INTRODUCTION TO BICATEGORIES

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Part I

Introduction. This is the first part of a work concerned with the study of the following type of structure: A family of categories $\underline{S}(A,B)$ (A,B in a set $\underline{S}_o$) together with pairing functors $c(A,B,C)$: $\underline{S}(A,B) \times \underline{S}(B,C) \to \underline{S}(A,C)$ which up to given coherent isomorphisms behave as if the $\underline{S}(A,B)$ were the $\mathrm{Hom}_?(B,A)$ for some "category" ?. The best known cases are perhaps $\underline{S}_o$ = one point, then we have a single category $\underline{S}$ with a multiplication in the sense of [B. 1], or a 2-category [B. 3] where the associativity isomorphisms are identities, or $\underline{S}_o$ = a set of rings, $\underline{S}(A,B)$ = category of (A,B)-Bimodules and $c(A,B,C) = \otimes_B$.

In §1 we formalise this situation in the definition of bicategory and show in §2 that many other cases considered by Epstein [E] or Yoneda [Y] fit in this pattern.

Even more important is the notion of morphisms defined in §4 where we do not require the functors $F(A,B)$: $\underline{S}(A,B) \to \overline{\underline{S}}(\overline{A},\overline{B})$ to commute with the $c(A,B,C)$, not even up to isomorphisms. The justification for such an apparently too complicated and unnecessarily general definition is in the number of examples (see §5) ranging from monads to pseudo-functors of [Gr] which can be handled and in the fact that most of the results ex-

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This research was supported in part by a grant from the Office of Naval Research.

pected for the strict homomorphisms, hold for general morphisms, and have meaningful interpretation (§6).

In §7 we define some of the invariants of a bicategory: the Poincaré and classifying categories and the Picard groupoid which will be used in Part II. Finally §8 is devoted to the construction of the analogue of the path space, namely the bicategory of cylinders, which gives the possibility to define transformations between morphisms (similar to natural transformations, or homotopies). For this construction we have used heavily the geometrical analogy without which definitions and results seem artificial and are incomprehensible. In many cases we have even replaced the proofs -- essentially setting up very big commutative diagrams-- by more suggestive pictures.

In Part II, we will first complete the construction of the 3-dimensional part of Bicat, by defining "modifications" between transformations, then study the notions of representability, adjointness and equivalence, which are quite different in the two-dimensional case from their ordinary analogue. Then we will examine the case when the functors $c(A,B,C)$ have a right adjoint, and finally study many examples of bicategories, devoting the greatest time to bicategories of "Profunctors".

§1. Bicategories

(1.1) Local definition. A bicategory $\underline{S}$ is determined by the following data:

(i) A set $\underline{S}_o = \mathrm{Ob}(\underline{S})$ called set of objects, or vertices of $\underline{S}$.

(ii) For each pair (A, B) of objects, a category $\underline{S}(A, B)$.

An object S of $\underline{S}(A, B)$ is called an edge or arrow of $\underline{S}$, and written $A \overset{S}{\longleftarrow} B$; the composition sign $\circ$ of maps in $\underline{S}(A, B)$ will usually be omitted. A map s from S to S' will be called a 2-cell and written $s: S \Longrightarrow S'$, or better, will be represented by : $\begin{tikzcd} A & B \arrow[l, bend right, "S"'] \arrow[l, bend left, "S'"] \arrow[l, phantom, "\Downarrow"] \end{tikzcd}$, the composition will thus correspond to the pasting:

$$\begin{tikzcd} A & & B \arrow[ll, bend right=50, "S"'] \arrow[ll, "S'" description] \arrow[ll, bend left=50, "S''"] \arrow[ll, phantom, shift right=3, "\Downarrow s"] \arrow[ll, phantom, shift left=3, "\Downarrow s'"] \end{tikzcd} \quad \longrightarrow \quad \begin{tikzcd} A & & B \arrow[ll, bend right=50, "S"'] \arrow[ll, bend left=50, "S''"] \arrow[ll, phantom, "\Downarrow s's"] \end{tikzcd} ,$$

the identity maps of the categories $\underline{S}(A, B)$ will be called degenerate 2-cells. (We shall in particular use this representation with categories as vertices, functors as arrows and natural transformations as 2-cells in (v) and (vi) below.)

(iii) For each triple (A, B, C) of objects of $\underline{S}$, a composition functor:

$$c(A, B, C): \underline{S}(A, B) \times \underline{S}(B, C) \longrightarrow \underline{S}(A, C).$$

We write $S \circ T$ and $s \circ t$ instead of $c(A, B, C)(S, T)$ and $c(A, B, C)(s, t)$ for (S, T) and (s, t) objects and maps of $\underline{S}(A, B) \times \underline{S}(B, C)$, and abbreviate $\mathrm{Id}_S \circ t$ and $s \circ \mathrm{Id}_T$ into $S \circ t$ and $s \circ T$. This composition corresponds to to the pasting:

$$\begin{tikzcd} A & B \arrow[l, bend right, "S"'] \arrow[l, bend left, "S'"] \arrow[l, phantom, "\Downarrow s"] & C \arrow[l, bend right, "T"'] \arrow[l, bend left, "T'"] \arrow[l, phantom, "\Downarrow t"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} A & C \arrow[l, bend right, "S \circ T"'] \arrow[l, bend left, "S' \circ T'"] \arrow[l, phantom, "\Downarrow s \circ t"] \end{tikzcd}$$

(iv) For each object A of $\underline{S}$ an object I_A of $\underline{S}(A,A)$ called <u>identity arrow</u> of A. The identity map of I_A in $\underline{S}(A,A)$ is denoted $i_A : I_A \Longrightarrow I_A$ and called <u>identity 2-cell</u> of A.

(v) For each quadruple (A,B,C,D) of objects of $\underline{S}$, a natural isomorphism $a(A,B,C,D)$, called <u>associativity isomorphism</u>, between the two composite functors bounding the diagram:

$$\begin{tikzcd} \underline{S}(A,B) \times \underline{S}(B,C) \times \underline{S}(C,D) \arrow[rr, "\mathrm{Id} \times c(B,C,D)"] \arrow[d, "c(A,B,C) \times \mathrm{Id}"'] \arrow[rr, phantom, bend right=20, "\overset{\sim}{\Nearrow} a(A,B,C,D)"] & & \underline{S}(A,B) \times \underline{S}(B,D) \arrow[d, "c(A,B,D)"] \\ \underline{S}(A,C) \times \underline{S}(C,D) \arrow[rr, "c(A,C,D)"'] & & \underline{S}(A,D) \end{tikzcd}$$

Explicitly:

$$a(A,B,C,D) : c(A,C,D) \circ (c(A,B,C) \times \mathrm{Id}) \longrightarrow c(A,B,D) \circ (\mathrm{Id} \times c(B,C,D))$$

If (S,T,U) is an object of $\underline{S}(A,B) \times \underline{S}(B,C) \times \underline{S}(C,D)$ the isomorphism $a(A,B,C,D)(S,T,U) : (S \circ T) \circ U \xrightarrow{\ \sim\ } S \circ (T \circ U)$ in $\underline{S}(A,D)$ is called the <u>component</u> of $a(A,B,C,D)$ at (S,T,U) and is abbreviated into $a(S,T,U)$ or even a, except when confusions are possible (cf. §3 for example).

(vi) For each pair (A, B) of objects of $\underline{S}$, two natural isomorphisms $\ell(A, B)$ and $r(A, B)$, called <u>left</u> and <u>right</u> identities, between the functors bounding the diagrams:

$$\begin{tikzcd}
1 \times \underline{S}(A,B) \arrow[rr, "I_A \times \mathrm{Id}"] \arrow[dr, "\sim", "\text{canonical}"'] \arrow[rr, phantom, shift right=3ex, "\Nearrow \ell(A,B)"'] & & \underline{S}(A,A) \times \underline{S}(A,B) \arrow[dl, "c(A,A,B)"] \\
& \underline{S}(A,B) &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{S}(A,B) \times 1 \arrow[rr, "\mathrm{Id} \times I_B"] \arrow[dr, "\sim", "\text{canonical}"'] \arrow[rr, phantom, shift right=3ex, "\Nearrow r(A,B)"'] & & \underline{S}(A,B) \times \underline{S}(B,B) \arrow[dl, "c(A,B,B)"] \\
& \underline{S}(A,B) &
\end{tikzcd}$$

If S is an object of $\underline{S}(A, B)$, the isomorphism, component at S of $\ell(A, B)$,

$$\ell(A,B)(S) \colon I_A \circ S \xrightarrow{\sim} S$$

is abbreviated into $\ell(S)$ or even ℓ, and similarly we write:

$$r = r(S) = r(A,B)(S) \colon S \circ I_B \xrightarrow{\sim} S.$$

The families of natural isomorphisms $a(A, B, C, D)$, $\ell(A, B)$ and $r(A, B)$ are furthermore required to satisfy the following axioms:

(A. C.) Associativity coherence: If (S, T, U, V) is an object of $\underline{S}(A, B) \times \underline{S}(B, C) \times \underline{S}(C, D) \times \underline{S}(D, E)$ the following diagram commutes:

$$
\begin{tikzcd}
((S\circ T)\circ U)\circ V \arrow[rr, "a(S,T,U)\circ Id"] \arrow[d, "a(S\circ T,U,V)"'] & & (S\circ(T\circ U))\circ V \arrow[d, "a(S,T\circ U,V)"] \\
(S\circ T)\circ(U\circ V) \arrow[dr, "a(S,T,U\circ V)"'] & & S\circ((T\circ U)\circ V) \arrow[dl, "Id\circ a(T,U,V)"] \\
& S\circ(T\circ(U\circ V)) &
\end{tikzcd}
$$

(I. C.) Identity coherence: If (S, T) is an object of $\underline{S}(A, B) \times \underline{S}(B, C)$ the following diagram commutes:

$$
\begin{tikzcd}
(S\circ I_B)\circ T \arrow[rr, "a(S,I_B,T)"] \arrow[dr, "r(S)\circ Id"'] & & S\circ(I_B\circ T) \arrow[dl, "Id\circ \ell(T)"] \\
& S\circ T &
\end{tikzcd}
$$

<u>(1.2) Remark:</u> In order to avoid cumbersome notations, when the $\underline{S}(A, B)$'s shall not be disjoint, we will identify them with their canonical images in the disjoint union.

<u>(1. 3) Global definition</u>

(1.3.1) A <u>bigraph</u> (or <u>bidiagram scheme</u>) Σ is a diagram of sets and maps

$$
\Sigma_o \underset{\partial_1^{(1)}}{\overset{\partial_o^{(1)}}{\leftleftarrows}} \Sigma_1 \underset{\partial_1^{(2)}}{\overset{\partial_o^{(2)}}{\leftleftarrows}} \Sigma_2 \quad , \quad \text{such that:}
$$

(1.3.2) $$\partial_i^{(1)} \partial_o^{(2)} = \partial_i^{(1)} \partial_1^{(2)} \qquad (i = 0, 1) \ .$$

We usually omit the superscript. Elements of $\Sigma_o, \Sigma_1, \Sigma_2$ are called vertices $A, B, \ldots$, arrows $S, T, \ldots$, and 2-cells $s, t, \ldots$. A 2-cell is represented by:

$$\begin{tikzcd} A & B \arrow[l, bend right, "S_o"'] \arrow[l, bend left, "S_1"] \arrow[l, phantom, "\Downarrow s"] \end{tikzcd} \qquad \text{with :} \qquad \begin{array}{ll} S_o = \partial_o s & S_1 = \partial_1 s \\ \partial_o S_o = B = \partial_o S_1 \ , & \partial_1 S_o = A = \partial_1 S_1 \end{array}$$

For $n = 0, 1, 2$ we call <u>n-skeleton</u> the set $\Sigma^{[n]} = \bigcup_{i=0}^{n} \Sigma_i$. A bicategory $\underline{S}$ admits obviously an underlying bidiagram, which we usually also write $\underline{S}$, thus $S^{[k]}$ $k = 0, 1, 2$ makes sense. In particular, $Cat^{[1]}$ consists of "all" categories and functors (see (2. 2)).

(1. 3. 3) A <u>multiplication</u> μ on a bigraph Σ consists of maps:

$$\mu^{(2)}: \Sigma_2 \times_{\Sigma_1} \Sigma_2 \longrightarrow \Sigma_2 \qquad (s_1, s_2) \rightsquigarrow s_1 s_2$$

$$\mu_{\cdot}^{(1)}: \Sigma_1 \times_{\Sigma_o} \Sigma_1 \longrightarrow \Sigma_1 \qquad (S, T) \rightsquigarrow S \circ T$$

$$\mu_{\cdot}^{(2)}: \Sigma_2 \times_{\Sigma_o} \Sigma_2 \longrightarrow \Sigma_2 \qquad (s, t) \rightsquigarrow s \circ t$$

(by (1.3.2) there are only two maps from Σ_2 to Σ_o, thus $\Sigma_2 \times_{\Sigma_o} \Sigma_2$ is well defined) such that the diagrams (1.3.4) and (1.3.5) commute:

(1.3.4)
$$\begin{tikzcd} \Sigma_2 \arrow[d, "\partial_1"'] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[l, "pr_1"'] \arrow[r, "pr_2"] \arrow[d, "\mu^{(2)}"] & \Sigma_2 \arrow[d, "\partial_o"] \\ \Sigma_1 & \Sigma_2 \arrow[l, "\partial_1"] \arrow[r, "\partial_o"'] & \Sigma_1 \end{tikzcd} .$$

where $pr_i(s_1, s_2) = s_i$. That is, $\partial_o(s_o s_1) = \partial_o s_1$ and $\partial_1(s_o s_1) = \partial_1 s_o$.

