Projective crossed modules in semi-abelian categories

Maxime Culot

Université catholique de Louvain (Belgium)

September 2025

Dédié à l'occasion des "nonante" et des "quatre-vingt-dix" ans de Mme Andrée Ehresmann

Motivation

At the SIC in Lille (2024) and at CT (2024), I presented:

Theorem [CRVdL25]

Let $\mathcal{C}$ be a semi-abelian category with enough projectives that satisfy **Condition** (P). Let $\mathcal{E}$ be a semi-abelian category,
and let $F \colon \mathcal{C} \to \mathcal{E}$ be a protoadditive functor (i.e. it preserves split short exact sequences) that preserves binary coproducts and proper morphisms (i.e. the cokerel-kernel factorization).
Then the left-derived functors of F are defined as in the abelian context.

Motivation

At the SIC in Lille (2024) and at CT (2024), I presented:

Theorem [CRVdL25]

Let $\mathcal{C}$ be a semi-abelian category with enough projectives that satisfy **Condition** (P). Let $\mathcal{E}$ be a semi-abelian category,
and let $F\colon \mathcal{C} \to \mathcal{E}$ be a protoadditive functor (i.e. it preserves split short exact sequences) that preserves binary coproducts and proper morphisms (i.e. the cokerel-kernel factorization).
Then the left-derived functors of F are defined as in the abelian context.

Plan of today

- Recall some intuition for this theorem;
- Give an example: $\mathsf{XMod}(\mathcal{V})$ where $\mathcal{V}$ is a semi-abelian variety satisfying (P).

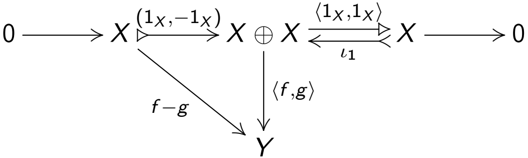
Table of contents

Classical left derived functor

In the usual abelian context, if we consider $F\colon \mathcal{C} \to \mathcal{E}$ an additive functor between two abelian categories with $\mathcal{C}$ has enough projectives, then for any $n \in \mathbb{Z}$, we can define the ***n*-th left derived functor of *F*** by setting

$$\mathrm{L}_n(F)(X) := \mathsf{H}_n(F(C_X))$$

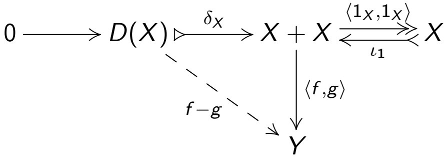
for all object $X \in \mathcal{C}$ and where C_X is a projective resolution of the object X. In such a context, we can encode the difference of two parallel morphisms via



This is important to define the notion of **homotopy** between chain morphisms. Indeed, we can always see the usual equations with only differences of parallel morphisms.

First step to non-additive version

In a pointed category $\mathcal{C}$ with kernels and binary coproducts, the previous diagram can be reformulated as



By setting, if we let $f - g := \langle f, g \rangle \delta_X \colon D(X) \to Y$.

Observation

If X is a projective object, then $X + X$ is as well. We also want that $D(X)$ (the kernel part of this split short exact sequence) to be projective!

Second step - (P)

Definition [CRVdL25]

We say that a pointed category satisfies **Condition (P)** when *the class of projective objects is closed under protosplit subobjects*: given $K \leq X$ a kernel of a split epimorphism with domain X, if X is projective, then K is projective.

$$K \overset{k}{\rightarrowtail} X \underset{s}{\overset{f}{\rightleftarrows}} Y$$

Some examples

- Any abelian category since $X \cong K \oplus Y$;
- Any Schreier variety (e.g. Gp, Ab, Mod_R if R is *P.I.D.*, $\mathrm{Lie}_{\mathbb{K}}$ if $\mathbb{K}$ a field);
- Not abelian and not Schreier: $\mathrm{Lie}_{\mathbb{K}}$ where $\mathbb{K}$ is a commutative ring, XMod(Gp), XMod($\mathcal{V}$) **for a semi-abelian variety of algebras $\mathcal{V}$ satisfying itself Condition** (P) [CRVdL25, Cul25]

Why (P) is important?

