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Coherent and Ideal Actions

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**Joint work with Manuel Mancini e Federica Piazza*

Plan of the talk

1. Actions: internal vs external
2. A convenient setting
3. Coherent and ideal actions
4. Some case studies

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2. A convenient setting
3. Coherent and ideal actions
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Group actions on groups

They are usually defined as *external operations*:

$$*\colon\ B \times X \longrightarrow X \qquad s.t.\ \ 1 * x = x,\ ab * x = a * (b * x),\ g * (x + y) = g * x + g * y$$

This is unfortunate, from a categorical perspective.

A classical solution:

$$\frac{*\colon\ B \times X \longrightarrow X \quad s.t.\ \dots}{\alpha_*\colon\ B \longrightarrow \mathbf{Aut}(X)} \qquad \begin{array}{l}\text{in } \mathbf{Set}\\ \text{in } \mathbf{Gp}\end{array}$$

Pros: - Actions are given in the internal language of the ambient category.

Cons: - The assignment $\mathbf{Aut}(-)$ is not functorial.
- Many otherwise well-behaved algebraic varieties don't have an $\mathbf{Aut}(-)$.

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Internal actions and semidirect products

Let us focus on another feature of (group) actions.

Fact. Semidirect product determines an equivalence of categories: B-**Act** $\simeq$ $\mathbf{Pt}(B)$

$$\boxed{B \times X \xrightarrow{\xi} X} \qquad \mapsto \qquad \boxed{\begin{tikzcd} X \rtimes_\xi B \arrow[r, shift left, "p_2"] & B \arrow[l, shift left, "i_2"] \end{tikzcd}}$$

Definition

$\mathbb{C}$ with split pullbacks has **semidirect products** if, for all $p \colon E \to B$, the functor

$$p^* \colon \mathbf{Pt}(B) \to \mathbf{Pt}(E)$$

has a left adjoint and is monadic (hence $\mathbb{C}$ protomodular!).

[BJ98] Bourn, Janelidze. *Protomodularty, descent and semi-direct products*

[BJK05] Borceux, Janelidze, Kelly. *Internal object actions*

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Internal actions: $\mathbb{C}$ **not pointed** vs $\mathbb{C}$ **pointed**

Generic $p\colon E \to B$

$$\begin{tikzcd}
 & (X,\xi) \arrow[dl, dashed] & \in & \mathbf{Pt}(E)^{T_p} \arrow[d] \\
(X,\xi) \rtimes (B,p) \in & \mathbf{Pt}(B) \arrow[urr, "K"] \arrow[rr, "p^*"', "\perp"] & & \mathbf{Pt}(E) \arrow[ll, bend right]
\end{tikzcd}$$

$\mathbb{C}$ pointed, choose $p\colon 0 \to B$

$$\begin{tikzcd}
 & (X,\xi) \arrow[dl, dashed] & \in & \mathbb{C}^{B\flat} \arrow[d] \\
(X,\xi) \rtimes B \ \in & \mathbf{Pt}(B) \arrow[urr, "K"] \arrow[rr, "\ker"', "\perp"] & & \mathbb{C} \arrow[ll, bend right]
\end{tikzcd}$$

where $B\flat X = \ker(\ B + X \xrightarrow{[1,0]} B\ )$

If $\mathbb{C}$ is **not** pointed, we can only describe

split epis over B as algebras on split epis over E

not a big deal, unless $\mathbf{Pt}(E)$ is somehow special...

If $\mathbb{C}$ is pointed, $\mathbf{Pt}(0) \simeq \mathbb{C}$, and we can describe

split epis over B as algebras on their kernels in $\mathbb{C}$

Internal actions in non-pointed categories

Now, non-pointed categories do not have kernels...

However, the *working mathematician* knows that, although in a non-pointed category kernels cannot be defined categorically, they are often replaced by ideals, i.e. kernels living in a bigger category.