And:

(1.3.5)
$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_o} \Sigma_1 \arrow[d, "\mu.^{(1)}"'] & \Sigma_2 \times_{\Sigma_o} \Sigma_2 \arrow[l, "\partial_o \times_{\Sigma_o} \partial_o"'] \arrow[r, "\partial_1 \times_{\Sigma_o} \partial_1"] \arrow[d, "\mu.^{(2)}"] & \Sigma_1 \times_{\Sigma_o} \Sigma_1 \arrow[d, "\mu.^{(1)}"] \\
\Sigma_1 & \Sigma_2 \arrow[l, "\partial_o"] \arrow[r, "\partial_1"'] & \Sigma_1
\end{tikzcd}$$

That is: $\partial_o(s \bullet t) = (\partial_o S \circ \partial_o T)$; $\partial_1(s \circ t) = (\partial_1 S \circ \partial_1 T)$.

(1.3.6) A <u>degeneracy</u> (system) σ on a bigraph Σ consists of a pair of maps:

$$\Sigma_o \xrightarrow{\sigma^{(1)}} \Sigma_1 \xrightarrow{\sigma^{(2)}} \Sigma_2$$

written $A \rightsquigarrow I_A = \sigma^{(1)}A$ and $S \rightsquigarrow i_S = \sigma^{(2)}S$, satisfying:

(1.3.7) $$\partial_j^{(i)}\sigma^{(i)} = \mathrm{Id} \qquad i = 1, 2 \; ; \quad j = 0, 1 \, .$$

Let Σ be a bigraph equipped with a multiplication μ. An <u>association</u> on (Σ, μ) is a map:

$$a : \Sigma_1 \times_{\Sigma_o} \Sigma_1 \times_{\Sigma_o} \Sigma_1 \longrightarrow \Sigma_2 \qquad (S, T, U) \rightsquigarrow a(S, T, U)$$

making commutative the diagram:

(1.3.8)
$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_o} \Sigma_1 \arrow[d, "\mu.^{(1)}"'] & \Sigma_1 \times_{\Sigma_o} \Sigma_1 \times_{\Sigma_o} \Sigma_1 \arrow[l, "\mu.^{(1)} \times_{\Sigma_o} \mathrm{Id}"'] \arrow[r, "\mathrm{Id} \times_{\Sigma_o} \mu.^{(1)}"] \arrow[d, "a"] & \Sigma_1 \times_{\Sigma_o} \Sigma_1 \arrow[d, "\mu.^{(1)}"] \\
\Sigma_1 & \Sigma_2 \arrow[l, "\partial_o"] \arrow[r, "\partial_1"'] & \Sigma_1
\end{tikzcd}$$

That is: $\partial_o a(S,T,U) = (S\circ T)\circ U$; $\partial_1 a(S,T,U) = S\circ(T\circ U)$.

If furthermore Σ is equipped with a degeneracy σ, <u>left</u> and <u>right</u> <u>identity (systems)</u> are maps:

$$\ell : \Sigma_1 \longrightarrow \Sigma_2 \qquad S \rightsquigarrow \ell(S)$$

$$r : \Sigma_1 \longrightarrow \Sigma_2 \qquad S \rightsquigarrow r(S)$$

making commutative the diagram:

(1.3.9)
$$\begin{tikzcd}
 & \Sigma_1 \arrow[dl, "Id"'] \arrow[d, "r"] \arrow[r, "{(\sigma^{(1)}, Id)}"] & \Sigma_1 \times_{\Sigma_o} \Sigma_1 \arrow[d, "\mu^{(1)}"] \arrow[r, "{(Id, \sigma^{(1)})}"] & \Sigma_1 \arrow[d, "\ell"] \arrow[dr, "Id"] & \\
\Sigma_1 \arrow[r, "\partial_1"'] & \Sigma_2 \arrow[r, "\partial_o"'] & \Sigma_1 \arrow[r, "\partial_o"'] & \Sigma_2 \arrow[r, "\partial_1"'] & \Sigma_1
\end{tikzcd}$$

That is, $\partial_o \ell(S) = I_A \circ S$; $\partial_1 \ell(S) = S$; $\partial_o r(S) = S\circ I_B$; $\partial_1 r(S) = S$.

The choice of notation is such that, if $\underline{S}$ is a bicategory, the underlying bigraph is clearly equipped with a canonical multiplication, degeneracy, association and identities, called underlying to $\underline{S}$.

In terms of $(\Sigma, \mu, \sigma, a, \ell, r)$ bicategories can be characterized by means of the following:

<u>(1.3.10) Proposition:</u> Let Σ be a bigraph equipped with μ, σ, a, ℓ, r. These data are underlying to a bicategory, then necessarily unique, iff they satisfy conditions (i) to (x) below:

(i) The following diagram is commutative:

$$\begin{tikzcd}
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mu^{(2)} \times_{\Sigma_1} Id"] \arrow[d, "Id \times_{\Sigma_1} \mu^{(2)}"'] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[d, "\mu^{(2)}"] \\
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mu^{(2)}"'] & \Sigma_2
\end{tikzcd} .$$

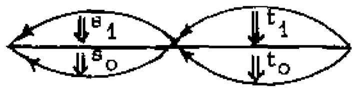
(ii) The following diagram is commutative:

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "{(\sigma\partial_1, \mathrm{Id})}"] \arrow[dr, "\mathrm{Id}"', "\sim"] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[d, "\mu^{(2)}"] & \Sigma_2 \arrow[l, "{(\mathrm{Id}, \sigma\partial_0)}"'] \arrow[dl, "\mathrm{Id}", "\sim"'] \\
& \Sigma_2 &
\end{tikzcd}$$

(iii) The following diagram is commutative:

$$\begin{tikzcd}
(\Sigma_2 \times_{\Sigma_0} \Sigma_2) \underset{\Sigma_1 \times_{\Sigma_0} \Sigma_1}{\times} (\Sigma_2 \times_{\Sigma_0} \Sigma_2) \arrow[rr, "\tau", "\sim"'] \arrow[d, "\mu.^{(2)} \times_{\Sigma_1} \mu.^{(2)}"'] & & (\Sigma_2 \times_{\Sigma_1} \Sigma_2) \times_{\Sigma_0} (\Sigma_2 \times_{\Sigma_1} \Sigma_2) \arrow[d, "\mu^{(2)} \times_{\Sigma_0} \mu^{(2)}"] \\
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mu^{(2)}"'] & \Sigma_2 & \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[l, "\mu.^{(2)}"]
\end{tikzcd} ,$$

where τ is the canonical map $((s_1, t_1), (s_0, t_0)) \rightsquigarrow ((s_1, s_0), (t_1, t_0))$ defined when s_i, t_i satisfy the incidence relations depicted by:



and $\mu.^{(2)} \times_{\Sigma_1} \mu.^{(2)}$ is the unique map, which exists because of (1.3.5), making commutative the diagram:

$$\begin{tikzcd}
\Sigma_2 \times_{\Sigma_0} \Sigma_2) \underset{\Sigma_1 \times_{\Sigma_0} \Sigma_1}{\times} (\Sigma_2 \times_{\Sigma_2} \Sigma_2) \arrow[r, hook] \arrow[d, dashed, "\mu.^{(2)} \times_{\Sigma_1} \mu.^{(2)}"'] & (\Sigma_2 \times_{\Sigma_0} \Sigma_2) \times (\Sigma_2 \times_{\Sigma_0} \Sigma_2) \arrow[d, "\mu.^{(2)} \times \mu.^{(2)}"] \\
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, hook] & \Sigma_2 \times \Sigma_2
\end{tikzcd} ,$$

where the horizontal arrows are the canonical monomorphisms of pull-backs into products.

(iv) The following diagram is commutative:

$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[r, "\mu.^{(1)}"] \arrow[d, "\sigma^{(2)} \times_{\Sigma_0} \sigma^{(2)}"'] & \Sigma_1 \arrow[d, "\sigma^{(2)}"] \\
\Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[r, "\mu.^{(2)}"'] & \Sigma_2
\end{tikzcd} .$$

From (1.3.8) and (1.3.5) it follows that the exterior of the following diagram is commutative, thus there exists a unique map φ making the whole diagram commutative

$$\begin{tikzcd}
 & \Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[rr, "a"] & & \Sigma_2 \arrow[dr, "\partial_0"] & \\
\Sigma_2 \times_{\Sigma_0} \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[ur, "\partial_1 \times_{\Sigma_0} \partial_1 \times_{\Sigma_0} \partial_1"] \arrow[dr, "\mu.^{(2)} \times_{\Sigma_0} \mathrm{Id}"'] \arrow[rr, dashed, "\varphi"] & & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[ur, "\mathrm{proj}_1"] \arrow[dr, "\mathrm{proj}_2"] & & \Sigma_1 \\
 & \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[rr, "\mu.^{(2)}"'] & & \Sigma_2 \arrow[ur, "\partial_1"'] &
\end{tikzcd} .$$

Similarly, let φ' be the unique map making commutative:

$$\begin{tikzcd}
 & \Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[rr, "a"] & & \Sigma_2 \arrow[dr, "\partial_1"] & \\
\Sigma_2 \times_{\Sigma_0} \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[ur, "\partial_0 \times_{\Sigma_0} \partial_0 \times_{\Sigma_0} \partial_0"] \arrow[dr, "\mathrm{Id} \times_{\Sigma_0} \mu.^{(2)}"'] \arrow[rr, dashed, "\varphi'"] & & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[ur, "\mathrm{proj}_2"] \arrow[dr, "\mathrm{proj}_1"] & & \Sigma_1 \\
 & \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[rr, "\mu.^{(2)}"'] & & \Sigma_2 \arrow[ur, "\partial_0"'] &
\end{tikzcd} .$$

(v) The following diagram is commutative:

$$\begin{tikzcd}
\Sigma_2 \times_{\Sigma_0} \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[r, "\varphi"] \arrow[d, "\varphi'"'] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[d, "\mu^{(2)}"] \\
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mu^{(2)}"'] & \Sigma_2
\end{tikzcd}$$

From (1.3.9) it follows that the exterior of the following diagram commutes, hence there is a unique ψ_ℓ making the whole diagram commute:

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "\partial_0"] \arrow[dr, dashed, "\psi_\ell"'] \arrow[ddr, "\mathrm{Id}"', "\wr"] & \Sigma_1 \arrow[dr, "\ell"] & \\
& \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mathrm{proj}_1"'] \arrow[d, "\mathrm{proj}_2"] & \Sigma_2 \arrow[d, "\partial_1"] \\
& \Sigma_2 \arrow[r, "\partial_0"'] & \Sigma_1
\end{tikzcd}$$

Similarly let ψ_ℓ' be the unique map making commutative:

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "\partial_1\partial_1"] \arrow[dr, dashed, "\psi_\ell'"'] \arrow[ddr, "\mathrm{Id}"', "\wr"] & \Sigma_0 \arrow[dr, "\sigma^{(2)}\sigma^{(1)}"] & \\
& \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[r, "\mathrm{proj}_1"'] \arrow[d, "\mathrm{proj}_2"] & \Sigma_2 \arrow[d, "\partial_0\partial_0"] \\
& \Sigma_2 \arrow[r, "\partial_1\partial_1"'] & \Sigma_0
\end{tikzcd}$$

And ψ''_ℓ the unique map making commutative:

$$\begin{tikzcd}
\Sigma_2 \arrow[rr, "\psi'_\ell"] \arrow[dr, dashed, "\psi''_\ell"'] \arrow[dd, "\partial_1"'] & & \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[dr, "\mu.^{(2)}"] & \\
& \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[rr, "proj_2"'] \arrow[dd, "proj_1"] & & \Sigma_2 \arrow[dd, "\partial_1"] \\
\Sigma_1 \arrow[dr, "\ell"'] & & & \\
& \Sigma_2 \arrow[rr, "\partial_0"'] & & \Sigma_1
\end{tikzcd} .$$

$(vi)_\ell$: The following diagram is **commutative:**

$$\begin{tikzcd}
\Sigma_2 \arrow[r, "\psi''_\ell"] \arrow[d, "\psi_\ell"'] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[d, "\mu^{(2)}"] \\
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mu^{(2)}"'] & \Sigma_2
\end{tikzcd} .$$

We let the reader write the analogous diagram $(vi)_r$ for r.