A priori, the condition (P) seems too strong ... However, we can say:

- a homological category with binary coproducts satisfies (P) if and only if the "*Half Horseshoe Lemma*" holds;
- as a consequence of the previous point: with (P), we can expect a long exact sequence in homology relating the derived functors of the objects in a given short exact sequence;
- with the assumption of the previous theorem (i.e. including (P)). Let $X \in \mathrm{dom}(F)$, $C(X)$ a chain resolution and $\mathbb{S}(X)$ a simplicial resolution

$$\mathsf{H}_n(F(C(X))) = \mathsf{H}_n(F(\mathrm{N}(\mathbb{S}(X)))) = \mathsf{H}_n(\mathrm{N}(F(\mathbb{S}(X))))$$

where N is the Moore normalization functor.
Moreover, suppose we deal with varieties of algebras. In that case, we have an isomorphism with $\mathsf{H}_{n+1}(-, F)_{\mathbb{G}}$ the $(n+1)$st simplicially derived functor of F in the sense of Barr–Beck [BB69, EVdL04].

Crossed modules over groups - XMod

J.H.C. Whitehead introduced the notion of **crossed module** (1949):

Definition

A **crossed module** (X, A, ∂) consists of a homomorphism of the group $\partial\colon X \to A$ (the **boundary map**), together with a group action of A on X (denoted ${}^{a}x$) satisfying

- $\partial({}^{a}x) = a\partial(x)a^{-1}$ (**precrossed module condition**);
- ${}^{\partial(x)}y = xyx^{-1}$ (**Peiffer condition**)

for all $y, x \in X$ and $a \in A$.

Definition

A **crossed module morphism** $f = (f_X, f_A)\colon (X, A, \partial) \to (X', A', \partial')$ is a pair of group homomorphism $f_X\colon X \to X'$ and $f_A\colon A \to A'$, such that

- $\partial' f_X = f_A \partial$ ("compatibility condition w.r.t. the boundary maps");
- for all $a \in A$ and $x \in X$: $f_X({}^{a}x) =^{f_A(a)} f_X(x)$ ("compatibility condition w.r.t. the actions").

Classical internal actions and equivalences of categories

Let $A, X \in \mathcal{C}$ (a semi-abelian category), an internal action of A on X is defined as an algebra over a monad $A \flat X := \operatorname{Ker}(\langle 1_A, 0 \rangle A + X \to A)$.

Motivation for the next definitions

$$\begin{array}{ccccc}
\mathsf{SSES}(\mathcal{C}) & \hookleftarrow & \mathsf{RG}(\mathcal{C}) & \hookleftarrow & \mathsf{Grpd}(\mathcal{C}) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathsf{Act}(\mathcal{C}) & \hookleftarrow & \mathsf{PXMod}(\mathcal{C}) & \hookleftarrow & \mathsf{XMod}(\mathcal{C})
\end{array}$$

Classical internal actions and equivalences of categories

Let $A, X \in \mathcal{C}$ (a semi-abelian category), an internal action of A on X is defined as an algebra over a monad $A\flat X := \mathrm{Ker}\,(\langle 1_A, 0\rangle A + X \to A)$.

Motivation for the next definitions

$$\begin{array}{ccccc}
\mathrm{SSES}(\mathcal{C}) & \hookleftarrow & \mathrm{RG}(\mathcal{C}) & \hookleftarrow & \mathrm{Grpd}(\mathcal{C}) \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
\mathrm{Act}(\mathcal{C}) & \hookleftarrow & \mathrm{PXMod}(\mathcal{C}) & \hookleftarrow & \mathrm{XMod}(\mathcal{C})
\end{array}$$

$$0 \longrightarrow X \overset{k}{\rightarrowtail} E \underset{c}{\overset{d}{\underset{\iota}{\leftrightarrows}}} A \longrightarrow 0$$

- $(X, E, A, k, d, \iota) \in \mathrm{SSES}(\mathcal{C})$ when $k = \ker(d)$ and $d\iota = 1_A$;
- $(X, E, A, k, d, c, \iota) \in \mathrm{RG}(\mathcal{C})$ when $(X, E, A, k, d, \iota) \in \mathrm{SSES}(\mathcal{C})$ and $c\iota = 1_A$;
- $(X, E, A, k, d, c, \iota) \in \mathrm{Grpd}(\mathcal{C})$ when the reflexive graph associated admits an internal groupoid structure.