Example

If we consider a homomorphism of unitary commutative rings

$$p\colon A \to B$$

its kernel $X = \{x \in A \mid p(x) = 0\}$ is not a sub(unitary)ring of A in general, but rather an ideal, i.e. a kernel in the category of (non necess. unitary) commutative rings.

All this can be generalized, but first let us examine the case of rings in more detail.

Unitary actions for commutative rings

$$\begin{tikzcd} \mathbf{CRing} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathbf{CRng} \arrow[l, bend right, "F"'] \end{tikzcd}$$

- **CRing** exact protomodular
- **CRng** semi-abelian
- U monadic

where $F(R) := R \rtimes \mathbb{Z}$ has multiplication
$(r,n)(r',n') \;= (n'r + nr' + rr', nn')$

Definition

An ideal X of A is a kernel
$X = \ker\left(U(p)\colon U(A) \to U(B)\right)$

Now, for B in **CRng**,

$$\mathbf{Pt}_{\mathbf{CRng}}(B) \simeq B\flat(-)\text{-}\mathbf{Alg} \simeq B\text{-actions}$$

Definition (B-actions in CRng)

An action of the comm. ring B on the comm. ring X is a ring homomorphism $\alpha\colon B \otimes X \to X$ s.t.

$$b' \cdot (b \cdot x) = bb' \cdot x, \qquad b \cdot xx' = (b \cdot x)x' = x(b \cdot x')$$

where $B \otimes X = B \otimes_{\mathbb{Z}} X$ with $(a \otimes x)(a' \otimes x') = aa' \otimes xx'$ and $b \cdot x := \alpha(b \otimes x)$.

Unitary actions for commutative rings

$$\begin{tikzcd} \mathbf{CRing} \arrow[r, "U"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathbf{CRng} \arrow[l, bend right, "F"'] \end{tikzcd}$$

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The equivalence $\mathbf{Pt}_{\mathbf{CRng}}(B) \simeq B$-actions

- Given a B-action on X, define the split epi $X \rtimes B \underset{}{\overset{p_2}{\rightleftarrows}} B$ with $X \rtimes B = (X \oplus B, +, \;)$, multiplication: $(x, b)(x', b') = (xx' + b' \cdot x + b \cdot x', bb')$.
- Given a split epi $A \underset{s}{\overset{p}{\rightleftarrows}} B$, $X = \ker(p)$ is a B ring with $b \cdot x = s(b)x$.

Q: *If B is a* **unitary** *commutative ring, and X is a B-ring, when is the split epi*

$$X \rtimes U(B) \underset{i_2}{\overset{p_2}{\rightleftarrows}} U(B)$$

a split epi of unitary commutative rings?

A: *This happens precisely when the B-action satisfies*

$$1_B \cdot x = x, \qquad \textit{for all } x \in X$$

and then $(0, 1_B) \in X \rtimes U(B)$ is the multiplicative unit.

Aim of this talk: provide a framework to deal with Q&A w.r.t. internal actions.

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A convenient setting for relative U-ideals

[LMS2024] Lapenta, M., Spada. *Relative ideals in homological categories...*

$$\text{Basic Setting} = \begin{tikzcd} \mathbb{U} \arrow[r, "U"'] & \mathbb{V} \arrow[l, bend right, "F"'] \arrow[l, phantom, "\perp"] \end{tikzcd} \quad \text{s.t.}$$

- $\mathbb{V}$ homological (i.e. pp + reg.)
- U conservative (+ faithful)

Definition

$k\colon X \to U(A)$ is a **relative U-ideal** if $\exists$ $\begin{tikzcd} A \arrow[d, "f"] \\ B \end{tikzcd}$ s.t. $\begin{tikzcd} X \arrow[r, "k"] \arrow[d] & U(A) \arrow[d, "U(f)"] \\ 0 \arrow[r] & U(B) \end{tikzcd}$ is a pullback.

$\eta_X\colon X \to UF(X)$ is an **augmentation U-ideal** if $\eta_X = \ker(U(p_X))$,

where $p_X\colon F(X) \to 0$ is the unique morphism s.t. $U(p_X) \circ \eta_X = 0$.

