Twisted Commutators

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Classical commutativity

$f\colon X \to Y$ and $g\colon B \to Y$ **commute/cooperate** (Bourn, Huq) whenever they admit a **cooperator** $\varphi_{f,g}\colon X \times B \to Y$.

$$\begin{array}{ccccc} X & \xrightarrow{(1_X,0)} & X \times B & \xleftarrow{(0,1_B)} & B \\ & {\scriptstyle f}\searrow & \Big\downarrow{\scriptstyle \varphi_{f,g}} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array}$$

The **commutator** of f and g (Mantovani, Metere, Higgins) is constructed as the image below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & X \diamond B & \xrightarrow{\iota_{X,B}} & X + B & \xrightarrow{\langle (1_X,0),(0,1_B)\rangle} & X \times B \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle \langle f,g\rangle} & & \\ & & [f,g] & \rightarrowtail & Y & & \end{array}$$

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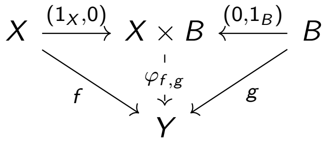

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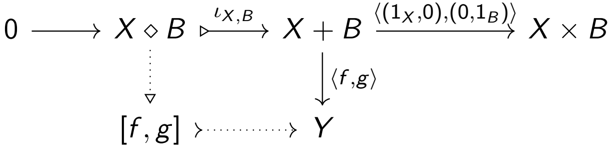

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Relative commutativity

f and g **commute relatively to a cospan** $X \xrightarrow{k} A \xleftarrow{s} B$ whenever they admit a (k,s)-**cooperator** $\varphi_{f,g} \colon A \to Y$.

$$\begin{array}{ccccc} X & \xrightarrow{k} & A & \xleftarrow{s} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \varphi_{f,g}} & \swarrow{\scriptstyle g} & \\ & & Y & & \end{array}$$

The (k,s)-**commutator** of f and g is constructed as the image below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & X \diamond_{k,s} B & \xrightarrow{\iota_{k,s}} & X + B & \xrightarrow{\langle k,s \rangle} & A \\ & & \downarrow & & \downarrow{\scriptstyle \langle f,g \rangle} & & \\ & & [f,g]_{k,s} & \rightarrowtail & Y & & \end{array}$$

Internal actions

In a semi-abelian category, an **internal action** $\xi \colon B\flat X \to X$ can be seen as a **split short exact sequence (SSES)**

$$0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \underset{s}{\overset{p}{\rightleftarrows}} B \longrightarrow 0$$

where $X \rtimes_\xi B$ is the **semi-direct product** of X by B (relatively to the action ξ).

In particular, the **trivial action** can be represented by the SSES

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Link between commutators and commutativity

Theorem

Two arrows $f\colon X \to Y$ and $g\colon B \to Y$ commute relatively to an extremally epic cospan $X \xrightarrow{k} A \xleftarrow{s} B$ if and only if their (k,s)-commutator $[f,g]_{k,s}$ is trivial.

Proof.

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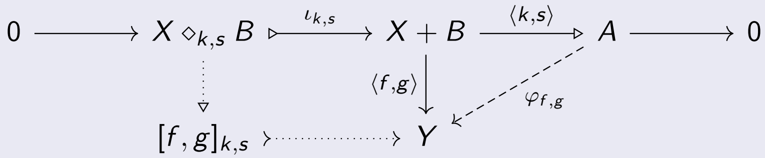

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Composition properties

Proposition

- *If f and g (k, s)-commute, then so do $h \circ f$ and $h \circ g$. Moreover, the converse holds if h is monic and the cospan (k, s) is extremally epic.*
- $[h \circ f, h \circ g]_{k,s} = h([f, g]_{k,s})$.

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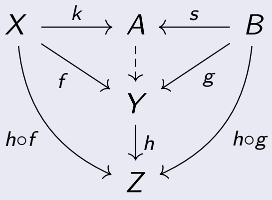

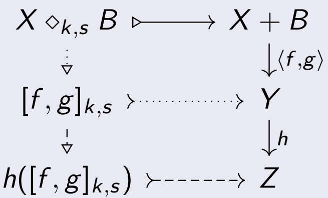

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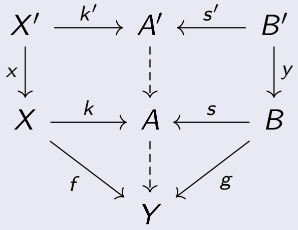
Composition properties

Proposition

If $k \circ x$ *and* $s \circ y$ (k', s')*-commute and* f *and* g (k, s)*-commute, then so do* $f \circ x$ *and* $g \circ y$.



Twisted commutativity and equivariance

Proposition

Let ξ, ξ' be two actions and (k, s), (k', s') be their associated cospans.