Internal action core

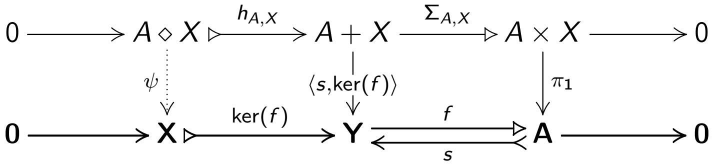
In this talk, I will not use the "classical internal action" (i.e. $A \flat X$). Today, we can see an action as the (**bold**) bottom split short exact sequence

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A \diamond X & \xrightarrow{h_{A,X}} & A + X & \xrightarrow{\Sigma_{A,X}} & A \times X & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \langle s, \ker(f) \rangle} & & \Big\downarrow{\scriptstyle \pi_1} & & \\
\mathbf{0} & \longrightarrow & \mathbf{X} & \xrightarrow{\ker(f)} & \mathbf{Y} & \underset{s}{\overset{f}{\rightleftarrows}} & \mathbf{A} & \longrightarrow & \mathbf{0}
\end{array}
$$

where $A \diamond X$ is called the **binary cosmash product of** A **and** X.

Internal action core

In this talk, I will not use the "classical internal action" (i.e. $A\flat X$). Today, we can see an action as the (**bold**) bottom split short exact sequence



where $A \diamond X$ is called the **binary cosmash product of** A **and** X.
The original definition of **internal crossed modules** (G. Janelidze [Jan03]) is expressed in terms of an algebra over the monad $A\flat -$.
Today, I make use of "ψ" (which codifies the above split short exact sequence via a semi-direct product construction), which leads to an alternative characterization (M. Hartl and T. Van der Linden [HVdL13]).
This approach leads to shorter proofs.

Definition of crossed modules

Definition

A **internal crossed module** is given by $(X \in \mathcal{C}, A \in \mathcal{C}, \partial \colon X \to A, \psi \colon A \diamond X \to X)$ where ψ is an **action core** and where ∂ is called the **boundary morphism**, satisfying three conditions:

$$
\begin{array}{ccc}
A \diamond X & \xrightarrow{1_A \diamond \partial} & A \diamond A \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \overline{\chi_A}} \\
X & \xrightarrow[\partial]{} & A
\end{array}
\qquad
\begin{array}{ccc}
X \diamond X & \xrightarrow{\partial \diamond 1_X} & A \diamond X \\
{\scriptstyle \overline{\chi_X}}\downarrow & & \downarrow{\scriptstyle \psi} \\
X & \xrightarrow[1_X]{} & X
\end{array}
\qquad
\begin{array}{ccc}
A \diamond X \diamond X & \xrightarrow{\psi^{A,X}_{1,2}} & X \\
{\scriptstyle 1_A \diamond \partial \diamond 1_X}\downarrow & & \downarrow{\scriptstyle 1_X} \\
A \diamond A \diamond X & \xrightarrow[\psi^{A,X}_{2,1}]{} & X
\end{array}
$$

where $\overline{\chi_A} \coloneqq \langle 1_A, 1_A \rangle h_{A,A}$ is the **conjugation action core**, $\psi^{A,X}_{1,2} \coloneqq \psi S^{A,X}_{1,2}$ and $\psi^{A,X}_{2,1} \coloneqq \psi S^{A,X}_{2,1}$.
The diagrams are called, respectively, the **precrossed module**, the **Peiffer condition** and the **ternary commutator condition**.

Projective crossed modules

Proposition [CCRG02]

In XMod(Gp), if P is a projective group and Q is a projective P-group then the inclusion morphism $Q \to Q \rtimes P$ is a projective crossed module.

Projective crossed modules

Proposition [CCRG02]

In XMod(Gp), if P is a projective group and Q is a projective P-group then the inclusion morphism $Q \to Q \rtimes P$ is a projective crossed module.

Theorem [Cul25]

If P is a projective object in $\mathcal{C}$ and if the split extension

$$0 \longrightarrow Q \overset{\partial'}{\rightarrowtail} Z \underset{s}{\overset{p}{\rightleftarrows}} P \longrightarrow 0$$

is a projective object in the category of split extensions of P,
then the kernel ∂', viewed as an internal crossed module,
is a projective object in XMod($\mathcal{C}$).