Theorem

If, $\forall X$ in $\mathbb{V}$, η_X augmentation ideal, then $\ker\colon (\mathbb{U} \downarrow 0) \to \mathbb{V}$ *is an equivalence.*

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Relative U-ideals: not only rings...

unitary (non-associative) K-algebras: $\begin{tikzcd} \mathbf{UAlg}_K \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathbf{Alg}_K \arrow[l, bend right=40, "F"'] \end{tikzcd}$ (K field)

$$F(X) = X \rtimes K \quad (x,k)(x',k') = (k'x + kx' + xx', kk') \quad k(x,k') = (kx, kk')$$

unitary C^*-algebras: $\begin{tikzcd} \mathbf{UCStar} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathbf{CStar} \arrow[l, bend right=40, "F"'] \end{tikzcd}$

$$F(X) = X \oplus \mathbb{C} \quad \text{with multiplication as above, and } (x,z)^* = (x^*, \bar{z})$$

MV-algebras and $\mathbf{Set}^{op}$: we will discuss them later...

A convenient setting for varieties: $\begin{tikzcd} \mathbb{U} \arrow[r, "U"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathbb{V} \arrow[l, bend right=40, "F"'] \end{tikzcd}$, *where U forgetful functor between $\mathbb{U} = (\mathbb{U}, \Sigma_{\mathbb{U}}, Z_{\mathbb{U}})$ and $\mathbb{V} = (\mathbb{V}, \Sigma_{\mathbb{V}}, Z_{\mathbb{V}})$ semi-abelian, with $\Sigma_{\mathbb{V}} \subseteq \Sigma_{\mathbb{U}}$, $Z_{\mathbb{V}} \subseteq Z_{\mathbb{U}}$.*

We are particularly interested in U forgetting only constants (and the corresponding equations).

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Ideally exact categories [J2024] Janelidze. *Ideally exact categories.*

Definition

$\mathbb{U}$ is **ideally exact** if
- $\mathbb{U}$ is Barr-exact
- $\mathbb{U}$ has finite coproducts
- $\mathbb{U}$ is protomodular and $0 \to 1$ reg. epi

Recall from [BJ03] that a variety $\mathbb{U}$ is protomodular iff there exist terms s.t.:

$$\theta(\alpha_1(x,y),\ldots,\alpha_n(x,y),y) = x, \quad \alpha_i(x,x) = e_i \quad \text{for } i = 1,\ldots,n$$

$\mathbb{U}$ is **classically ideally determined** if equations above hold for a specified constant $0 = e_1 = \cdots = e_n$. In this case, $U \colon \mathbb{U} \to \mathbb{V}$ forgets all constants but 0, and U-ideal are 0-ideal a lá Ursini.

A **non-trivial** algebraic variety $\mathbb{U}$ is ideally exact iff it is protomodular.

Ideally exact contexts

Theorem ([J2024])

$\mathbb{U}$ *is* **ideally exact** *iff*
$\mathbb{U}$ *is Barr-exact*
$\mathbb{U}$ *has finite coproducts*
$\exists U \colon \mathbb{U} \to \mathbb{V}$ *monadic, with* $\mathbb{V}$ *semi-abelian*

Fact. The functor U can be chosen s.t. η is cartesian.
Fact. η cartesian iff η_X augmentation ideal, for all X in $\mathbb{V}$.

Here is our **convenient setting**:

Definition (Ideally Exact Context)

$$\begin{tikzcd} \mathbb{U} \arrow[r, "U"', ""{name=A}] & \mathbb{V} \arrow[l, bend right, "F"', ""{name=B}] \arrow[from=A, to=B, phantom, "\perp"] \end{tikzcd}$$

$\mathbb{U}$ ideally exact, $\mathbb{V}$ semi-abelian
$F \dashv U$ monadic, η cartesian

From now on, we stick to an **ideally exact context**.