From the definitions of $a, \mu.^{(2)}$ and $\mu.^{(1)}$ it follows that the exterior of the diagram below is commutative, hence there is a unique map θ_1 making the whole diagram commutative:

$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[rr, "a \times_{\Sigma_0} \sigma^{(1)}"] \arrow[dr, dashed, "\theta_1"] \arrow[dd, "Id \times_{\Sigma_0} \mu.^{(1)} \times_{\Sigma_0} Id"'] & & \Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[dr, "\mu.^{(2)}"] & \\
& \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[rr, "proj_2"'] \arrow[dd, "proj_1"] & & \Sigma_2 \arrow[dd, "\partial_1"] \\
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[dr, "a"'] & & & \\
& \Sigma_2 \arrow[rr, "\partial_0"'] & & \Sigma_1
\end{tikzcd} .$$

In a similar way, let θ_2 be the unique map making commutative the diagram:

$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[r, "\theta_1"] \arrow[dr, dashed, "\theta_2"] \arrow[dd, "\sigma^{(1)} \times_{\Sigma_0} a"'] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[dr, "\mu^{(2)}"] & \\
 & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"] & \Sigma_2 \arrow[d, "\partial_1"] \\
\Sigma_2 \times_{\Sigma_0} \Sigma_2 \arrow[r, "\mu^{(1)}"'] & \Sigma_2 \arrow[r, "\partial_0"'] & \Sigma_1
\end{tikzcd}$$

and θ_3 be the unique map making commutative:

$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[r, "\mu^{(1)} \times_{\Sigma_0} \mathrm{Id} \times_{\Sigma_0} \mathrm{Id}"] \arrow[dr, dashed, "\theta_3"] \arrow[dd, "\mathrm{Id} \times_{\Sigma_0} \mathrm{Id} \times_{\Sigma_0} \mu^{(1)}"'] & \Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[dr, "a"] & \\
 & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mathrm{proj}_2"] \arrow[d, "\mathrm{proj}_1"] & \Sigma_2 \arrow[d, "\partial_1"] \\
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[r, "a"'] & \Sigma_2 \arrow[r, "\partial_0"'] & \Sigma_1
\end{tikzcd}$$

(vii) The following diagram is commutative:

$$\begin{tikzcd}
\Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[r, "\theta_2"] \arrow[d, "\theta_3"'] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[d, "\mu^{(2)}"] \\
\Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[r, "\mu^{(2)}"'] & \Sigma_2 .
\end{tikzcd}$$

Again one can check the commutativity of the exterior of the diagram below, and define α to be the unique map making commutative:

$$\begin{tikzcd}
\Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \arrow[rr, "(\mathrm{Id}, \sigma^{(2)}\partial_0) \underset{\Sigma_0}{\times} \mathrm{Id}"] \arrow[dr, dashed, "\alpha"] \arrow[dd, "\sigma^{(2)} \underset{\Sigma_0}{\times} \ell"'] & & (\Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1) \underset{\Sigma_0}{\times} \Sigma_1 \arrow[d, "a"] \\
& \Sigma_2 \underset{\Sigma_1}{\times} \Sigma_2 \arrow[r, "\mathrm{proj}_2"'] \arrow[d, "\mathrm{proj}_1"] & \Sigma_2 \arrow[d, "\partial_1"] \\
\Sigma_2 \underset{\Sigma_0}{\times} \Sigma_2 \arrow[r, "\mu.^{(2)}"'] & \Sigma_2 \arrow[r, "\partial_0"'] & \Sigma_1
\end{tikzcd}$$

(viii) The following diagram is commutative:

$$\begin{tikzcd}
\Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \arrow[r, "\alpha"] \arrow[d, "r \underset{\Sigma_0}{\times} \sigma^{(2)}"'] & \Sigma_2 \underset{\Sigma_1}{\times} \Sigma_2 \arrow[d, "\mu^{(2)}"] \\
\Sigma_2 \underset{\Sigma_0}{\times} \Sigma_2 \arrow[r, "\mu.^{(2)}"'] & \Sigma_2
\end{tikzcd}$$

(ix) There exists a map, (necessarily unique)

$$\bar{a}\colon \Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \longrightarrow \Sigma_2$$

such that $\partial_0 \bar{a} = \partial_1 a$, $\partial_1 \bar{a} = \partial_0 a$, making commutative the diagram (where a map into a pullback is denoted by its two components):

$$\begin{tikzcd}
& \Sigma_2 \underset{\Sigma_1}{\times} \Sigma_2 \arrow[dl, "\mu^{(2)}"'] & & \Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \arrow[ll, "(a, \bar{a})"'] \arrow[rr, "(\bar{a}, a)"] \arrow[ddl, "\mathrm{Id} \underset{\Sigma_0}{\times} \mu.^{(1)}"'] \arrow[ddr, "\mu.^{(1)} \underset{\Sigma_0}{\times} \mathrm{Id}"] & & \Sigma_2 \underset{\Sigma_1}{\times} \Sigma_2 \arrow[dr, "\mu^{(2)}"] & \\
\Sigma_2 & & & & & & \Sigma_2 \\
& \Sigma_1 \arrow[ul, "\sigma^{(2)}"] & \Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \arrow[l, "\mu.^{(1)}"] & & \Sigma_1 \underset{\Sigma_0}{\times} \Sigma_1 \arrow[r, "\mu.^{(1)}"'] & \Sigma_1 \arrow[ur, "\sigma^{(2)}"'] &
\end{tikzcd}$$

$(x)_\ell$ There exists a map, again unique

$$\bar{\ell} : \Sigma_1 \longrightarrow \Sigma_2$$

such that $\partial_0 \bar{\ell} = \partial_1 \ell$, $\partial_1 \bar{\ell} = \partial_0 \ell$, making commutative:

$$\begin{tikzcd}
& \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[dl, "\mu^{(2)}"'] & \Sigma_1 \arrow[l, "(\bar{\ell},\ell)"'] \arrow[r, "(\ell,\bar{\ell})"] \arrow[d, "(\sigma^{(1)}\partial_1, \mathrm{Id})"'] \arrow[dr, "\sigma^{(2)}"] & \Sigma_2 \times_{\Sigma_1} \Sigma_2 \arrow[d, "\mu^{(2)}"] \\
\Sigma_2 & \Sigma_1 \arrow[l, "\sigma^{(2)}"] & \Sigma_1 \times_{\Sigma_0} \Sigma_1 \arrow[l, "\mu.^{(1)}"] & \Sigma_2
\end{tikzcd}$$

Again there is a similar diagram $(x)_r$ for r, left to the reader to provide.

We make the following comments about the proof, the details of which are left to the reader. Given $(\Sigma, \mu, \dots)$ satisfying (i) to (x), we define a bicategory $\underline{S}$ as follows:

It has Σ_0 as set of objects. If $A, B \in \Sigma_0$, $\underline{S}(A, B)$ has as objects the elements S of Σ_1 such that $\partial_1 S = A$, $\partial_0 S = B$ and as arrows the elements s of Σ_2 such that $\partial_0 \partial_0 s = B$, $\partial_1 \partial_1 s = A$; the domain and codomain of s are $\partial_0 s$ and $\partial_1 s$, the composition is $(s_1, s_2) \rightsquigarrow s_1 s_2$; the conditions (i) and (ii) state precisely that $\underline{S}(A, B)$ is a category. For each triple (A, B, C), $c(A, B, C)$ is given on objects (resp. maps) by the restrictions of $\mu.^{(1)}$ (resp. $\mu.^{(2)}$) to $\mathrm{Ob}\,\underline{S}(A, B) \times \mathrm{Ob}\,\underline{S}(B, C) \subset \Sigma_1 \times_{\Sigma_0} \Sigma_1$ (resp. ...). Then (iii) and (iv) mean that the $c(A, B, C)$'s are bifunctors. For each

A, B, C, D, the restriction of a to $\mathrm{Ob}\,\underline{S}(A,B) \times \mathrm{Ob}\,\underline{S}(B,C) \times \mathrm{Ob}\,\underline{S}(C,D) \subset \Sigma_1 \times_{\Sigma_o} \Sigma_1 \times_{\Sigma_o} \Sigma_1$ is a(A, B, C, D) whose naturality follows from (v). For each A, B the restriction of ℓ to $\mathrm{Ob}\,\underline{S}(A,B) \subset \Sigma_1$ is according to $(vi)_\ell$ a natural transformation $\ell(A,B)$ (same thing for r). The coherence (A.C) is expressed by (vii), (I.C) by (vii), and finally (ix) and $(x)_\ell$ state that the a's and ℓ's are isomorphisms with inverses the restrictions of $\bar{a}$ and $\bar{\ell}$ (and $(x)_r$ gives r^{-1}).

<u>(1.4) Remark:</u> The proposition (1.3.10) makes it possible to define a bicategory in terms of $\Sigma_o, \Sigma_1, \Sigma_2$, the maps $\partial, \mu, a, \ldots$ satisfying (i) to (x). This definition, although much longer and less intuitive than (1.1) has the following advantages:

(i) It is purely "diagrammatic", and can be stated with Σ_i objects of any category with pullbacks, giving such examples as topological, or ordered, bicategories. (The only place where elements were used was, for the sake of brevity, in defining τ which obviously exists in any category with pullbacks.)

(ii) Even in the case of sets, it shows that bicategories are "algebraic", i. e. , defined in terms of finite inverse limits, and makes available all the general theorems on algebraic structures (see e. g. (7.4.1) below).

2. Examples of bicategories

The following examples are designed to fix the terminology for further reference and, hopefully, to provide the reader with intuitive support and motivation for the forthcoming nonsense:

(2.1) 2-Categories: A 2-category is defined in [B. 3] (example 2) by data identical to (i), (ii), (iii), and (iv) of (1.1), such that the diagrams of functors bounding the 2-cells of (v) and (vi) are commutative. If we take $a(A, B, C, D)$, $\ell(A, B)$, $r(A, B)$ to be the identity natural isomorphisms, the axioms (A. C) and (I. C) are obviously satisfied. Thus, the 2-categories (also called Hypercategories in [E. K]) can be identified with the bicategories where c is strictly associative, with I_A's as strict identities for c; and a, ℓ, r are identity natural transformations. We shall see however that the notion of morphisms of bicategories, even when restricted to 2-categories, gives a wider and more interesting class than the 2-functors (cf. (5.3)).

In particular, we will denote by Tac(*) the 2-category with objects "all" categories, $\mathrm{Tac}(A, B)$ being the category of all functors from B to A, and if S, S', S'', T, T' and s, s', t are functors and natural transformations satisfying the incidence relations represented in the "bidiagram"

$$\begin{tikzcd}[column sep=large]
A & B \arrow[l, bend right=60, "S"'] \arrow[l, "S'" description] \arrow[l, bend left=60, "S''"] \arrow[l, phantom, shift right=3, "\Downarrow s"] \arrow[l, phantom, shift left=3, "\Downarrow s'"] & C \arrow[l, bend right=60, "T"'] \arrow[l, bend left=60, "T'"] \arrow[l, phantom, "\Downarrow t"]
\end{tikzcd},$$

(*) The notation $\mathrm{Moh}(A, B)$ for $\mathrm{Hom}(B, A)$, A, B objects of any category, is due to Epstein. See (3.4.1) for the "transpose" Cat of Tac.

then $S \circ T$ is the composite functor ST, $s' \circ s$ is the usual composite of natural transformations, and $s \circ t = (S' * t) \circ (s * T) = (s * T') \circ (S * t)$ with the notatation of [G] (p. 269). In this case we shall write $s * t$ instead of $s \circ t$.

(2.2) Multiplicative categories. Let $\underline{M} = (\underline{A}, \otimes, \Lambda, \theta, \gamma, \delta)$ be a category with multiplication (c. m.) as defined in [B. 1]. Take $\underline{S}_o$ to be a set having a single element, say 0. Define $\underline{S}(0,0) = \underline{A}$, $c(0,0,0) = \otimes$, $I_o = \Lambda$, $a(0,0,0,0) = \theta$, $\ell(0,0) = \gamma$, $r(0,0) = \delta$. These data satisfy (A. C) and (I. C) and thus define a bicategory $\underline{S}$ with one object. Conversely, every bicategory with one object "is" a c. m. . More generally we have:

Proposition (2.2.1). Let $\underline{S}$ be a bicategory and A an object of $\underline{S}$, then $c(A,A,A) = \otimes$, $I_A = \Lambda$, $a(A,A,A,A) = \theta$, $\ell(A,A) = \gamma$, $r(A,A) = \delta$ determine on the category $\underline{A} = \underline{S}(A,A)$ a multiplicative structure called induced by $\underline{S}$.

The proposition follows from the general coherence theorem of [B. 4]. See also [M], and compare with the "one-dimensional" case: a monoid "is" a category with one object, and for any category $\underline{C}$ and object A of $\underline{C}$, $\mathrm{Hom}(A,A)$ is a monoid.

Note furthermore that if $\underline{S}$ is a 2-category, then $\underline{S}(A,A)$ is a strictly associative c. m.. In particular, taking $\underline{S} = \mathrm{Tac}$, we get the multiplicative structure of the category of endofunctors of any category defined in [B. 1].

(2.3) Actions of Multiplicative Categories. Let $\underline{M}$ be a c.m. and $\underline{X}$ any category. A left action of $\underline{M}$ on $\underline{X}$ is defined by:

(i) A functor: $\underline{A} \times \underline{X} \longrightarrow \underline{X}$; $(A, X) \rightsquigarrow A \otimes X$,

(ii) Natural isomorphisms:

$$\alpha: (A_1 \otimes A_2) \otimes X \xrightarrow{\sim} A_1 \otimes (A_2 \otimes X) \quad \text{and} \quad \eta: \Lambda \otimes X \xrightarrow{\sim} X,$$

satisfying "obvious" coherence conditions.

Such a left action can be identified with the bicategory $\underline{S}$ described by: $\mathrm{Ob}(\underline{S}) = \{0, 1\}$, $\underline{S}(0,0) = \underline{A}$, $\underline{S}(0,1) = \underline{X}$, $\underline{S}(1,1) = 1$, $\underline{S}(0,1) = \mathbf{0}$, $c(0,0,0) = \otimes$, $c(0,0,1) = \otimes$, ... The reader will provide the rest of the data.