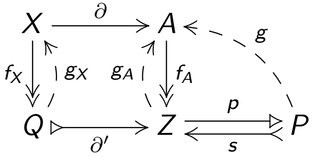
Any kernel can be endowed with a (unique) crossed module structure: the action is the **conjugation action core** (denoted $\overline{\chi}$), and the boundary map is the **inclusion** ∂'.

Sketch of the proof

What are the morphisms in $\mathrm{XMod}(\mathcal{C})$?

An **internal crossed module morphism** $(f_X, f_A)\colon (X, A, \psi, \partial) \to (X', A', \psi', \partial')$ is a pair of morphisms $f_X\colon X \to X'$, $f_A\colon A \to A'$ in $\mathcal{C}$ compatible with the action cores and with the boundary morphisms.

Consider a regular epimorphism $(f_X, f_A)\colon (X, A, \phi, \partial) \to (Q, Z, \overline{\chi}, \partial')$ in $\mathrm{XMod}(\mathcal{C})$:



1. Lifting of s along f_A (P is projective);
2. A section of f_X (the bottom is projective object in $\mathrm{SSES}_P(\mathcal{C})$);
3. A section of f_A (the construction of $Z \cong Q \rtimes_\psi P$);
4. The pair of sections is a morphism in $\mathrm{XMod}(\mathcal{C})$ ("⋄" characterization).

“Size” of the proof

P. Carrasco et al. / Journal of Pure and Applied Algebra 168 (2002) 147–176 153

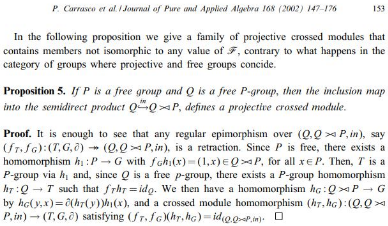
In the following proposition we give a family of projective crossed modules that contains members not isomorphic to any value of $\mathscr{F}$, contrary to what happens in the category of groups where projective and free groups concide.

Proposition 5. *If P is a free group and Q is a free P-group, then the inclusion map into the semidirect product $Q \overset{in}{\hookrightarrow} Q \rtimes P$, defines a projective crossed module.*

Proof. It is enough to see that any regular epimorphism over $(Q, Q \rtimes P, in)$, say $(f_T, f_G) : (T, G, \partial) \twoheadrightarrow (Q, Q \rtimes P, in)$, is a retraction. Since P is free, there exists a homomorphism $h_1 : P \to G$ with $f_G h_1(x) = (1, x) \in Q \rtimes P$, for all $x \in P$. Then, T is a P-group via h_1 and, since Q is a free p-group, there exists a P-group homomorphism $h_T : Q \to T$ such that $f_T h_T = id_Q$. We then have a homomorphism $h_G : Q \rtimes P \to G$ by $h_G(y, x) = \partial(h_T(y))h_1(x)$, and a crossed module homomorphism $(h_T, h_G) : (Q, Q \rtimes P, in) \to (T, G, \partial)$ satisfying $(f_T, f_G)(h_T, h_G) = id_{(Q, Q \rtimes P, in)}$. $\square$

Figure: Proof in XMod

"Size" of the proof



P. Carrasco et al. / Journal of Pure and Applied Algebra 168 (2002) 147–176 153

In the following proposition we give a family of projective crossed modules that contains members not isomorphic to any value of $\mathscr{F}$, contrary to what happens in the category of groups where projective and free groups concide.

Proposition 5. *If P is a free group and Q is a free P-group, then the inclusion map into the semidirect product $Q \overset{in}{\hookrightarrow} Q \rtimes P$, defines a projective crossed module.*

Proof. It is enough to see that any regular epimorphism over $(Q, Q \rtimes P, in)$, say $(f_T, f_G):(T,G,\partial) \twoheadrightarrow (Q, Q \rtimes P, in)$, is a retraction. Since P is free, there exists a homomorphism $h_1 : P \to G$ with $f_G h_1(x) = (1,x) \in Q \rtimes P$, for all $x \in P$. Then, T is a P-group via h_1 and, since Q is a free p-group, there exists a P-group homomorphism $h_T : Q \to T$ such that $f_T h_T = id_Q$. We then have a homomorphism $h_G : Q \rtimes P \to G$ by $h_G(y,x) = \partial(h_T(y))h_1(x)$, and a crossed module homomorphism $(h_T, h_G):(Q, Q \rtimes P, in) \to (T,G,\partial)$ satisfying $(f_T, f_G)(h_T, h_G) = id_{(Q,Q\rtimes P,in)}$. □