- *f and g commute with respect to ξ if and only if they are equivariant with respect to ξ and the conjugation action $c^{Y,Y}$ of Y on itself.*
- *f and g are equivariant with respect to two actions ξ and ξ' if and only if $k' \circ f$ and $s' \circ g$ commute with respect to ξ.*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{k} & X \rtimes_\xi B & \underset{s}{\overset{p}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & \downarrow f & & \vdots & & \downarrow g & & \\
0 & \longrightarrow & Y & \xrightarrow[(1_Y,0)]{} & Y \times Y & \underset{\Delta_Y}{\overset{\pi_2}{\rightleftarrows}} & Y & \longrightarrow & 0
\end{array}
\qquad
\begin{array}{ccccc}
X & \xrightarrow{k} & X \rtimes_\xi B & \xleftarrow{s} & B \\
 & {\scriptstyle k' \circ f} \searrow & \vdots & \swarrow {\scriptstyle s' \circ g} & \\
 & & X' \rtimes_{\xi'} B' & &
\end{array}
$$

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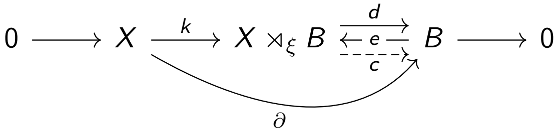
Definition

A **precrossed module** is a couple (∂, ξ), where $\partial\colon X \to B$ is an arrow and ξ is an action of B on X with induced SSES $0 \longrightarrow X \xrightarrow{k} X \rtimes_\xi B \overset{d}{\underset{e}{\rightleftarrows}} B \longrightarrow 0$, such that there exists an arrow $c\colon X \rtimes_\xi B \to B$ satisfying the equations $c \circ e = 1_B$ and $c \circ k = \partial$.



Since k can be recovered as a kernel of d, a precrossed module is essentially just a **reflexive graph**.

A **crossed module** is a precrossed module whose corresponding reflexive graph admits a groupoid structure.

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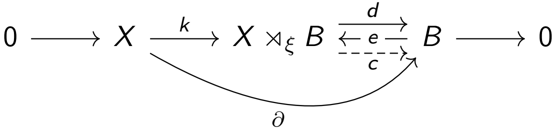

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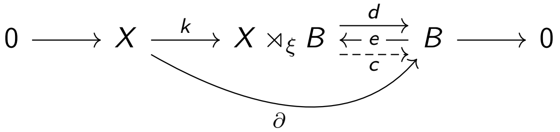

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Characterisation

Theorem (Reformulation of results of [MM10b])

- (∂, ξ) *is a precrossed module if and only if it satisfies the* **Precrossed Module Condition (PCM)** $[\partial, 1_B]_\xi = 0$.

$$\begin{array}{ccccc} X & \xrightarrow{k} & X \rtimes_\xi B & \xleftarrow{e} & B \\ & \searrow^{\partial} & \downarrow{\scriptstyle c} & \swarrow_{1_B} & \\ & & B & & \end{array}$$

- *Under (SH),* (∂, ξ) *is a crossed module if and only if it also satisfies the* **Peiffer Condition (PFF)** $[k, e \circ \partial]_{c^{X,X}} = 0$.

$$\begin{array}{ccccc} X & \xrightarrow{(1_X,0)} & X \times X & \xleftarrow{\Delta_X} & X \\ & \searrow_{k} & \downarrow & \swarrow_{e\circ\partial} & \\ & & X \rtimes_\xi B & & \end{array} \qquad \begin{array}{ccc} X \flat X & \xrightarrow{c^{X,X}} & X \\ {\scriptstyle \partial\flat 1_X}\downarrow & & \downarrow{\scriptstyle 1_X} \\ B \flat X & \xrightarrow[\xi]{} & X \end{array}$$

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Monic ∂

If ∂ is monic, then (∂, ξ) is a precrossed module if and only if ∂ is the inclusion of a normal subobject and ξ is the action of conjugation of B on X.

$$\begin{array}{ccc} B\flat X & \overset{\xi}{\dashrightarrow} & X \\ {\scriptstyle B\flat\partial}\downarrow & & \downarrow{\scriptstyle \partial} \\ B\flat B & \underset{c^{B,B}}{\longrightarrow} & B \end{array}$$

Moreover, when this is the case, (PFF) automatically holds so (∂, ξ) is even a crossed module.

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Regularly epic ∂

If ∂ is regularly epic, then (∂, ξ) is a crossed module if and only if ∂ is a central extension, i.e. a regular epimorphism that is central.

Lemma

If (∂, ξ) satisfies (PFF), then ∂ is central, i.e. its kernel and its domain commute.

For the other implication, we can show that there exists an action such that (PFF) is satisfied and, once we have this action, (PCM) comes for free.

$$\begin{array}{ccc} X\flat X & \xrightarrow{c^{X,X}} & X \\ {\scriptstyle \partial\flat 1_X}\downarrow & & \downarrow{\scriptstyle 1_X} \\ B\flat X & \overset{}{\underset{\xi}{\dashrightarrow}} & X \end{array}$$

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