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Coherent actions and coherent points

Given an action $\xi\colon U(B)\flat X \to X$ assoc. with the split epi $A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$, and $\xi_0\colon UF(0)\flat X \to X$ the action assoc. with the canonical $UF(X) \underset{UF(\iota)}{\overset{UF(\tau)}{\rightleftarrows}} UF(0)$, T.F.A.E.

(C1) The diagram
$$\begin{tikzcd} UF(0)\flat X \arrow[r, "U(\iota)\flat 1"'] \arrow[rr, bend left, "\xi_0"] & U(B)\flat X \arrow[r, "\xi"'] & X \end{tikzcd}$$
commutes,

(C2) $\exists f$ s.t.
$$\begin{tikzcd} UF(X) \arrow[r, shift left, "UF(\tau)"] \arrow[d, "f"'] & UF(0) \arrow[l, shift left, "UF(\iota)"] \arrow[d, "U(\iota)"] \\ A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"] \end{tikzcd}$$
is a pullback

Definition

An action ξ is **coherent** if it satisfies (C1), a point (p, s) is **coherent** if it satisfies (C2).

Example. An action of a unitary ring B on a non-unitary ring X is **coherent** when the multiplicative unit acts *coherently* as in $F(X) = X \rtimes \mathbb{Z}$.

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Ideal points and ideal actions

Definition

A split epi $A \underset{s}{\overset{p}{\rightleftarrows}} U(B)$ is **ideal** if there exist a split epi $A' \underset{s'}{\overset{p'}{\rightleftarrows}} B$, and an isomorphism $\sigma\colon U(A') \to A$ that induces an isomorphism of points.

A morphism h of split epis over $U(B)$ is ideal if there exists h' s.t. $U(h') = h$.

Fact. U full on isos $\Rightarrow (p', s')$ essentially unique.

Definition

An action $\xi\colon U(B)\flat X \to X$ is **ideal** if it determines an ideal split epi.

Example. An action of a unitary ring B on a non-unitary ring X is **ideal** precisely when the corresponding split epi is unitary.

Ideal vs Coherent

For rings, **coherent** and **ideal** actions coincide. What can be said in general?

Theorem

In an ideally exact context, all ideal actions are coherent.

We do not know if the converse of this statement holds in the ideally exact context.

However, when it holds, it establishes a convenient setting to study a well-behaved notion of action.

Definition

An ideally exact context admits a **good theory of actions** (it is **BAT**, for short) if all coherent actions are ideal, and all morphisms of such actions are ideal.

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Ideal actions as algebras.

[J2024] G. Janelidze *Semidirect products in ideally exact categories*

Proposition (J2024)

An ideally exact context $F \dashv U \colon \mathbb{U} \to \mathbb{V}$, induces a monadic adjunction:

$$\begin{tikzcd} \mathbf{Pt}_{\mathbb{U}}(B) \arrow[r, "U^B"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbb{V} \arrow[l, bend right, "F^B"'] \end{tikzcd} \quad \textit{where} \quad U^B \colon X \mapsto \begin{tikzcd} B+F(X) \arrow[d, shift right, "{[1,\iota\circ F\tau]}"'] \\ B \arrow[u, shift right, "i_1"'] \end{tikzcd} \qquad F^B \colon \begin{tikzcd} A' \arrow[d, shift right, "p'"'] \\ B \arrow[u, shift right, "s'"'] \end{tikzcd} \mapsto \ker(U(p'))$$

$\Rightarrow$ one can describe points over B as algebras for the monad $B\#(-) = U^B \circ F^B$.