Conversely, if $\underline{S}$ is any bicategory, and A, B are two objects of $\underline{S}$, the c.m. $\underline{S}(A, A)$ inherits from $\underline{S}$ a canonical left action on $\underline{S}(A, B)$ given by: $S \circ T = S \otimes T$, $\alpha = a\colon (S_1 \otimes S_2) \otimes T \xrightarrow{\sim} S_1 \otimes (S_2 \otimes T)$, $\eta = \ell$, $I_A \otimes T \xrightarrow{\sim} T$.

For example, if $\underline{X}$ is any category and $\underline{M} = \mathrm{Tac}(\underline{X}, \underline{X})$ is the category of endofunctors of $\underline{X}$, it acts on $\underline{X}$ by $(F, X) \rightsquigarrow F(X)$. Or again: If $\underline{X}$ is any abelian category with arbitrary colimits (resp. any category with arbitrary products) and $\underline{M}$ is the c.m. of abelian groups (resp. Sets*) with $\otimes$ (resp $\times$) as multiplication, a choice of colimits (resp. products) determines a canonical left action by $(A, X) \rightsquigarrow A \otimes X$ (resp. $(A, X) \rightsquigarrow X^A = \prod_{a \in A} X$).

Similarly, we can define a right action, or a biaction: $\underline{M}$ and $\overline{\underline{M}}$ are c. m., $\underline{M}$ acts on the left on $\underline{X}$, $\overline{\underline{M}}$ on the right, and both actions "commute" up to coherent isomorphisms:

$$(A\otimes X)\otimes \overline{A} \xrightarrow{\sim} A\otimes(X\otimes\overline{A}).$$

All these data and axioms can be reduced to: a bicategory $\underline{S}$ with two objects, say 0 and 1, such that $\underline{S}(1,0) = \emptyset$.

(2.4). In [E], Epstein considers the following situation: Categories $\underline{A}, \underline{B}, \underline{C}, \underline{M}, \underline{N}, \underline{O}$, functors, denoted by $\otimes$:

$$\underline{A}\times\underline{B} \longrightarrow \underline{M}\ ;\ \underline{B}\times\underline{C}\longrightarrow \underline{N}\ ;\ \underline{M}\times\underline{C}\longrightarrow\underline{O}\ ;\ \underline{A}\times\underline{N}\longrightarrow\underline{O}$$

and a natural isomorphism $\alpha:(A\otimes B)\otimes C \xrightarrow{\sim} A\otimes(B\otimes C)$. This reduces to: A bicategory $\underline{S}$ having four objects 0, 1, 2, 3, such that

$$\underline{S}(i,j) = \emptyset \quad \text{for } i>j \quad \text{and} \quad \underline{S}(i,i) = 1.$$

(Take then $\underline{S}(0,1) = \underline{A}$, $\underline{S}(1,2) = \underline{B}$, $\underline{S}(2,3) = \underline{C}$, $\underline{S}(0,2) = \underline{M}$, $\underline{S}(1,3) = \underline{N}$, $\underline{S}(0,3) = \underline{O}$, etc.)

(2.5) Bimodules. A basic example, to be generalized and studied in Part II, is the bicategory Bim of bimodules defined as follows: The objects are the rings with identity. If A and B are rings, $\mathrm{Bim}(A,B) = {}_A\underline{M}_B$ is the category of (A, B)-bimodules. If $M \in {}_A\underline{M}_B$ and $N \in {}_B\underline{M}_C$, $M\circ N$ is the (A, C)-bimodule $M\otimes_B N$. The ring A, as an (A, A)-bimodule is I_A. Finally, a, ℓ, r are the usual isomorphisms of the tensor.

With this definition, an "arrow" between two rings A and B is an (A, B)-bimodule, composition being the tensor. Note that the usual arrows, i.e., ring homomorphisms $f: B \longrightarrow A$ determine (A, B)-bimodules $M_f = A$ (viewed as (A, B)-bimodule through f) and that, if $f': B \longrightarrow A$, M_f and $M_{f'}$ are isomorphic in ${}_A\underline{M}_B$ iff $f = f'$. Thus ring homomorphisms "are" arrows of Bim. Furthermore, if $g: C \longrightarrow B$, we have obviously a canonical isomorphism $M_f \circ M_g \longrightarrow M_{fg}$.

<u>(2.6) Spans.</u> Let $\underline{C}$ be any category with pullbacks (*). Choosing for each diagram $U \longrightarrow V \longleftarrow W$ in $\underline{C}$ a pull back diagram:

$$\begin{tikzcd} U \underset{V}{\pi} W \arrow[r] \arrow[d] & W \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

we now define "the" bicategory $Sp\,\underline{C}$ of spans of $\underline{C}$ (another choice of pullbacks would give a bicategory isomorphic in an obvious sense). The objects of $Sp\,\underline{C}$ are the objects of $\underline{C}$. If A and B are two objects, the category $Sp\,\underline{C}(A, B)$ has as objects, i.e., arrows of $Sp\,\underline{C}$, all diagrams $s: A \xleftarrow{\alpha} X \xrightarrow{\beta} B$ in $\underline{C}$. A map s in $Sp\,\underline{C}(A, B)$ from S to S': $A \xleftarrow{\alpha'} X' \xrightarrow{\beta'} B$ is a commutative diagram in $\underline{C}$

$$\begin{tikzcd} & X \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] \arrow[dd, "\xi"] & \\ s : A & & B \\ & X' \arrow[ul, "\alpha'"] \arrow[ur, "\beta'"'] & \end{tikzcd} .$$

(*) The notion of span was introduced by Yoneda in [Y] for the case $\underline{C} = Cat^{[1]}$, the category of categories.

Composition in $Sp\,\underline{C}(A,B)$ is the obvious one. The pairings:

$$Sp\,\underline{C}(A,B) \times Sp\,\underline{C}(B,C) \longrightarrow Sp\,\underline{C}(A,C) \;;\; (S,T) \rightsquigarrow S\circ T$$

are defined by pullback. Explicitely, if $T\colon B \xleftarrow{\gamma} Y \xrightarrow{\delta} C$,

$$S\circ T \text{ is } \quad A \xleftarrow{\alpha p_1} X \times_B Y \xrightarrow{\delta p_2} C\,,$$

where $p_1\colon X\times_B Y \longrightarrow X$ and $p_2\colon X\times_B Y \longrightarrow Y$ are the canonical projections of the pullback. The identity arrow of A is:

$I_A\colon A \xleftarrow{Id} A \xrightarrow{Id} A$. Finally, a, ℓ, r are given by the usual isomorphisms of associativity and identity of pullbacks.

Note that if $\underline{C}$ has a final object, say 1, and thus finite products, the multiplicative structure on $\underline{C}$ defined by the product, is isomorphic to $Sp\,\underline{C}(1,1)$ with the induced structure. Dually, if $\underline{C}$ has pushouts, define the bicategory $Cosp\,\underline{C}$ of Cospans in $\underline{C}$, isomorphic with $Sp(\underline{C}^*)$.

<u>(2.7) Local properties of bicategories.</u> Let P be a property of categories, a bicategory $\underline{S}$ is <u>locally P</u> if all the categories $\underline{S}(A,B)$ satisfy P. For example, Bim is locally abelian.

If in the data of $\underline{S}$ all the $\underline{S}(A,B)$ are partially ordered sets, identified to categories, the coherence conditions are automatically satisfied, thus $\underline{S}$ is a bicategory, called locally ordered. The extreme types of partially ordered sets are the <u>discrete</u> ($x \le y$ iff $x = y$) and the <u>anti-discrete</u> (for all x and for all y, $x \le y$).

The <u>locally discrete</u> bicategories, are always 2-categories. Moreover, all their 2-cells are degenerate, we will therefore call them 1-dimensional, and identify categories $\underline{C}$ with 1-dimensional bicategories (by identifying the sets $\mathrm{Hom}_{\underline{C}}(A, B)$ with discrete categories). Thus we will speak of morphisms of a category $\underline{C}$ into a bicategory $\underline{S}$ or of the empty bicategory $\mathbb{0}$, and the punctual bicategory $\mathbb{1}$ with one object $\mathbb{0}$, and $\mathbb{1}(\mathbb{0}, \mathbb{0}) = \{\mathbb{0}\} = \mathbb{1}$, etc .

The locally antidiscrete $\underline{S}$'s can be identified with families of sets $\underline{S}(A, B)$ equipped with maps $\underline{S}(A, B) \times \underline{S}(B, C) \longrightarrow \underline{S}(A, C)$ and base points $I_A \in \underline{S}(A, A)$ satisfying no axioms, and thus are not in general 2-categories.

The bicategory $\underline{R}$ of relations provides a good example of locally ordered bicategory: The objects are sets. If A and B are two sets $\underline{R}(A, B)$ is the power set of $A \times B$, ordered by inclusion. The pairings c are given by the composition of relations and the identity of A is the image Δ_A of A under the diagonal map $\Delta : A \longrightarrow A \times A$. It is clearly a 2-category.

This example can obviously be extended by replacing the category of sets by an abelian category $\underline{A}$ to get the category of additive relations of $\underline{A}$, or by a category with finite limits and some exactness properties which we won't list.

(2.8) Extensions. Let $\underline{A}$ be an abelian category. For each integer n, let $\underline{\mathrm{Ext}}^n_{\underline{A}}(A, B)$ be the category with objects the n-fold extensions

$$S:\ 0 \longleftarrow A \longleftarrow E_1 \longleftarrow \dots\ E_n \longleftarrow B \longleftarrow 0$$

and maps the ordinary maps with endpoints fixed, i.e., the commutative diagrams:

$$\begin{array}{ccccccccccccc}
S & : & 0 & \longleftarrow & A & \longleftarrow & E_1 & \longleftarrow \cdots & E_n & \longleftarrow & B & \longleftarrow & 0 \\
{\scriptstyle s}\downarrow & & & & \| & & \downarrow & & \downarrow & & \| & & \\
S' & : & 0 & \longleftarrow & A & \longleftarrow & E'_1 & \longleftarrow \cdots & E'_n & \longleftarrow & B & \longleftarrow & 0
\end{array}$$

Define: (i) $\underline{\mathrm{Ext}}_{\underline{A}}(A, B)$ to be the union of the categories $\underline{\mathrm{Ext}}^n_{\underline{A}}(A, B)$;

(ii) the composition pairings:

$$\underline{\mathrm{Ext}}_{\underline{A}}(A, B) \times \underline{\mathrm{Ext}}_{\underline{A}}(B, C) \longrightarrow \underline{\mathrm{Ext}}_{\underline{A}}(A, C)$$

to be the Yoneda composition of exact sequences.

(iii) For each A in $\underline{A}$, I_A to be $0 \longleftarrow A \xleftarrow{\mathrm{Id}} A \longleftarrow 0$

(iv) The a, ℓ, r to be the identity natural isomorphisms.

We obtain thus the <u>bicategory of extensions of</u> $\underline{A}$, written $\underline{\mathrm{Ext}}_{\underline{A}}$. The same construction can be performed when $\underline{A}$ is a relative abelian category.

3. Dualities.

For a category $\underline{C}$ there is only one kind of symmetric, namely the dual $\underline{C}^*$ of $\underline{C}$. For a bicategory $\underline{S}$, there are three such, all having the same objects, arrows and 2-cells as $\underline{S}$, described as follows:

(3.1). The conjugate $\underline{S}^c$ defined by:

$$\underline{S}^c(A,B) = [\underline{S}(A,B)]^* \quad ; \quad I_A^c = I_A$$

$$c^c(A,B,C) = [c(A,B,C)]^* : \underline{S}^c(A,B) \times \underline{S}^c(B,C) \longrightarrow \underline{S}^c(A,C)$$

$$a^c(A,B,C,D) = [a(A,B,C,D)]^{-1} \; ; \; \ell^c(A,B) = [\ell(A,B)]^{-1} ,$$

$$r^c(A,B) = [r(A,B)]^{-1}$$

(3.2). The transpose $\underline{S}^t$, defined by:

$$\underline{S}^t(A,B) = \underline{S}(B,A) \quad , \quad I_A^t = I_A .$$

$c^t(A,B,C)$ makes the following diagram of functors commutative:

$$\begin{tikzcd}
\underline{S}^t(A,B) \times \underline{S}^t(B,C) \arrow[r, "c^t(A,B,C)"] \arrow[d, equal] & \underline{S}^t(A,C) = \underline{S}(C,A) \\
\underline{S}(B,A) \times \underline{S}(C,B) \arrow[r, "\sim", "\text{canonical}"'] & \underline{S}(C,B) \times \underline{S}(B,A) \arrow[u, "c(C,B,A)"']
\end{tikzcd}$$

$$a^t(A,B,C,D)(S,T,U) = [a(D,C,B,A)(U,T,S)]^{-1}$$

$$\ell^t(A,B)(S) = r(B,A)(S) \quad \text{and} \quad r^t(A,B)(S) = \ell(B,A)(S).$$

(3.3). The symmetric $\underline{S}^s$, defined by $\underline{S}^s = \underline{S}^{ct}$.

