- Recall that in a protomodular category $\mathbb{E}$, any fibrant morphism of equivalence relations $f : S \to R$ is cocartesian (B+Gran 2002, B+Metere 2021).

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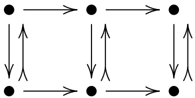

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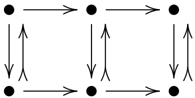

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- **Proof.**

Suppose $[R, S] = 0$. Then, any square in the here below left hand side diagram being a pullback, the morphism $s_0^R : S \rightarrowtail W$ is fibrant.

Conversely, when the morphism $s_0^R : S \rightarrowtail W$ is cocartesian, it induces the morphisms $d_i^R : W \to S$ which are fibrant as well, since $\mathbb{E}$ is proto. □

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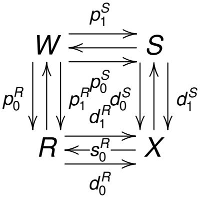

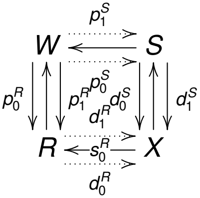

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Theorem

In any Equ*-saturating protomodular category* $\mathbb{E}$*, any equivalence relation* R *has a centralizer. So, it is the case for any protomodular variety* $\mathbb{V}$*.*

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Let R be an equivalence relation on the object X, then consider the equivalence relation $\forall_{s_0^R}\nabla_X$ on R and the induced saturated monomorphism above s_0^R:

According to the previous proposition, we get $[R, \Sigma] = 0$. Now, if we have $[R, S] = 0$ for some S with some W as in the previous proposition, the universal property of $\forall_{s_0^R}\nabla_X$ implies the inclusion $W \subset \forall_{s_0^R}\nabla_X$ and thus $S \subset \Sigma$. $\square$

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Outline

In the pointed protomodular context, there is a direct translation of the Equ-saturating property:

Proposition

Let $\mathbb{E}$ be a pointed protomodular category. It is Equ*-saturating if and only if, given any monomorphism $u : U \rightarrowtail X$, any normal subobject $v : V \rightarrowtail U$ determines a largest normal subobject $w : W \rightarrowtail X$ such that $W \subset V$.*

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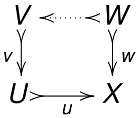
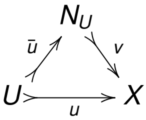
- 

J. Gray (2014) introduces the notion of abstract normalizer of a mono u in a pointed protomodular category, namely the largest subobject N_U of X in which u is normal:



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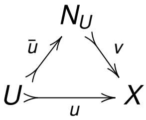

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Let $\mathbb{E}$ be an exact Equ*-saturating pointed protomodular category. Any commutative square of vertical split epimorphisms with horizontal regular epimorphisms (h, k) has a universal dotted decomposition where the left hand side square is a pullback above a regular epimorphism $\bar{h}$, as on the following left hand side diagram:*

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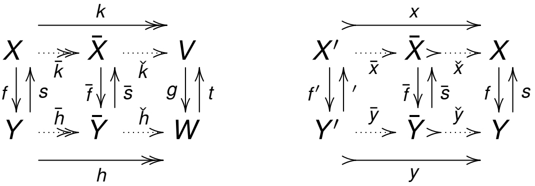
- 

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Outline

There are other examples of Equ-saturating categories.

Let Cat$\mathbb{E}$ be the category of internal categories, Grd$\mathbb{E}$ the subcategory of internal groupoids and $(\ )_0 : \mathrm{Cat}\mathbb{E} \to \mathbb{E}$ the associated fibration whose cartesian maps are the fully faithful internal functors.

We denote by $\mathrm{Cat}_X\mathbb{E}$ and $\mathrm{Grd}_X\mathbb{E}$ the respective fibers. The respective fibers above the singleton 1 are Mon$\mathbb{E}$ and Gp$\mathbb{E}$. Similarly to Gp$\mathbb{E}$, any fiber $\mathrm{Grd}_X\mathbb{E}$ is protomodular.

- **Proposition**

  The fibers $\mathrm{Cat}_X\mathbb{E}$ are Eq-saturating in the three following cases.
  1) $\mathbb{E} = \mathrm{Set}$;
  2) $\mathbb{E}$ is any Mal'tsev category;
  3) $\mathbb{E}$ is a Equ-saturating Gumm category in the sense of B+Gran,2004;
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Let $\gamma : \mathbb{C} \rightarrowtail \mathbb{D}$ be a bijective on objects inclusion ($C_0 = X = D_0$), and S an internal equivalence relation on $\mathbb{C}$ in the fiber Cat_X, namely an equivalence relation on parallel pairs of morphims. Define the following generalized syntactic relation $\forall_{\mathbb{C}} S$ on the parallel pairs $(f, f') : a \rightrightarrows b$ of morphisms in $\mathbb{D}$:
$f(\forall_{\mathbb{C}} S)f'$ when $\forall (g, k)$ pair of maps in $\mathbb{D}$ with $dom(g) = b$ and $cod(k) = a$,

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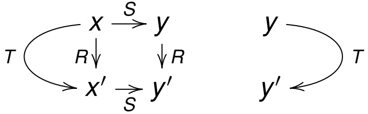
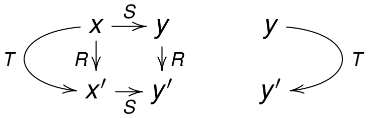


- Then $\mathsf{Cat}_X\mathbb{E}$ is Equ-saturating,

  and $\mathsf{Grd}_X\mathbb{E}$ is a Equ-saturating protomodular category.

Case 2: $\mathbb{E}$ Mal'tsev (any reflexive relation is an equivalence relation, Carboni+Lambek+Peddichio, 1990).
Then $\mathsf{Cat}_X\mathbb{E} = \mathsf{Grd}_X\mathbb{E}$ is a protomodular Equ-saturating category.

- Case 3: $\mathbb{E}$ Equ-saturating and Gumm. Shifting Lemma: given any triple (T, S, R) of equivalence relations on an object X such that $R \cap S \subset T$, *the following left hand side situation implies the right hand side one:*



- Then $\mathsf{Cat}_X\mathbb{E}$ is Equ-saturating,

  and $\mathsf{Grd}_X\mathbb{E}$ is a Equ-saturating protomodular category.