Fact. There is a morphism of monads $\gamma_{B,X}$ given by:

$$\begin{tikzcd} U(B)\flat X \arrow[r] \arrow[d, dashed, "\gamma_{B,X}"] & U(B)+X \arrow[r, "{[1,0]}"] \arrow[d, "{[U(i_1),U(i_2)\circ\eta_X]}"] & U(B) \arrow[d, equal] \\ B\#X \arrow[r] & U(B+F(X)) \arrow[r, "{U([1,\iota\circ F\tau])}"'] & U(B). \end{tikzcd}$$

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[J2024] G. Janelidze *Semidirect products in ideally exact categories*

Proposition (J2024)

An ideally exact context $F \dashv U \colon \mathbb{U} \to \mathbb{V}$, *induces a monadic adjunction:*

$$
\begin{tikzcd}
\mathbf{Pt}_{\mathbb{U}}(B) \arrow[r, "\perp"', "U^B"', shift right] & \mathbb{V} \arrow[l, bend right, "F^B"']
\end{tikzcd}
\quad \textit{where} \quad
U^B \colon X \mapsto
\begin{tikzcd}
B + F(X) \arrow[d, shift right, "{[1,\iota\circ F\tau]}"'] \\
B \arrow[u, shift right, "i_1"']
\end{tikzcd}
\qquad
F^B \colon
\begin{tikzcd}
A' \arrow[d, shift right, "p'"'] \\
B \arrow[u, shift right, "s'"']
\end{tikzcd}
\mapsto \ker(U(p'))
$$

$\Rightarrow$ one can describe points over B as algebras for the monad $B\#(-) = U^B \circ F^B$.

Fact. There is a morphism of monads $\gamma_{B,X}$ given by:

$$
\begin{tikzcd}
U(B)\flat X \arrow[r] \arrow[d, dashed, "\gamma_{B,X}"] & U(B)+X \arrow[r, "{[1,0]}"] \arrow[d, "{[U(i_1),U(i_2)\circ\eta_X]}"] & U(B) \arrow[d, equal] \\
B\#X \arrow[r] & U(B+F(X)) \arrow[r, "{U([1,\iota\circ F\tau])}"'] & U(B).
\end{tikzcd}
$$

Lemma

An action $\xi\colon U(B)\flat X \to X$ is ideal if and only if there exists a $B\#$-algebra $\xi'\colon B\#X \to X$ s.t.

$$\begin{tikzcd} U(B)\flat X \arrow[r, "\gamma_{B,X}"'] \arrow[rr, bend left, "\xi"] & B\#X \arrow[r, dashed, "\xi'"'] & X \end{tikzcd}$$

Theorem

For an ideally exact context $F \dashv U\colon \mathbb{U} \to \mathbb{V}$, with U full on isos, T.F.A.E.

(i) *All the coherent B-actions are ideal and all the morphisms of coherent B-actions are ideal.*

(ii) *The following diagram is a pullback:*

$$\begin{tikzcd} \mathbb{V}^{B\#} \arrow[r, "(-)\circ\gamma_B"] \arrow[d, "S"'] & \mathbb{V}^{U(B)\flat} \arrow[d, "(-)\circ((U\iota)\flat 1)"'] \\ \mathbb{V} \arrow[r, "{[\xi_0]}"'] & \mathbb{V}^{UF(0)\flat} \end{tikzcd}$$

where S forgetful and $[\xi_0](X) = \xi_{0,X}$.

Corollary

For an ideally exact context $F \dashv U \colon \mathbb{U} \to \mathbb{V}$, with U full on isos,

T.F.A.E.

(i) *The context is BAT*

(ii) *For all B in $\mathbb{U}$, this diagram is a pullback:*

$$\begin{tikzcd} \mathbb{V}^{B\#} \arrow[r, "(-)\circ\gamma_B"] \arrow[d, "s"'] & \mathbb{V}^{U(B)\flat} \arrow[d, "(-)\circ((U\iota)\flat 1)"] \\ \mathbb{V} \arrow[r, "{[\xi_0]}"'] & \mathbb{V}^{UF(0)\flat} \end{tikzcd}$$

(iii) *The natural transformation $\overline{U}\colon \#\text{-}alg \Rightarrow \flat\text{-}alg\colon \mathbb{U}^{op} \to$ **Cat** is cartesian, where $\overline{U}$ is the functor between the categories of algebras induced by the morphism of monads $\gamma = \gamma_{B,X}$.*

Plan of the talk

1. Actions: internal vs external
2. A convenient setting
3. Coherent and ideal actions.
4. **Some case studies**

BAT contexts

In the last part of my talk, I will examine some BAT ideally exact context, i.e. with a good theory of actions.