If we make the convention to represent objects A, arrows S and 2-cells s by A^t, S^t, s^t (resp. $A^c, \ldots$) when considered as belonging to $\underline{S}^t$ (resp. $\underline{S}^c, \ldots$) the unpalatable formulae defining the different dualities have simple geometric pictures: The typical 2-cell of

$$\underline{S} : \begin{tikzcd} A & B \arrow[l, bend right, "S"'] \arrow[l, bend left, "\bar{S}"] \arrow[l, phantom, "\Downarrow s"] \end{tikzcd}$$

is represented in $\underline{S}^c$, $\underline{S}^t$ and $\underline{S}^s$ respectively by

$$\begin{tikzcd} A^c & B^c \arrow[l, bend right, "S^c"'] \arrow[l, bend left, "\bar{S}^c"] \arrow[l, phantom, "\Uparrow s^c"] \end{tikzcd} , \quad \begin{tikzcd} A^t \arrow[r, bend left, "S^t"] \arrow[r, bend right, "\bar{S}^t"'] \arrow[r, phantom, "\Downarrow s^t"] & B^t \end{tikzcd} , \quad \begin{tikzcd} A^s \arrow[r, bend left, "S^s"] \arrow[r, bend right, "\bar{S}^s"'] \arrow[r, phantom, "\Uparrow s^s"] & B^s \end{tikzcd}$$

The definitions of $\circ^c, c^c, I^c, a^c, \ell^c, r^c$ (resp. ...) are "forced" by these pictures. And the equations $\underline{S}^{cc} = \underline{S}^{tt} = \underline{S}^{ss} = \underline{S}$, and $\underline{S}^{ct} = \underline{S}^{tc}$ which can be directly checked, become "geometrically obvious."

3.4 Examples.

(3.4.1). If $\underline{S}$ is a 2-category, so are $\underline{S}^c, \underline{S}^t$ and $\underline{S}^s$. In particular, the transpose Tac^t of Tac as defined in (2.1) will be denoted Cat.

(3.4.2). If $\underline{M} = (\underline{A}, \otimes, \ldots)$ is a c.m., then $\underline{M}^c$ is the category $\underline{A}^*$ dual of $\underline{A}$, equipped with "the same" multiplication as $\underline{A}$, $\underline{M}^t$ is the category $\underline{A}$ equipped with the opposite multiplication $(A, B) \rightsquigarrow B \otimes A$, and $\underline{M}^s$ is $\underline{A}^*$ with the opposite multiplication. All this obviously extends to actions of c.m. on categories (e.g., transposition transforms right action into left action ...).

(3.4.3). If $\underline{S}$ is locally ordered (discrete, antidiscrete) so are $\underline{S}^c, \underline{S}^t$, and $\underline{S}^s$. In the discrete case, i.e., when S is a category, we have furthermore: $\underline{S}^c = \underline{S}$, $\underline{S}^t = \underline{S}^s = \underline{S}^*$.

(3.4.4). Clearly every statement about bicategories contains really four statements: If a proposition P is true for $\underline{S}$, then there are conjugate, transpose and symmetric propositions P^c, P^t, P^s, true for $\underline{S}^c, \underline{S}^t$, $\underline{S}^s$ which we will omit most of the time.

4. Morphisms of bicategories

(4.1). Definition: Let $\underline{S} = (\underline{S}_o, c, I, a, \ell, r)$ and $\overline{\underline{S}} = (\overline{\underline{S}}_o, \overline{c}, \ldots)$ be two bicategories. A morphism $\Phi = (F, \varphi)$ from $\underline{S}$ to $\overline{\underline{S}}$ is determined by the following:

(i) A map $F: \underline{S}_o \longrightarrow \overline{\underline{S}}_o \quad , \quad A \rightsquigarrow FA.$

(ii) A family of functors

$$F(A,B): \underline{S}(A,B) \longrightarrow \overline{\underline{S}}(FA,FB) \quad , \quad S \rightsquigarrow FS \quad , \quad s \rightsquigarrow Fs$$

(iii) For each object A of $\underline{S}$, an arrow of $\underline{S}(FA,FA)$ (i. e. a 2-cell of $\underline{S}$)

$$\varphi_A: \overline{I}_{FA} \longrightarrow F(I_A)$$

(iv) A family of natural transformations:

$$\varphi(A,B,C): \overline{c}(FA,FB,FC) \circ (F(A,B) \times F(B,C)) \longrightarrow F(A,C) \circ c(A,B,C).$$

$$\begin{tikzcd}
\underline{S}(A,C) \arrow[d, "F(A,C)"'] & & \underline{S}(A,B) \times \underline{S}(B,C) \arrow[ll, "c(A,B,C)"'] \arrow[d, "F(A,B) \times F(B,C)"] \arrow[dll, phantom, "\Nwarrow \varphi(A,B,C)"] \\
\overline{\underline{S}}(FA,FC) & & \overline{\underline{S}}(FA,FB) \times \overline{\underline{S}}(FB,FC) \arrow[ll, "\overline{c}(FA,FB,FC)"]
\end{tikzcd}$$

If (S,T) is an object of $\underline{S}(A,B) \times \underline{S}(B,C)$ the (S,T)-component of $\varphi(A,B,C)$

$$F(S \circ T) \xleftarrow{\varphi(A,B,C)(S,T)} FS \circ FT \quad (= FS \,\overline{\circ}\, FT) \qquad (*)$$

shall usually be abbreviated into $\varphi(S,T)$ or even φ.

(*) As usual in algebra, corresponding operations as c and $\overline{c}$ are in the abbreviated notation denoted by the same symbol, when no confusion is likely.

These data are required to satisfy the following coherence axioms:

(M. 1) If (S, T, U) is an object of $\underline{S}(A, B) \times \underline{S}(B, C) \times \underline{S}(C, D)$ the following diagram, where indices A, B, C, D have been omitted, is commutative:

$$\begin{tikzcd} FS\circ(FT\circ FU) \arrow[d, "\mathrm{Id}\circ\varphi(T,U)"'] & & (FS\circ FT)\circ FU \arrow[ll, "\bar{a}(FS,FT,FU)"', "\sim"] \arrow[d, "\varphi(S,T)\circ \mathrm{Id}"] \\ FS\circ F(T\circ U) \arrow[d, "\varphi(S,T\circ U)"'] & & F(S\circ T)\circ FU \arrow[d, "\varphi(S\circ T,U)"] \\ F(S\circ(T\circ U)) & & F((S\circ T)\circ U) \arrow[ll, "\sim", "F(a(S,T,U))"'] \end{tikzcd}$$

(M. 2) If S is an object of $\underline{S}(A, B)$ the following diagrams commute:

$$\begin{tikzcd} FS & F(S\circ I_B) \arrow[l, "Fr"', "\sim"] \\ FS\circ \bar{I}_{FB} \arrow[u, "\bar{r}", "\wr"'] \arrow[r, "\mathrm{Id}\circ\varphi_B"'] & FS\circ FI_B \arrow[u, "\varphi(S, I_B)"'] \end{tikzcd} \qquad \begin{tikzcd} F(I_A\circ S) \arrow[r, "F\ell", "\sim"'] & FS \\ FI_A\circ FS \arrow[u, "\varphi(I_A, S)"] & \bar{I}_{FA}\circ FS \arrow[l, "\varphi_A\circ \mathrm{Id}"] \arrow[u, "\wr", "\bar{\ell}"'] \end{tikzcd}$$

<u>(4. 2) Remark:</u> The usual devices of universal algebra would have suggested the following "natural" notion of maps between bicategories $\underline{S}$ and $\underline{\bar{S}}$:

(i) A map $F: \underline{S}_o \longrightarrow \underline{\bar{S}}_o$, $A \rightsquigarrow FA$

(ii) A family of functors $F(A, B): \underline{S}(A, B) \longrightarrow \underline{\bar{S}}(FA, FB)$, commuting with the compositions; that is, $F(S\circ T) = FS\circ FT$ and $F(s\circ t) = F(s)\circ F(t)$, with the identities: $FI_A = \bar{I}_{FA}$, and with the a, ℓ, r :

$F(a(S, T, U)) = \bar{a}(FS, FT, FU)$, $F(\ell(S)) = \bar{\ell}(FS)$ and $F(r(S)) = \bar{r}(FS)$.

Such a map, called <u>strict homomorphism</u> can be, and will be, identified with the morphism $\Phi = (F, \varphi)$ defined by (i), (ii), and $\varphi_A = \mathrm{Id}\colon FI_A \xrightarrow{=} \overline{I}_{FA}$, and $\varphi(A,B,C)(S,T) = \mathrm{Id}\colon FS \circ FT \xrightarrow{=} F(S \circ T)$. We do not even require in (4.1) that F should commute up to isomorphisms with the compositions and units, i.e., that the φ_A and $\varphi(A,B,C)(S,T)$ should be isomorphisms. If this is satisfied, we say that $\Phi = (F, \varphi)$ is a <u>homomorphism.</u> If only the φ_A are isomorphisms, we say that Φ is a <u>unitary morphism</u>; if the φ_A are identities we say that Φ is a <u>strictly unitary morphism</u>.

The fact that all the desired results hold in the more general context, and, even more, the numerous mathematical examples where we have morphisms which are not homomorphisms, let alone strict ones, will be the essential justification of the definition (4.1). (See § 5)

<u>(4.3) Composition of morphisms.</u> Let $\underline{S} = (\underline{S}_o, c, \ldots)$, $\overline{\underline{S}} = (\overline{\underline{S}}_o, \overline{c}, \ldots)$ and $\overline{\overline{\underline{S}}} = (\overline{\overline{\underline{S}}}_o, \overline{\overline{c}}, \ldots)$ be bicategories, $\Phi = (F, \varphi)\colon \underline{S} \longrightarrow \overline{\underline{S}}$ and $\overline{\Phi} = (\overline{F}, \overline{\varphi})\colon \overline{\underline{S}} \longrightarrow \overline{\overline{\underline{S}}}$ be morphisms: Construct the following

(i) A map $G = \overline{F} \circ F : \underline{S}_o \longrightarrow \overline{\overline{\underline{S}}}_o$.

(ii) A family of functors $G(A, B)$ as the composite:

$$\underline{S}(A,B) \xrightarrow{F(A,B)} \overline{\underline{S}}(FA,FB) \xrightarrow{\overline{F}(FA,FB)} \overline{\overline{\underline{S}}}(\overline{F}FA, \overline{F}FB) = \overline{\overline{\underline{S}}}(GA,GB)$$

(iii) For each object A of $\underline{S}$, an arrow ψ_A in $\underline{S}(GA, GA)$ as the composite:

$$\overline{\overline{I}}_{GA} = \overline{\overline{I}}_{\overline{F}FA} \xrightarrow{\overline{\varphi}_{FA}} \overline{F}\,\overline{I}_{FA} \xrightarrow{\overline{F}\varphi_A} \overline{F}FI_A = GI_A \,.$$

(iv) A family of natural transformations:

$$\psi(A,B,C)\colon \overline{\overline{c}}(GA,GB,GC) \circ (G(A,B) \times G(B,C)) \longrightarrow G(A,C) \circ c(A,B,C)$$

by components $\psi(A,B,C)(S,T)$ for (S,T) objects of $\underline{S}(A,B) \times \underline{S}(B,C)$

making commutative the diagram:

$$\begin{tikzcd}
GS \circ GT \arrow[rr, "\psi(A,B,C)(S,T)"] \arrow[d, equal] & & G(S \circ T) \arrow[d, equal] \\
\overline{F}FS \circ \overline{F}FT \arrow[r, "\overline{\varphi}(FS,FT)"'] & \overline{F}(FS \circ FT) \arrow[r, "\overline{F}(\varphi(S,T))"'] & \overline{F}F(S \circ T)
\end{tikzcd}$$

(The fact that the $\psi(A,B,C)(S,T)$ are natural follows from the 2-dimensional definition of $\psi(A,B,C)$ as the composite:

$$G \circ c = \overline{F} \circ F \circ c \xleftarrow{\overline{F} \circ \varphi} \overline{F} \circ \overline{c} \circ (F \times F) \xleftarrow{\overline{\varphi} \circ (F \times F)} \overline{\overline{c}} \circ (\overline{F} \times \overline{F}) \circ (F \times F) = \overline{\overline{c}} \circ (G \times G)$$

where again indices A, B, C are omitted.)

<u>(4.3.1) Theorem.</u> With the previous notation

(i) The data $(G, G(A,B), \psi_A, \psi(A,B,C))$ define a morphism $(G, \psi) = \Psi$ from $\underline{S}$ to $\underline{\overline{\overline{S}}}$, called composite of Φ and $\overline{\Phi}$, and written $\overline{\Phi}\Phi$.

(ii) With this composition, bicategories and their morphisms form a category, which we will denote (*) $\mathrm{Bicat}^{[1]}$.

<u>Proof of (i).</u> To show that Ψ satisfies (M.1) we must prove that the exterior of the following diagram is commutative:

(*) Later on we will define a "trigraph" having $\mathrm{Bicat}^{[1]}$ as one-dimensional skeleton.