Figure: Proof in XMod

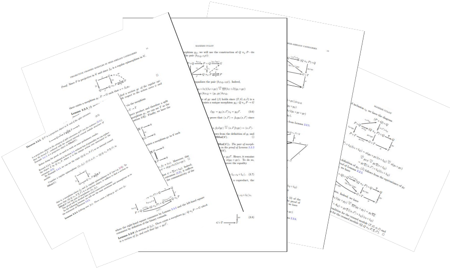
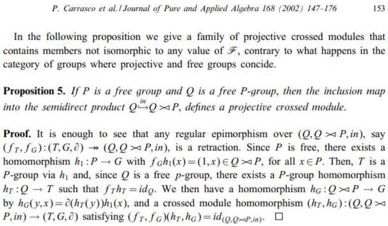


Figure: Proof in XMod($\mathcal{C}$) - 5 pages

"Size" of the proof



P. Carrasco et al. / Journal of Pure and Applied Algebra 168 (2002) 147–176 153

In the following proposition we give a family of projective crossed modules that contains members not isomorphic to any value of $\mathscr{F}$, contrary to what happens in the category of groups where projective and free groups concide.

Proposition 5. *If P is a free group and Q is a free P-group, then the inclusion map into the semidirect product $Q \overset{in}{\hookrightarrow} Q \rtimes P$, defines a projective crossed module.*

Proof. It is enough to see that any regular epimorphism over $(Q, Q \rtimes P, in)$, say $(f_T, f_G) : (T, G, \partial) \twoheadrightarrow (Q, Q \rtimes P, in)$, is a retraction. Since P is free, there exists a homomorphism $h_1 : P \to G$ with $f_G h_1(x) = (1, x) \in Q \rtimes P$, for all $x \in P$. Then, T is a P-group via h_1 and, since Q is a free p-group, there exists a P-group homomorphism $h_T : Q \to T$ such that $f_T h_T = id_Q$. We then have a homomorphism $h_G : Q \rtimes P \to G$ by $h_G(y, x) = \partial(h_T(y))h_1(x)$, and a crossed module homomorphism $(h_T, h_G) : (Q, Q \rtimes P, in) \to (T, G, \partial)$ satisfying $(f_T, f_G)(h_T, h_G) = id_{(Q, Q \rtimes P, in)}$. □

Figure: Proof in XMod

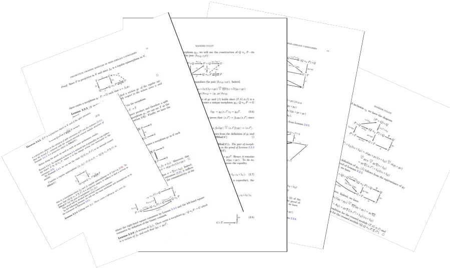


Figure: Proof in XMod($\mathcal{C}$) - 5 pages

A comment

In Gp, the proof is shorter since it satisfies *Smith is Huq* (i.e. a commutator condition). In a category satisfying (SH), some "internal structures" behave better.

Moreover, with (SH), we can "drop" the *ternary commutator condition* in the previous definition.

Free crossed modules in variety $\mathcal{V}$

Consider a semi-abelian **variety of algebras** $\mathcal{V}$ with $F_r\colon \mathsf{Set} \to \mathcal{V}$ the associated free functor. All free internal crossed modules are of the form

$$(F_r(S)\flat F_r(S), F_r(S) + F_r(S), \overline{\chi}, \kappa_{F_r(S),F_r(S)})$$

where $\kappa_{F_r(S),F_r(S)}\colon F_r(S)\flat F_r(S) \to F_r(S) + F_r(S)$, for some $S \in \mathsf{Set}$.

Free crossed modules in variety $\mathcal{V}$

Consider a semi-abelian **variety of algebras** $\mathcal{V}$ with $F_r \colon \mathrm{Set} \to \mathcal{V}$ the associated free functor. All free internal crossed modules are of the form

$$(F_r(S) \flat F_r(S), F_r(S) + F_r(S), \overline{\chi}, \kappa_{F_r(S),F_r(S)})$$

where $\kappa_{F_r(S),F_r(S)} \colon F_r(S) \flat F_r(S) \to F_r(S) + F_r(S)$, for some $S \in \mathrm{Set}$.