(i) (Unitary) non-associative K-algebras.

(ii) (Bounded) Wajsberg hoops, but it also applies to product hoops.

(iii) $\mathbf{Set}^{op} \xrightarrow{U} \mathbf{Set}_*^{op}$

(i) Non-associative K-algebras.

Recall. A non-associative algebra on a field K is a K-vector space A endowed with a bilinear operation. A variety $\mathbb{V}$ of non-associative algebras is a class of such algebras that satisfy specified identities.

A variety $\mathbb{V}$ of non-associative algebras is **unit-closed** if, for any algebra X,

$$\langle X, 1\rangle / \{x1 = x = 1x\}$$

is an algebra of the variety (eg. K-**Lie** is not).

Fact. Varieties of non-associative algebras are semi-abelian.

Proposition

If a variety of non-associative algebras $\mathbb{V}$ is unit-closed, and $\mathbb{U}$ is the corresponding variety of unitary algebras, then the free/forgetful adjunction

$$\begin{tikzcd} \mathbb{U} \arrow[r, "U"'] \arrow[r, phantom, shift left=3, "\perp"] & \mathbb{V} \arrow[l, bend right, "F"'] \end{tikzcd}$$

gives a BAT ideally exact context.

Proof of the proposition for non-associative K-algebras.

proof (outline). Given f s.t. the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, equal] & U(X \rtimes K) \arrow[r, shift left, "U(p_2)"] \arrow[d, "f"] & U(K) \arrow[l, shift left, "U(i_2)"] \arrow[d, "U(\iota)"] \\ X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"] \end{tikzcd}$$

the K-algebra A has unit $s(1_B)$ preserved by p.

Actually, given that $a = k(x) + s(b) \in A$, one has

$$a \cdot s(1_B) = \big(k(x) + s(b)\big) \cdot s(1_B) = k(x) \cdot s(1_B) + s(b)$$

but

$$\begin{aligned} k(x) \cdot s(1_B) &= f\eta_X(x) \cdot fU(i_2)(1_K) = f\left(\eta_X(x) \cdot U(i_2)(1_X)\right) \\ &= f\left((x,0) \cdot (0,1_K)\right) = f(x,0) = k(x) \end{aligned}$$

(ii) Wajsberg Hoops

A **hoop** is an algebra $(A; \cdot, \to, 1)$ s.t. $(A; \cdot, 1)$ is a commutative monoid and

(H1) $x \to x = 1$

(H2) $x \cdot (x \to y) = y \cdot (y \to x)$

(H3) $x \cdot y \to z = x \to (y \to z)$

A **Wajsberg hoop** is a hoop s.t.

(W) $(x \to y) \to y = (y \to x) \to x$

A **bounded hoop** is a hoop $(A; \cdot, \to, 1)$ with a constant $0 \in A$ s.t.

(B) $0 \to x = 1$

Facts:

- Hoops are $\wedge$-semilattices, with $x \wedge y := x \cdot (x \to y)$.
- Hoops are partially ordered, with $x \leqslant y$ iff $x \to y = 1$ iff $\exists\ u$ s.t. $x = u \cdot y$.
- Bounded Wajsberg hoops are term-equivalent to MV-algebras.

The categories **Hoop**, **WHoop**, **BWHoop** *are defined.*

(ii) Whoops: the free bounded hoop.