(Where S, T, U are objects of $\underline{S}(A,B)$, $\underline{S}(B,C)$, $\underline{S}(C,D)$)

$$
\begin{tikzcd}
 & \overline{F}FS\circ(\overline{F}FT\circ\overline{F}FU) \arrow[d, "\mathrm{Id}\circ\overline{\varphi}(FT,FU)"] \arrow[ddl, "\mathrm{Id}\circ\psi(T,U)"'] \arrow[dl, phantom, "1"] & & & & (\overline{F}FS\circ\overline{F}FT)\circ\overline{F}FU \arrow[llll, "a(\overline{F}FS,\overline{F}FT,\overline{F}FU)"'] \arrow[d, "\overline{\varphi}(FS,FT)\circ\mathrm{Id}"'] \arrow[ddr, "\psi(S,T)\circ\mathrm{Id}"] \arrow[dr, phantom, "1'"] & \\
 & \overline{F}FS\circ\overline{F}(FT\circ FU) \arrow[dl, "\mathrm{Id}\circ\overline{F}(\varphi(T,U))"'] \arrow[dr, "\overline{\varphi}(FS,FT\circ FU)"] \arrow[rrrr, phantom, "4"] & & & & \overline{F}(FS\circ FT)\circ\overline{F}FU \arrow[dl, "\overline{\varphi}(FS\circ FT,FU)"'] \arrow[dr, "\overline{F}(\varphi(S,T))\circ\mathrm{Id}"] & \\
\overline{F}FS\circ(\overline{F}F(T\circ U)) \arrow[dr, "\overline{\varphi}(FS,F(T\circ U))"] \arrow[ddr, "\psi(S,T\circ U)"'] \arrow[rr, phantom, "2"] & & \overline{F}(FS\circ(FT\circ FU)) \arrow[dl, "\overline{F}(\mathrm{Id}\circ\varphi(T,U))"] & & \overline{F}((FS\circ FT)\circ FU) \arrow[ll, "\overline{F}(\overline{a}(FS,FT,FU))"] \arrow[dr, "\overline{F}(\varphi(S,T)\circ\mathrm{Id})"'] \arrow[rr, phantom, "2'"] & & (\overline{F}F(S\circ T))\circ\overline{F}FU \arrow[dl, "\overline{\varphi}(F(S\circ T),FU)"'] \arrow[ddl, "\psi(S\circ T,U)"] \\
 & \overline{F}(FS\circ F(T\circ U)) \arrow[d, "\overline{F}(\varphi(S,T\circ U))"] \arrow[l, phantom, "3"] \arrow[rrrr, phantom, "4'"] & & & & \overline{F}(F(S\circ T)\circ FU) \arrow[d, "\overline{F}(\varphi(S\circ T),U)"'] \arrow[r, phantom, "3'"] & \\
 & \overline{F}F(S\circ(T\circ U)) & & & & \overline{F}F((S\circ T)\circ U) \arrow[llll, "\overline{F}F(a(S,T,U))"] &
\end{tikzcd}
$$

But the triangles 1 and 1' commute by definition of ψ and by the fact that $\overline{\overline{c}}$ is a bifunctor ; 3 and 3' commute by definition of ψ; 2 and 2' by naturality of $\overline{\varphi}$; 4 is the axiom (M. 1) for φ applied on the object (FS, FT, FU) and 4' is the image by the functor $\overline{F}(FA,FC)$ of the commutative diagram (M. 1) for φ.

To prove (M.2) we have to show that for any object S of $\underline{S}(A,B)$ the exteriors of the following two diagrams commute:

$$\begin{tikzcd}
\bar F FS \arrow[rrr, phantom, "(1)", shift right=6ex] & & & \bar F F(S\circ I_B) \arrow[lll, "\bar F F(r)"'] \\
& \bar F(FS\circ \bar I_{FB}) \arrow[ul, "\bar F(\bar r)"] \arrow[r, "\bar F(\mathrm{Id}\circ\varphi_B)"] \arrow[dr, phantom, "(2)"] & \bar F(FS\circ FI_B) \arrow[ur, "\bar F(\varphi(S,I_B))"'] \arrow[dr, phantom, "(3)"] & \\
& (\bar F FS)\circ(\bar F \bar I_{FB}) \arrow[u, "\bar\varphi(FS,\bar I_{FB})"] \arrow[drr, "\mathrm{Id}\circ\bar F(\varphi_B)"] \arrow[l, phantom, "(1')"] \arrow[dr, phantom, "(3')"'] & & \\
(\bar F FS)\circ(\bar I_{\bar F FB}) \arrow[uuu, "\bar{\bar r}"] \arrow[ur, "\mathrm{Id}\circ\bar\varphi_{FB}"] \arrow[rrr, "\mathrm{Id}\circ\psi_B"'] & & & (\bar F FS)\circ(\bar F FI_B) \arrow[uul, "\bar\varphi(FS,FI_B)"] \arrow[uuu, "\psi(S,I_B)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\bar F FS & & & \bar F F(I_A\circ S) \arrow[lll, "\bar F F(\ell)"'] \\
& \bar F(\bar I_{FA}\circ FS) \arrow[ul, "\bar F(\bar\ell)"] \arrow[r, "\bar F(\varphi_A\circ \mathrm{Id})"] & \bar F(FI_A\circ FS) \arrow[ur, "\bar F(\varphi(I_A,S))"'] & \\
& (\bar F\bar I_{FA})\circ(\bar F FS) \arrow[u, "\bar\varphi(\bar I_{FA},FS)"] \arrow[drr, "\bar F(\varphi_A)\circ \mathrm{Id}"] & & \\
(\bar I_{\bar F FA})\circ(\bar F FS) \arrow[uuu, "\bar\ell"] \arrow[ur, "\bar\varphi_{FA}\circ \mathrm{Id}"] \arrow[rrr, "\psi_A\circ \mathrm{Id}"'] & & & (\bar F FI_A)\circ(\bar F FS) \arrow[uul, "\bar\varphi(FI_A,FS)"] \arrow[uuu, "\psi(I_A,S)"']
\end{tikzcd}$$

Now in the first diagram, the region 1 is the image by $\bar{F}$ of the diagram (M. 2) for r and $\bar{r}$, the region 1' is (M. 2) for $\bar{r}$ and $\bar{\bar{r}}$ applied on the object FS, then 2 commutes by naturality of φ, 3 by definition of $\psi(S, I_B)$ and 3' by definition of ψ_B and the fact that $\bar{\bar{c}}$ is a bifunctor. The commutativity of the second diagram can be proved similarly, or better, follows by transposition (cf. (3. 4. 4)) from the commutativity of the first.

<u>Proof of (ii).</u> Let $\bar{\bar{\Phi}} = (\bar{\bar{F}}, \bar{\bar{\varphi}}): \bar{\underline{S}} \longrightarrow \bar{\bar{\underline{S}}}$ be a third morphism. Denote by $\bar{\Psi} = (\bar{G}, \bar{\psi})$ the composite $\bar{\bar{\Phi}}\bar{\Phi}$, by $\Lambda = (L,\lambda)$ the composite $(\bar{\bar{\Phi}}\bar{\Phi})\Phi$ and by $(L', \lambda') = \Lambda'$ the composite $\bar{\bar{\Phi}}(\bar{\Phi}\Phi)$. The equations $L = L'$ and $L(A,B) = L'(A,B)$ for A, B objects of $\underline{S}_o$ are obvious. By definition of composition $\lambda_A = \lambda'_A$ is equivalent to the commutativity of the exterior of the diagram:

$$\begin{tikzcd}[column sep=huge]
\bar{\bar{F}}\bar{I}_{\bar{F}FA} \arrow[d, "\bar{\bar{F}}(\psi_A)"'] \arrow[drr, "\bar{\bar{F}}(\bar{\varphi}_{FA})"] \arrow[dr, phantom, "2"'] & & \bar{\bar{I}}_{\bar{\bar{F}}\bar{F}FA} \arrow[ll, "\bar{\bar{\varphi}}_{\bar{F}FA}"'] \arrow[d, "\bar{\psi}_{FA}"] \arrow[dl, phantom, "1"] \\
\bar{\bar{F}}\bar{F}FI_A & & \bar{\bar{F}}\bar{F}\bar{I}_{FA} \arrow[ll, "\bar{\bar{F}}\bar{F}(\varphi_A)"]
\end{tikzcd}$$

but 1 commutes by definition of $\bar{\psi}_{FA}$, and 2 is the image by $\bar{\bar{F}}$ of the commutative diagram defining ψ_A.

Similarly, for S, T objects of $\underline{S}(A, B)$, $\underline{S}(B, C)$, $\lambda(S, T) = \lambda'(S, T)$ is equivalent to the commutativity of the exterior of :

$$\begin{tikzcd}[column sep=huge]
\overline{\overline{F}}(\overline{F}FS \circ \overline{F}FT) \arrow[d, "\overline{\overline{F}}(\psi(S,T))"'] \arrow[drr, "\overline{\overline{F}}(\overline{\varphi}(FS,FT))"] \arrow[dr, phantom, pos=0.25, "2"'] & & (\overline{\overline{F}}\overline{F}FS)\circ(\overline{\overline{F}}\overline{F}FT) \arrow[ll, "\overline{\overline{\varphi}}(\overline{F}FS,\overline{F}FT)"'] \arrow[d, "\overline{\psi}(FS,FT)"] \arrow[dl, phantom, pos=0.25, "1"] \\
\overline{\overline{F}}\overline{F}F(S\circ T) & & \overline{\overline{F}}\overline{F}(FS\circ FT) \arrow[ll, "\overline{\overline{F}}\overline{F}(\varphi(S,T))"]
\end{tikzcd}$$

which follows again from the definitions of $\overline{\psi}(FS,FT)$ and $\psi(S,T)$. Thus composition of morphisms is associative. Finally, the data:

$Id\colon \underline{S}_0 \longrightarrow \underline{S}_0$, $Id(A,B) = Id\colon \underline{S}(A,B) \longrightarrow \underline{S}(A,B)$, $i_A = Id(I_A)$ and $i(A,B,C) = Id(c(A,B,C))$ obviously define a morphism $Id_{\underline{S}}\colon \underline{S} \longrightarrow \underline{S}$ which is an identity for the composition .

<u>(4.3.2) Proposition.</u> If Φ and $\overline{\Phi}$ are unitary, or strictly unitary, or homomorphisms or strict homomorphisms, so is their composite $\overline{\Phi}\Phi$.

Straightforward.

We shall denote the respective subcategories of $\mathrm{Bicat}^{[1]}$ by $\text{U-Bicat}^{[1]}$, $\text{S}\circ\text{U-Bicat}^{[1]}$, $\text{H-Bicat}^{[1]}$ and $\text{S}\circ\text{H-Bicat}^{[1]}$.

<u>(4.3.3) Remark:</u> One can define, as in (1.3), morphisms of bicategories in a global diagrammatic way. This is left to the reader.

5. Examples

(5.1) Functors. Let $\underline{C}$ and $\overline{\underline{C}}$ be categories and $F: \underline{C} \longrightarrow \overline{\underline{C}}$ be a functor. Call $D\underline{C}$ and $D\overline{\underline{C}}$ the locally discrete bicategories associated with $\underline{C}$ and $\overline{\underline{C}}$. Then F obviously determines a morphism $DF: D\underline{C} \longrightarrow D\overline{\underline{C}}$. Conversely every morphism from $D\underline{C}$ to $D\overline{\underline{C}}$ comes from a unique such F, and obviously we get a full and faithful functor, called degeneracy

$$D: \mathrm{Cat}^{[1]} \longrightarrow \mathrm{Bicat}^{[1]}.$$

Moreover, DF is always a strict homomorphism, and more generally, if $\underline{S}$ is any bicategory and $\underline{C}$ a category, any morphism $\Phi: \underline{S} \longrightarrow D\underline{C}$ is a strict homomorphism. $D\underline{C}$ is called degenerate of $\underline{C}$.

(5.2) Multiplicative Categories. Let $\underline{M}$ and $\overline{\underline{M}}$ be c.m. and $\Phi = (F, \varphi, \lambda)$ a morphism from $\underline{M}$ to $\overline{\underline{M}}$ as defined in [B.1]. Call $\underline{S} = I\underline{M}$ and $\overline{\underline{S}} = I\overline{\underline{M}}$ the bicategories with a single object associated with $\underline{M}$ and $\overline{\underline{M}}$ in (2.2) and, with the same notations, define a map $\hat{F}: \underline{S}_o \longrightarrow \overline{\underline{S}}_o$, $0 \rightsquigarrow \overline{0}$; a functor $\hat{F}(0,0) = F$, a map $\hat{\varphi}_o = \lambda$ and a natural transformation $\hat{\varphi}(0,0,0) = \varphi$. Then $(\hat{F}, \hat{\varphi})$ is a morphism $I\Phi: I\underline{M} \longrightarrow I\overline{\underline{M}}$. If $\overline{\Phi}: \overline{\underline{M}} \longrightarrow \overline{\overline{\underline{M}}}$ is another morphism of c.m., and $\overline{\Phi}\,\Phi$ is the composite (in the sense of [B.1]) we have $I(\overline{\Phi}\Phi) = (I\overline{\Phi})(I\Phi)$; and thus a functor:

$$I: \mathrm{Mult}^{[1]} \longrightarrow \mathrm{Bicat}^{[1]}.$$

It is clear that I is full and faithful, and that Φ is a homomorphism or a strict homomorphism of c.m. iff $I\Phi$ has the same property in $\mathrm{Bicat}^{[1]}$. (This full and faithful embedding obviously extends to actions of c.m. on categories or to the situation considered by Epstein. cf (2.4)). If $\Phi^* = (F^*, \varphi^*, \lambda^*)$ is a comorphism of c.m. it can be, according to [B.1], identified with a morphism of the duals $\underline{M}^*$ and $\overline{\underline{M}}^*$, these in turn can be identified with the conjugates $(I\underline{M})^c$ and $(I\overline{\underline{M}})^c$ by (3.4.2). Thus the notion of comorphism is reduced, via a suitable duality to that of morphism.