Corollary [Cul25]

For any non-trivial semi-abelian variety $\mathcal{V}$, the variety $\mathrm{XMod}(\mathcal{V})$ is not a Schreier variety (free objects are not stable under subobjects).

Sketch of the proof

Consider two different projectives objects P and X in $\mathcal{V}$, then

$$0 \longrightarrow P \flat X \xrightarrow{\kappa_{P,X}} P + X \underset{\iota_1}{\overset{\langle 1_P, 0 \rangle}{\rightleftarrows}} P \longrightarrow 0$$

the kernel part is projective in $\mathrm{XMod}(\mathcal{V})$ but not free since $P \neq X$.

XMod($\mathcal{V}$) satisfied (P) as soon as $\mathcal{V}$ does

XMod($\mathcal{V}$) satisfied (P) as soon as $\mathcal{V}$ does

Proposition [CRVdL25]

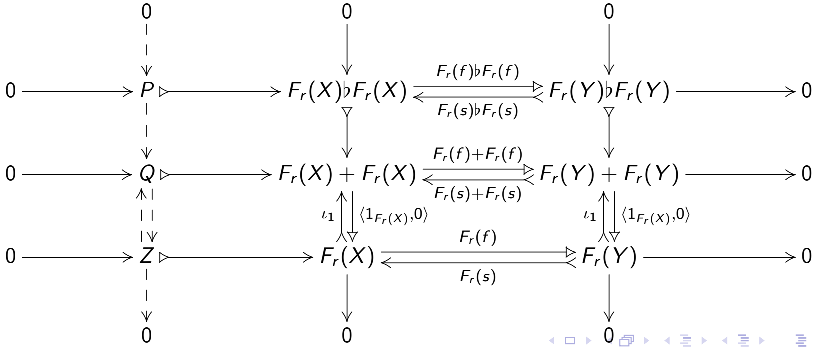
Let $\mathcal{V}$ be a pointed Mal'tsev variety of algebras, with forgetful functor $U\colon \mathcal{V} \to \mathsf{Set}$ and its left adjoint $G\colon \mathsf{Set} \to \mathcal{V}$. The variety $\mathcal{V}$ satisfies (P) if and only if the kernel of $G(f)$ (for $f\colon X \to Y$ a split epimorphism of sets) is a projective object in $\mathcal{V}$.

XMod($\mathcal{V}$) satisfied (P) as soon as $\mathcal{V}$ does

Proposition [CRVdL25]

Let $\mathcal{V}$ be a pointed Mal'tsev variety of algebras, with forgetful functor $U\colon \mathcal{V} \to \mathsf{Set}$ and its left adjoint $G\colon \mathsf{Set} \to \mathcal{V}$. The variety $\mathcal{V}$ satisfies (P) if and only if the kernel of $G(f)$ (for $f\colon X \to Y$ a split epimorphism of sets) is a projective object in $\mathcal{V}$.

In our situation, let $f\colon X \to Y$ a split epimorphism of sets with its section s



Sketch of the proof

- the middle vertical sequence and the right vertical sequence are, respectively, the free object on the set X and Y, and there are (split) short exact sequences by construction;
- the morphisms in the left-hand vertical sequence are restrictions to the kernels;
- the free objects $F_r(X)$ and $F_r(Y)$ are projective in $\mathcal{V}$, and therefore $F_r(X) + F_r(X)$ and $F_r(Y) + F_r(Y)$ are projective as well;
- since $\mathcal{V}$ satisfies (P), we have respectively that
  - the kernels $F_r(X) \flat F_r(X)$ and $F_r(Y) \flat F_r(Y)$ of the solid vertical split short exact sequences are also projective;
  - the object $Z := \ker(F_r(f))$ is projective in $\mathcal{V}$;
  - the object $Q := \ker(F_r(f) + F_r(f))$ is projective in $\mathcal{V}$;
  - the object $P := \ker(F_r(f) \flat F_r(f))$ is projective in $\mathcal{V}$;
- the left-hand vertical sequence with the dashed arrows is also a (split) short exact sequence.