Proposition (LMS2024)

Hoop *and* ***WHoop*** *are semi-abelian,* ***BWHoop*** *is protomodular.*

The adjunction $\mathbf{BWHoop} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{WHoop}$ gives an ideally exact context, where

$F(X) = (X \times L_2; \cdot, \to, 0, 1)$ with operations you don't really want to read:

$$(x, \mathbf{0})(y, \mathbf{0}) = ((x \to xy) \to y, \mathbf{0})$$
$$(x, \mathbf{0})(y, \mathbf{1}) = (y \to x, \mathbf{0})$$
$$(x, \mathbf{1})(y, \mathbf{0}) = (x \to y, \mathbf{0})$$

$$(x, \mathbf{1})(y, \mathbf{1}) = (xy, \mathbf{1})$$
$$(x, \mathbf{0}) \to (y, \mathbf{0}) = (y \to x, \mathbf{1})$$
$$(x, \mathbf{0}) \to (y, \mathbf{1}) = ((x \to xy) \to y, \mathbf{1})$$

$$(x, \mathbf{1}) \to (y, \mathbf{0}) = (xy, \mathbf{0})$$
$$(x, \mathbf{1}) \to (y, \mathbf{1}) = (x \to y, \mathbf{1})$$

[ACD2010] Abad, Castaño, Díaz Varela. *MV-closures of Wajsberg hoops...*

Proposition

The ideally exact context $\mathbf{BWHoop} \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{WHoop}$ *is BAT.*

Proof of the proposition for Wajsberg hoops.

proof outline. Given f s.t. the following diagram commutes (i.e. (p, s) is coherent)

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, equal] & U(X \rtimes L_2) \arrow[r, shift left, "U(p_2)"] \arrow[d, "f"] & U(L_2) \arrow[l, shift left, "U(i_2)"] \arrow[d, "U(\iota)"] \\
X \arrow[r, "k"'] & A \arrow[r, shift left, "p"] & U(B) \arrow[l, shift left, "s"]
\end{tikzcd}$$

A has bottom element $s(0_B)$ preserved by p (i.e. (p, s) is ideal).

Actually, for all $a \in A$, $s(0_B) \to a \in X$, since

$$p(s(0_B) \to a) = p(s(0_B)) \to p(a) = 0_B \to p(a) = 1$$

however

$$\begin{aligned}
1 = f(1,1) = f\big((1,0) \to (s(0_B) \to a, 1)\big) &= f(1,0) \to f\big((s(0_B) \to a, 1)\big) \\
&= (f \circ U(i_2))(0) \to (f \circ \eta_X)(s(0) \to a) \\
&= s(0_B) \to \big(s(0_B) \to a\big) = s(0_B) \cdot s(0_B) \to a \\
& s(0_B \cdot 0_B) \to a = s(0_B) \to a
\end{aligned}$$

i.e. $s(0_a) \leqslant a$.

(iii) A non varietal example: $\mathbf{Set}^{op}$

Dualize the classical **pointed sets adjunction** in order to obtain an **ideally exact context**:

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "1+(-)"'] \arrow[r, phantom, shift left=2, "\top"] & \mathbf{Set}_* \arrow[l, bend right, "\text{forgetful}"'] \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} \mathbf{Set}^{op} \arrow[r, "U"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathbf{Set}_*^{op} \arrow[l, bend right, "\mathsf{F}"'] \end{tikzcd}$$

where $\mathbf{Set}_*^{op}$ is semi-abelian and $\mathbf{Set}_*^{op}$ is ideally exact, and

$$U\colon A \mapsto (1+A, 1) \qquad F\colon (X,\star) \mapsto X$$

Notice that unit and counit, as functions, go backward!

$$\eta_{(X,\star)}\colon (X,\star) \to^{op} UF(X,\star) = (1+X,1) \text{ is given by } [\star, id_X]\colon 1+X \to X$$

$$\epsilon_A\colon 1+A = FU(A) \to^{op} A \text{ is given by } i_2\colon A \to 1+A$$

Split extensions in $\mathbf{Set}_*^{op}$

[D2022] Deval. *A categorical approach to internal actions and semidirect products*

$$\begin{tikzcd} B \arrow[r, shift left, tail, "p"] & A \arrow[l, shift left, two heads, "s"] \arrow[r, two heads, "k"] & X \end{tikzcd}$$

Where:

- we represent arrows of $\mathbf{Set}_*^{op}$ via functions in $\mathbf{Set}_*$, basepoint often omitted,
- s is a split epi, $p = \ker(k)$ is a split mono,
- $k = \mathrm{coker}(p)$ is a normal epi, and $X = \big(1 + (A \smallsetminus p(B)), 1\big)$.