(5.2.1) Remark: One should note that this identification is contravariant: the comorphisms from $\underline{M}$ to $\overline{\underline{M}}$ and the morphisms from $\underline{M}^*$ to $\overline{\underline{M}}^*$ can both be made, in a natural way, the objects of categories: $\mathrm{Comor}(\underline{M}, \overline{\underline{M}})$ and $\mathrm{Mor}(\underline{M}^*, \overline{\underline{M}}^*)$ dual to each other. Using the dualities of §3 one could define eight(!!!) different variances of morphisms between bicategories. The only way to avoid a cumbersome terminology is to consider always morphisms, and specify in each case the suitable dual categories for the domain and range.

(5.3) 2-Functors. Let $\underline{A}$ and $\overline{\underline{A}}$ be 2-categories and $F: \underline{A} \longrightarrow \overline{\underline{A}}$ a 2-functor as defined in [B.3]. Calling $J\underline{A}$ and $J\overline{\underline{A}}$ the strictly associative bicategories associated with $\underline{A}$ and $\overline{\underline{A}}$, F determines obviously a morphism $JF: J\underline{A} \longrightarrow J\overline{\underline{A}}$, and we get a functor:

$$J:\ 2\text{-}\mathrm{Cat}^{[1]} \longrightarrow \mathrm{Bicat}^{[1]}$$

which is faithful but no longer full. Explicitely the morphisms from $J\underline{A}$ to $J\overline{\underline{A}}$ which are of form JF are exactly the strict homomorphisms.

In the rest of the paper we shall usually identify categories, c.m.'s and 2-categories with bicategories via the functors D, I, and J. All these examples have nothing surprising since the definition of bicategories was clearly devised to contain them. The following ones are of a completely different nature.

<u>(5.4) Monads.</u> Let $\underline{S}$ be a bicategory.

<u>(5.4.1) Definition.</u> A <u>monad in</u> $\underline{S}$ (or $\underline{S}$-monad) is a morphism from $\mathbb{1}$ to $\underline{S}$. An <u>$\underline{S}$-comonad</u> is an $\underline{S}^c$-monad.

Interpreting (4.1), a monad $\Phi = (F, \varphi) : 1 \longrightarrow \underline{S}$ is determined by:

(i) One object $F(O) = X$ of $\underline{S}$; Φ is called an $\underline{S}$-monad <u>on</u> X or <u>over</u> X.

(ii) One functor $F(O, O) : 1 \longrightarrow \underline{S}(X, X)$, i.e., an object T of $\underline{S}(X, X)$.

(iii) One arrow $\varphi_O = \eta : I_X \longrightarrow T$ in $\underline{S}(X, X)$.

(iv) One natural transformation $\varphi(O, O, O)$ identified with its unique component $\varphi(O, O, O)(O) = \mu : T \circ T \longrightarrow T$ in $\underline{S}(X, X)$.

The axiom (M.1) is equivalent to the commutativity of:

$$\begin{tikzcd}
T \circ (T \circ T) \arrow[d, "T \circ \mu"'] & (T \circ T) \circ T \arrow[l, "a(T,T,T)", "\sim"'] \arrow[d, "\mu \circ T"] \\
T \circ T \arrow[d, "\mu"'] & T \circ T \arrow[d, "\mu"] \\
T & T \arrow[l, "\sim", "F(\mathrm{Id}) = \mathrm{Id}"']
\end{tikzcd}$$

And (M. 2) to the commutativity of the diagrams:

$$\begin{tikzcd} T & T \arrow[l, "\mathrm{Id}", "\sim"'] \\ T \circ I_X \arrow[u, "r"] \arrow[r, "T\circ\eta"'] & T\circ T \arrow[u, "\mu"'] \end{tikzcd} \qquad \begin{tikzcd} T & T \arrow[l, "\mathrm{Id}", "\sim"'] \\ I_X \circ T \arrow[u, "\ell"] \arrow[r, "\eta\circ T"'] & T\circ T \arrow[u, "\mu"'] \end{tikzcd}$$

By suitably chosing $\underline{S}$ we will have many examples:

<u>(5. 4. 1) Monoids:</u> Take $\underline{S} = \underline{M} = (\underline{A}, \otimes, \ldots)$ to be a c. m. ; as a bicategory it has a unique object, say 0, thus X is determined. An $\underline{M}$-monad will therefore be defined by: an object T of $\underline{A}$, two arrows $\mu : T \otimes T \longrightarrow T$ and $\eta : \Lambda \longrightarrow T$. The commutativity of the previous diagrams is exactly the requirement that (T, μ, η) should be a monoid in $\underline{M}$, in the sense of [G]. (*) Dually, the $\underline{M}$-comonads are the comonoids in $\underline{M}$. In particular, for $\underline{M} = (\mathrm{Sets}, \times, \ldots)$ we have the ordinary monoids, for $\underline{M} = (\underline{A}, \times, \ldots)$ where $\underline{A}$ is a category with finite products, we have the monoid-like objects of [E. H], for further examples see [B. 2].

<u>(5. 4. 2) Standard constructions:</u> Take $\underline{S}$ = Cat, then X is a category, $T : X \longrightarrow X$ a functor, $\eta : \mathrm{Id}_X \longrightarrow T$ and $\mu : TT \longrightarrow T$ natural transformations, a, ℓ, r are identities and the commutative diagrams express Godement's axioms for a <u>standard construction</u>, also called <u>triple</u> in [E. M]. By conjugation, Cat-comonads are identified with categories equipped with a co-construction or cotriple. We will usually abbreviate Cat-monads and Cat-comonads, to monads and comonads.

(*) Our choice of "monad" comes from this example and the definition (5. 5).

(5.4.3) Categories inside a category: Let $\underline{C}$ be any category with pull-backs $Sp\underline{C}$ the bicategory of spans of $\underline{C}$, of (2.6). We define a category inside $\underline{C}$ to be a monad of $Sp\underline{C}$.

Explicitly, such a category is defined by:

(i) An object A of $\underline{C}$, written X_o and called object of objects

(ii) A diagram T: $X_o \xleftarrow{d_1} X_1 \xrightarrow{d_o} X_o$; X_1 is the object of arrows, d_o and d_1 are called domain and codomain maps

(iii) A commutative diagram:

$$\eta :\quad \begin{tikzcd} & X_o \arrow[dl, "Id"'] \arrow[dr, "Id"] \arrow[d, "i"] & \\ X_o & X_1 \arrow[l, "d_1"] \arrow[r, "d_o"'] & X_o \end{tikzcd}$$

is thus determined by i which is called degeneracy or identity.

(iv) A commutative diagram

$$\mu :\quad \begin{tikzcd} & X_1 \Pi_{X_o} X_1 \arrow[dl, "d_1 p_o"'] \arrow[dr, "d_o p_1"] \arrow[d, "c"] & \\ X_o & X_1 \arrow[l, "d_1"] \arrow[r, "d_o"'] & X_o \end{tikzcd}$$

; p_o, p_1 defined by the pullback:

$$\begin{tikzcd} & X_1 \Pi_{X_o} X_1 \arrow[dl, "p_o"'] \arrow[dr, "p_1"] & \\ X_1 \arrow[dr, "d_o"'] & & X_1 \arrow[dl, "d_1"] \\ & X_o & \end{tikzcd}$$

μ is determined by the previous maps and by c which is called multiplication or composition.

The maps (d_o, d_1, i, c) are required to make commutative three diagrams expressing the associativity of c and the fact that i is "an identity", which are left to the reader.

Taking $\underline{C}$ = Sets, we get the categories, $\underline{C} = \mathrm{Cat}^{[1]}$ we get the <u>double categories</u> of Ehresmann [Eh], $\underline{C}$ = Top = category of topological spaces we get the <u>topological categories</u>, etc. ...

Dually if $\underline{C}$ has pushouts, a <u>cocategory inside</u> $\underline{C}$ is a comonad of Cosp $\underline{C}$. Explicitly, it is defined by two objects of $\underline{C}$, X_o and X_1 together with maps in $\underline{C}$:

$$X_o \overset{\partial_o}{\underset{\partial_1}{\rightrightarrows}} X_1 \xrightarrow{\sigma} X_o \qquad\qquad X_1 \xrightarrow{\gamma} X_1 \coprod_{X_o} X_1$$

satisfying "well-known" axioms, which can best be visualized by looking, inside $\mathrm{Cat}^{[1]}$, at the <u>fundamental cocategory</u> described as follows:

$X_o = \mathbf{1}$, $X_1 = \mathbf{2} : 0 \longrightarrow 1$, $\sigma, \partial_o, \partial_1$ are the only possible distinct functors. Then $X_1 \coprod_{X_o} X_1$ is $\mathbf{3}$, i.e. $0 \longrightarrow 1 \longrightarrow 2$ and γ is the functor sending the non degenerate arrow of $\mathbf{2}$ on $0 \longrightarrow 2$ in $\mathbf{3}$.

<u>(5.4.4) Ordered sets:</u> If $\underline{S}$ is a locally ordered bicategory, in any of the categories $\underline{S}(A, B)$ all diagrams commute; thus a monad in $\underline{S}$ is determined by an object A of $\underline{S}$, an object T of $\underline{S}(A, A)$ such that $I_A \leq T$ and $T \circ T \leq T$ with no further conditions. In particular, in the bicategory $\underline{R}$ of relations, A is a set, T a subset of $A \times A$ such that $\Delta_A \subseteq T$ and $T \circ T \subset T$. Thus the monads of $\underline{R}$ are the partially ordered sets.

Note that all the examples of (5. 4) would have been reduced to objects of the different bicategories $\underline{S}$ involved, had we confined ourselves to strict homomorphisms, since the domain was $\mathbb{1}$.

(5. 5) Polyads. We now consider morphisms with domain slightly more general, namely a locally punctual bicategory, that is according to (2. 7) a bicategory $\underline{S}$ such that $\underline{S}(A, B) = \mathbb{1}$ for all objects A, B. Such a bicategory is clearly determined by the set $\mathrm{Ob}\,\underline{S}$ of its objects.

(5. 5. 1) Definition. Let $\overline{\underline{S}}$ be a bicategory. A polyad in $\overline{\underline{S}}$ (or $\overline{\underline{S}}$-polyad) is a morphism of bicategories $\Phi = (F, \varphi): \underline{S} \longrightarrow \overline{\underline{S}}$ where $\underline{S}$ is locally punctual. The set $\mathrm{Ob}\,\underline{S}$ is called set of objects or indices of the polyad. (The monads are obtained when $\mathrm{Ob}\,\underline{S} = \mathbb{1}$, hence the name of polyad.)

We will give three examples. By suitably choosing $\overline{\underline{S}}$ in the list of examples of §2, the reader can construct many more.

(5. 5. 2) Relative categories: Let $\underline{M} = (\underline{A}, \otimes, \Lambda, \ldots)$ be a multiplicative category. Let us recall our definition [B. 3] of an $\underline{M}$-category $\underline{C}$. It is given by

(1) A set $\mathrm{Ob}(\underline{C})$ whose elements $X, Y, \ldots$ are called objects of $\underline{C}$.

(2) For each X, Y in $\mathrm{Ob}(\underline{C})$ an object $\underline{C}(X, Y)$ of $\underline{A}$.

(3) For each X, Y, Z in $\mathrm{Ob}(\underline{C})$ a map of $\underline{A}$, $c(X, Y, Z)$ abbreviated in c,

$$c = c(X, Y, Z) : \underline{C}(X, Y) \otimes \underline{C}(Y, Z) \longrightarrow \underline{C}(X, Z)$$

(4) For each $X \in \mathrm{Ob}\,\underline{C}$, a map of $\underline{A}$, $i_{\underline{C}}(X)$ abbreviated in $i(X)$ or i,

$$i = i(X): \Lambda \longrightarrow \underline{C}(X, X)$$

such that, for all X, Y, Z, T in $\mathrm{Ob}(\underline{C})$ the following diagrams commute:

$$\begin{tikzcd}
(\underline{C}(X,Y)\otimes\underline{C}(Y,Z))\otimes\underline{C}(Z,T) \arrow[r, "c\otimes \mathrm{Id}"] \arrow[dd, "\theta"', "\wr"] & \underline{C}(X,Z)\otimes\underline{C}(Z,T) \arrow[dr, "c"] & \\
& & \underline{C}(X,T) \\
\underline{C}(X,Y)\otimes(\underline{C}(Y,Z)\otimes\underline{C}(Z,T)) \arrow[r, "\mathrm{Id}\otimes c"] & \underline{C}(X,Y)\otimes\underline{C}(Y,T) \arrow[ur, "c"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \underline{C}(X,X)\otimes\underline{C}(X,Y) \arrow[dr, "c"] & \\
\Lambda\otimes\underline{C}(X,Y) \arrow[ur, "i\otimes\mathrm{Id}"] \arrow[rr, "\sim", "\gamma"'] & & \underline{C}(X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& \underline{C}(X,Y)\otimes\underline{C}(Y,Y) \arrow[dr, "c"] & \\
\underline{C}(X,Y)\otimes\Lambda \arrow[ur, "\mathrm{Id}\otimes i"] \arrow[rr, "\sim", "\delta"'] & & \underline{C}(X,Y)
\end{tikzcd}$$

For examples we refer to [B. 3].