We can prove that the internal crossed module $(P, Q, \overline{\chi}, k)$ is projective as a retract of a projective internal crossed module.

Non-additive derived functor of π_0

For the moment, what do we know:

- we consider a semi-abelian variety $\mathcal{V}$;
- then $\mathsf{XMod}(\mathcal{V})$ is a semi-abelian category and it is a variety of algebras (and as a result has enough projectives);
- if $\mathcal{V}$ satisfies (P) then $\mathsf{XMod}(\mathcal{V})$ as well.

Non-additive derived functor of π_0

For the moment, what do we know:

- we consider a semi-abelian variety $\mathcal{V}$;
- then $\mathsf{XMod}(\mathcal{V})$ is a semi-abelian category and it is a variety of algebras (and as a result has enough projectives);
- if $\mathcal{V}$ satisfies (P) then $\mathsf{XMod}(\mathcal{V})$ as well.

Considering now the functor $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$ defines as

$$\pi_0(X, A, \partial, \psi) := \mathsf{Coker}(\partial).$$

Non-additive derived functor of π_0

For the moment, what do we know:

- we consider a semi-abelian variety $\mathcal{V}$;
- then XMod($\mathcal{V}$) is a semi-abelian category and it is a variety of algebras (and as a result has enough projectives);
- if $\mathcal{V}$ satisfies (P) then XMod($\mathcal{V}$) as well.

Considering now the functor $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$ defines as

$$\pi_0(X, A, \partial, \psi) := \mathsf{Coker}(\partial).$$

Assumptions on π_0

- protoadditive functor

Non-additive derived functor of π_0

For the moment, what do we know:

- we consider a semi-abelian variety $\mathcal{V}$;
- then $\mathsf{XMod}(\mathcal{V})$ is a semi-abelian category and it is a variety of algebras (and as a result has enough projectives);
- if $\mathcal{V}$ satisfies (P) then $\mathsf{XMod}(\mathcal{V})$ as well.

Considering now the functor $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$ defines as

$$\pi_0(X, A, \partial, \psi) := \mathsf{Coker}(\partial).$$

Assumptions on π_0

- protoadditive functor: done in [EG10] since $\mathcal{V}$ is semi-abelian;
- preserves binary coproducts:

Non-additive derived functor of π_0

For the moment, what do we know:

- we consider a semi-abelian variety $\mathcal{V}$;
- then XMod($\mathcal{V}$) is a semi-abelian category and it is a variety of algebras (and as a result has enough projectives);
- if $\mathcal{V}$ satisfies (P) then XMod($\mathcal{V}$) as well.

Considering now the functor $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$ defines as

$$\pi_0(X, A, \partial, \psi) \coloneqq \mathsf{Coker}(\partial).$$

Assumptions on π_0

- protoadditive functor: done in [EG10] since $\mathcal{V}$ is semi-abelian;
- preserves binary coproducts: it is a left-adjoint functor;
- preserves proper morphisms:

Non-additive derived functor of π_0

For the moment, what do we know:

- we consider a semi-abelian variety $\mathcal{V}$;
- then $\mathsf{XMod}(\mathcal{V})$ is a semi-abelian category and it is a variety of algebras (and as a result has enough projectives);
- if $\mathcal{V}$ satisfies (P) then $\mathsf{XMod}(\mathcal{V})$ as well.

Considering now the functor $\pi_0 \colon \mathsf{XMod}(\mathcal{V}) \to \mathcal{V}$ defines as

$$\pi_0(X, A, \partial, \psi) := \mathsf{Coker}(\partial).$$

Assumptions on π_0

- protoadditive functor: done in [EG10] since $\mathcal{V}$ is semi-abelian;
- preserves binary coproducts: it is a left-adjoint functor;
- preserves proper morphisms: it is a reflector from a variety to a subvariety.

Open questions/problems

- Today, we see the implication: $\mathcal{V}$ satisfies $(\mathrm{P}) \implies$ XMod($\mathcal{V}$) as well. Actually, the converse also holds.

Open questions/problems

- Today, we see the implication: $\mathcal{V}$ satisfies $(\mathrm{P}) \implies$ XMod($\mathcal{V}$) as well. Actually, the converse also holds.This brings me to this question: is the condition (P) stable under Birkhoff's subvarieties? Do we need a protoadditive reflector?
- Could we do the same for PXMod($\mathcal{V}$) with the same assumptions on $\mathcal{V}$?