Fact. The normal epi k has a unique splitting

$$\delta \colon X \to A \qquad \text{with } \ \delta(1) := \star_A \text{ and } \delta(x) := x \text{ if } x \neq 1$$

The case of $\mathbf{Set}_*^{op}$

The canonical split epi associated with a pointed set $(X, \star_X)$ is

$$UF(\{*\}, *) = (1 + \{*\}, 1) \underset{1+\tau_X}{\overset{1+\star_X}{\rightleftarrows}} UF(X, \star_X) = (1 + X, 1)$$

with cokernel $(X, \star_X)$.

Proposition

The ideally exact context $\mathbf{Set}^{op} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Set}_*^{op}$ *is BAT.*

Proof of the proposition for $\mathbf{Set}_*^{op}$

We prove that if (A, p, s) induces a coherent action (i.e. the diagram commutes), $\exists$ split mono (A', p', s') under B s.t. $A = 1 + A'$, $p = 1 + p'$ and $s = 1 + s'$.

$$\begin{tikzcd}
1+\{*\} \arrow[r, shift left, tail, "1+\star_X"] & 1+X \arrow[l, shift left, two heads, "1+\tau_X"] \arrow[r, two heads, "{[\star_X, id_X]}"] & X \arrow[d, equal] \\
1+B \arrow[u, "1+\tau_B"] \arrow[r, shift left, tail, "p"] & A \arrow[u, "f"] \arrow[l, shift left, two heads, "s"] \arrow[r, two heads, "k"] & X
\end{tikzcd}$$

For $a \in A$,

$$s(a) = 1 \Leftrightarrow a = p(1).$$

Indeed, $a = p(1) \Rightarrow s(a) = sp(1) = 1$. Viceversa, if $s(a) = 1$, $1 = (1 + \tau_B)(s(a)) = (1 + \tau_X)(f(a))$ so that $f(a) = 1$, and $a \in \ker(k)$, i.e. $\exists z \in 1 + B$ s.t. $p(z) = a$. Now, $z \notin B$. If so, $f(a) = \star_X$, contradiction. Hence $z = 1$.

Now we can define $A' = A \setminus \{p(1)\}$, and the split extension (k, p, s) becomes

$$\begin{tikzcd}
1+B \arrow[r, shift left, tail, "1+p'"] & 1+A' \arrow[l, shift left, two heads, "1+s'"] \arrow[r, two heads, "k'"] & X
\end{tikzcd} \qquad p' = p_{|_B}, \quad s' = s_{|_{A'}}.$$

THANK YOU FOR YOUR ATTENTION!

References

- M. Abad, D. N. Castaño, and J. P. Díaz Varela. MV-closures of Wajsberg hoops and applications. Algebra Universalis 64 (2010).
- F. Borceux, G. Janelidze and G. M. Kelly. Internal object actions. Commentationes Mathematicae Universitatis Carolinae 46 no. 2 (2005).
- D. Bourn. Protomodular aspect of the dual of a topos. Advances in Mathematics 187 no. 1 (2004).
- D. Bourn, G. Janelidze. Protomodularity, descent, and semidirect products. TAC 4 no. 2 (1998).
- D. Bourn, G. Janelidze. Characterization of protomodular varieties of universal algebras. TAC 11 no. 6 (2003).
- B. S. Deval. A categorical approach to internal actions and semidirect products. *Master Thesis, UCLouvain* (2022).
- G. Janelidze. Ideally exact categories. TAC 41 no. 11 (2024).
- G. Janelidze, Semidirect products in ideally exact categories, Adv. Studies: Euro-Tbilisi Math. J. 17 (2024).
- S. Lapenta, G. M., L. Spada. Relative ideals in homological categories. TAC 41 no. 27 (2024).
- M. Mancini, G. M., F. Piazza. Coherent and ideal actions in ideally exact categories, *preprint* (2025) available at: `arXiv:2507.06124`.