Open questions/problems

- Today, we see the implication: $\mathcal{V}$ satisfies $(\mathrm{P}) \implies$ XMod($\mathcal{V}$) as well. Actually, the converse also holds.This brings me to this question: is the condition (P) stable under Birkhoff's subvarieties? Do we need a protoadditive reflector?
- Could we do the same for PXMod($\mathcal{V}$) with the same assumptions on $\mathcal{V}$?
  My understanding of today:

Open questions/problems

- Today, we see the implication: $\mathcal{V}$ satisfies $(\mathrm{P}) \implies$ XMod($\mathcal{V}$) as well. Actually, the converse also holds.This brings me to this question: is the condition (P) stable under Birkhoff's subvarieties? Do we need a protoadditive reflector?
- Could we do the same for PXMod($\mathcal{V}$) with the same assumptions on $\mathcal{V}$?
  My understanding of today:
  - with the same definition, we have the functor $\pi_0 \colon \mathrm{PXMod}(\mathcal{V}) \to \mathcal{V}$ and it is still protoadditive;
  - π_0 is also a left adjoint (so preserves binary coproducts) and also it still preserves proper morphisms;

Open questions/problems

- Today, we see the implication: $\mathcal{V}$ satisfies $(\mathrm{P}) \implies$ XMod($\mathcal{V}$) as well. Actually, the converse also holds.This brings me to this question: is the condition (P) stable under Birkhoff's subvarieties? Do we need a protoadditive reflector?
- Could we do the same for PXMod($\mathcal{V}$) with the same assumptions on $\mathcal{V}$?
  My understanding of today:
  - with the same definition, we have the functor $\pi_0 \colon$ PXMod($\mathcal{V}$) $\to \mathcal{V}$ and it is still protoadditive;
  - π_0 is also a left adjoint (so preserves binary coproducts) and also it still preserves proper morphisms;
  - the construction of the free internal crossed module is actually the construction of the free internal (pre)crossed module;

Open questions/problems

- Today, we see the implication: $\mathcal{V}$ satisfies $(\mathrm{P}) \implies$ XMod($\mathcal{V}$) as well. Actually, the converse also holds.This brings me to this question: is the condition (P) stable under Birkhoff's subvarieties? Do we need a protoadditive reflector?
- Could we do the same for PXMod($\mathcal{V}$) with the same assumptions on $\mathcal{V}$?
  My understanding of today:
  - with the same definition, we have the functor $\pi_0 \colon \mathrm{PXMod}(\mathcal{V}) \to \mathcal{V}$ and it is still protoadditive;
  - π_0 is also a left adjoint (so preserves binary coproducts) and also it still preserves proper morphisms;
  - the construction of the free internal crossed module is actually the construction of the free internal (pre)crossed module;
  - **PROBLEM**: the proof for the particular projective internal crossed module is impossible in PXMod($\mathcal{V}$). Why? In the proof, we need the Peiffer condition to prove that the pair (g_X, g_A) is compatible with the actions.

Thank you!

Questions? Or comments?

References

[BB69] M. Barr and J. Beck. Homology and standard constructions. volume 80 of *Lecture Notes in Math.* Springer, 1969.

[CCRG02] P. Carrasco, A. M. Cegarra, and A. R.-Grandjeán. (Co)Homology of crossed modules. *J. Pure Appl. Algebra*, 168(2-3), 2002.

[CRVdL25] M. Culot, F. Renaud, and T. Van der Linden. Non-additive derived functors via chain resolutions. *Glasgow Math. J.*, 67(3), 2025.

[Cul25] M. Culot. Projective crossed modules in semi-abelian categories. *Appl. Categ. Structures*, 2025. to appear.

[EG10] T. Everaert and M. Gran. Homology of n-fold groupoids. *Theory Appl. Categ.*, 23(2), 2010.

[EVdL04] T. Everaert and T. Van der Linden. Baer invariants in semi-abelian categories II: Homology. *Theory Appl. Categ.*, 12(4), 2004.

[HVdL13] M. Hartl and T. Van der Linden. The ternary commutator obstruction for internal crossed modules. *Adv. Math.*, 232(1), 2013.

[Jan03] G. Janelidze. Internal crossed modules. *Georgian Math. J.*, 10(1), 2003.