Now if $\underline{C}$ is such an $\underline{M}$-category, take $\underline{S}$ to be the locally punctual bicategory having $\mathrm{Ob}(\underline{C})$ as set of objects, and $\overline{\underline{S}} = \mathrm{I}\underline{M}$ to be the bicategory with one object $\mathbb{O}$ (cf (5. 2)), and define:

(i) A map $F: \underline{S}_o \longrightarrow \overline{\underline{S}}_o$ as the unique map $\mathrm{Ob}(\underline{C}) \longrightarrow \mathbf{1}$

(ii) Functors $F(X,Y)$: $\underline{S}(X,Y) = \mathbf{1} \longrightarrow \overline{\underline{S}}(FX,FY) = \underline{A}$ by

$$F(X,Y)(\mathbb{O}) = \underline{C}(X,Y).$$

(iii) Arrows $\varphi_X : \overline{I}_{FX} = \Lambda \longrightarrow F(I_X) = F(X,X)(\mathbb{O}) = \underline{C}(X,X)$, by $\varphi_X = i_X$.

(iv) Natural transformations $\varphi(X,Y,Z)$ identified to their unique component $\varphi(X,Y,Z)(\mathbb{O},\mathbb{O})$ by

$$\varphi(X,Y,Z) = c(X,Y,Z): F(X,Y)(\mathbb{O})\otimes F(Y,Z)(\mathbb{O}) = \underline{C}(X,Y)\otimes\underline{C}(Y,Z) \longrightarrow \underline{C}(X,Z)$$
$$= F(X,Z)(\mathbb{O}).$$

One easily verifies that the commutativity of the previous diagrams is then equivalent to (M. 1) and (M. 2). Thus (F, φ) is a morphism $\Phi(\underline{C}) : \underline{S} \longrightarrow \overline{\underline{S}}$.

Conversely, given a polyad $\Phi : \underline{S} \longrightarrow \overline{\underline{S}}$ where $\overline{\underline{S}}$ has a single object $\mathbf{O}$, one defines a category $\underline{C}(\Phi)$ relative to the c.m. $\underline{S}(\mathbf{O}, \mathbf{O})$, having $\mathrm{Ob}(\underline{S})$ as set of objects, in an obvious manner.

(5.5.3) Proposition. The assignments $\underline{C} \rightsquigarrow \Phi(\underline{C})$ and $\Phi \rightsquigarrow \underline{C}(\Phi)$ establish a bijection between categories relative to multiplicative categories and morphisms of bicategories with domain locally punctual and codomain having a single object.

(5.5.4) Coherent families of isomorphisms. Frequently looking for objects X of a category $\underline{C}$ having some properties (e.g. universal properties, or objects obtained by iteration of a tensor product associative up to isomorphisms) one finds a whole family of such X_i, indexed by a set I, equipped with "canonical" isomorphisms $\varphi_{ij} : X_i \longleftarrow X_j$, such that $\varphi_{ii} = \mathrm{Id}$, $\varphi_{ij} \varphi_{jk} = \varphi_{ik}$. Such families can be obviously identified with $\underline{C}$-polyads indexed by I (where $\underline{C}$ is as usual identified with the degenerate bicategory $D\underline{C}$).

(5.5.5) Polyspans. Let $\underline{C}$ be a category with pullbacks, $\mathrm{Sp}\,\underline{C}$ the bicategory of spans of $\underline{C}$. A polyad in $\mathrm{Sp}\,\underline{C}$, indexed by a set I, is determined by:

(i) A map $F: I \longrightarrow \mathrm{Ob}\,\mathrm{Sp}\,\underline{C} = \mathrm{Ob}\,\underline{C}$, written $i \rightsquigarrow X_i$.

(ii) For each pair (i, j), a functor $F(i, j): \mathbb{1} \longrightarrow \mathrm{Sp}\,\underline{C}(X_i, X_j)$,
identified with an object S_{ij} of $\mathrm{Sp}\,\underline{C}(X_i, X_j)$, that is a diagram in $\underline{C}$:

$$S_{ij}: X_i \xleftarrow{\;g_{ij}\;} X_{ij} \xrightarrow{\;\overline{g}_{ij}\;} X_j \,.$$

(iii) For each $i \in I$, an arrow $\varphi_i: I_{X_i} \longrightarrow S_{ii}$ in $\mathrm{Sp}(X_i, X_i)$, that is a commutative diagram in $\underline{C}$:

$$\varphi_i: \quad \begin{tikzcd} & X_i \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] \arrow[dd, "g_i"] & \\ X_i & & X_i \\ & X_{ii} \arrow[ul, "g_{ii}"] \arrow[ur, "\overline{g}_{ii}"'] & \end{tikzcd}$$

(iv) For each (i, j, k) a natural transformation $\varphi(i, j, k)$ determined by its unique component $\varphi(i, j, k)(\mathbb{O}, \mathbb{O}): S_{ij} \circ S_{jk} \longrightarrow S_{ik}$, that is according to (2. 6) a commutative diagram in $\underline{C}$; where p and $\overline{p}$ are the projections of the pullback:

$$\begin{tikzcd} & & X_{ij} \prod_{X_j} X_{jk} \arrow[dl, "p_{ijk}"'] \arrow[dr, "\overline{p}_{ijk}"] \arrow[dd, "g_{ijk}"] & & \\ & X_{ij} \arrow[dl, "g_{ij}"'] & & X_{jk} \arrow[dr, "\overline{g}_{jk}"] & \\ X_i & & X_{ik} \arrow[ll, "g_{ik}"] \arrow[rr, "\overline{g}_{ik}"'] & & X_k \end{tikzcd}$$

Note that all these data are determined by the maps $g_i, g_{ij}, \overline{g}_{ij}, g_{ijk}$. The conditions (M. 1) and (M. 2) are expressed, in terms of these maps, by the commutativity of the three diagrams below, where the notation $X_{ij} \bullet X_{jk}$

stands for $X_{ij} \coprod_{X_j} X_{jk}$, $X_{ij} \circ X_j$ for $X_{ij} \coprod_{X_j} X_j$, and a, ℓ, r are the isomorphisms of associativity and identity of pullbacks.

$$
\begin{tikzcd}
(X_{ij} \circ X_{jk}) \circ X_{k\ell} \arrow[rr, "a", "\sim"'] \arrow[d, "g_{ijk} \circ Id_{k\ell}"'] & & X_{ij} \circ (X_{jk} \circ X_{k\ell}) \arrow[d, "Id_{ij} \circ g_{jk\ell}"] \\
X_{ik} \circ X_{k\ell} \arrow[d, "g_{ik\ell}"'] & & X_{ij} \circ X_{j\ell} \arrow[d, "g_{ij\ell}"] \\
X_{i\ell} \arrow[rr, "Id_{i\ell}"] & & X_{i\ell}
\end{tikzcd}
$$

And:

$$
\begin{tikzcd}
X_{ij} \circ X_j \arrow[r, "Id_{ij} \circ g_{jj}"] \arrow[d, "r", "\wr"'] & X_{ij} \circ X_{jj} \arrow[d, "g_{ijj}"] \\
X_{ij} & X_{ij} \arrow[l, "Id_{ij}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X_i \circ X_{ij} \arrow[r, "g_{ii} \circ Id_{ij}"] \arrow[d, "\ell", "\wr"'] & X_{ii} \circ X_{ij} \arrow[d, "g_{iij}"] \\
X_{ij} & X_{ij} \arrow[l, "Id_{ij}"']
\end{tikzcd}
$$

That is, neglecting the a, ℓ, r which is always possible according to Theorem (B. 4), the polyspans of $\underline{C}$ satisfy the cocycle conditions:

(P. 1) $\qquad g_{ik\ell}(g_{ijk} \circ Id_{k\ell}) = g_{ij\ell}(Id_{ij} \circ g_{jk\ell})$

(P. 2) $\qquad g_{ijj}(Id_{ij} \circ g_{jj}) = Id \quad , \quad g_{iij}(g_{ii} \circ Id_{ij}) = Id.$

The significance of these equations in descent theory and non-abelian cohomology shall be examined elsewhere.

(5. 6) Pseudo-functors. In [Gr], Grothendieck defines a pseudo-functor $\underline{E}^* \longrightarrow Cat$, where $\underline{E}$ is a category, as:

(a) A map $S \rightsquigarrow \underline{F}_S$ from $\mathrm{Ob}\,\underline{E}$ to Cat.

(b) For each $f\colon T \longrightarrow S$ in $\underline{E}$, a functor $f^*\colon \underline{F}_S \longrightarrow \underline{F}_T$.

(c) For each pair (f, g) of maps of $\underline{E}$ such that fg is defined, a natural transformation $c_{f,g}\colon g^*f^* \longrightarrow (fg)^*$.

(d) For each object S of $\underline{E}$, a natural transformation $\alpha_S\colon (\mathrm{Id}_S)^* \longrightarrow \mathrm{Id}_{\underline{F}_S}$.

These data are required to satisfy:

(A) $$\begin{cases} c_{f,\mathrm{id}_T}(\xi) = \alpha_T(f^*(\xi)) \\ c_{\mathrm{id}_S, f}(\xi) = f^*(\alpha_S(\xi)) \end{cases}$$

(B) $$c_{f,gh}(\xi) \circ c_{g,h}(f^*(\xi)) = c_{fg,h}(\xi) \circ h^*(c_{f,g}(\xi))$$

for any maps $f\colon T \longrightarrow S$, $g\colon U \longrightarrow T$, $h\colon V \longrightarrow U$ in $\underline{E}$, and object ξ of $\underline{F}_S$.

He also considers the following special cases:

(i) For all S, $(\mathrm{id}_S)^* = \mathrm{id}_{\underline{F}_S}$ and the α_S are identities. (A) reduces to:

(A') $$c_{f,\mathrm{id}_T} = \mathrm{id}_{f^*}, \quad c_{\mathrm{id}_S, f} = \mathrm{id}_{f^*}$$

which he calls <u>normalized</u>.

(ii) All the $c_{f,g}$ are isomorphisms (this corresponds to fibered categories).

(iii) For all f, g, $(fg)^* = g^*f^*$ and $c_{fg} = \mathrm{Id}$ (this corresponds to split-fibrations, or functors $\underline{E}^* \longrightarrow \mathrm{Cat}$).

Define, given such a pseudo-functor $\underline{P} = (\underline{F}, *, c, \alpha)$, the following:

(1) A map $F\colon \mathrm{Ob}\,\underline{DE}^* = \mathrm{Ob}\,\underline{E} \longrightarrow \mathrm{Ob}\,\mathrm{Cat}$, by $FS = \underline{F}_S$. (Where $\underline{DE}^*$ is the degenerate bicategory defined by $\underline{E}^*$.)

(2) If f is an object of the discrete category $D\underline{E}^*(T,S)$, $F(T,S)(f) = f^*$.

(3) $\varphi(U,T,S)$ to be the natural transformation having as components the natural transformations $\varphi(U,T,S)(g,f) = c_{f,g}$.

(4) Arrows φ_S in $\mathrm{Cat}(\underline{F}_S, \underline{F}_S)$ to be the natural transformation α_S.

(5. 6. 1) Theorem: With the previous notations:

(a) $(F, F(S,T), \varphi_S, \varphi(U,T,S))$ define a morphism $\Phi(\underline{P})$: $D\underline{E}^* \longrightarrow \mathrm{Cat}$.

(b) The correspondence $\underline{P} \rightsquigarrow \Phi(\underline{P})$ is a bijection between pseudo-functors and morphisms of bicategories with domain a category and co-domain Cat.

(c) Under this correspondence the pseudo-functors satisfying (i), (ii), and (iii) become respectively the strictly unitary morphisms, the homo-morphisms and the strict homomorphisms.

The proof is a straightforward and tedious verification that the requirements (A) and (B) for pseudo-functors, are equivalent in this case to (M. 2) and (M. 1) of (4. 11) respectively, and then (c) is a rephrasing of the definitions.

In Part II, the construction of [Gr] assigning to each morphism, i. e., pseudo-functor, $\underline{E}^* \longrightarrow \mathrm{Cat}$ a category $\underline{F}$ equipped with a functor $p: \underline{F} \longrightarrow \underline{E}$ together with a cleavage of p, shall be extended by replacing Cat by the bigger bicategory Prof of profunctors. Then all the properties of categories over a category $\underline{E}$ -- fibrations, cofibrations, cleavages, splittings, ... -- will have simple interpretations in terms of morphisms $\underline{E}^* \longrightarrow \mathrm{Prof}$. We will also extend the construction to the case where the domain is any bicategory, not necessarily one-dimensional.

(5.7) Bimodules and Rings. Let Ring be the category of rings with identity, Bim the bicategory defined in (2.6). With the same notations, define:

(i) A map $F = \mathrm{Id} : \mathrm{Ob}(\mathrm{Ring}) \longrightarrow \mathrm{Ob}(\mathrm{Bim})$.

(ii) Functors $F(A,B)\colon \mathrm{Ring}(A,B) \longrightarrow \mathrm{Bim}(A,B) = {}_A\underline{M}_B$, $f \rightsquigarrow M_f$.

(iii) For each A, a map of bimodules $\varphi_A = \mathrm{Id}\colon A \longrightarrow A$.

(iv) Natural transformations $\varphi(A,B,C)$ by their components:

$$\varphi(A,B,C)(f,g)\colon\ M_f \circ M_g \xrightarrow{\ \sim\ } M_{fg}\ .$$

Then $(F,\varphi)\colon \mathrm{Ring} \longrightarrow \mathrm{Bim}$ is a homomorphism, strictly unitary. (Here, of course, Ring has been identified with the degenerate bicategory D Ring.) Moreover, the correspondence $A \rightsquigarrow A$, $f \rightsquigarrow M_f$ embeds the category of rings in the bicategory of bimodules.

(5.8) Functoriality of spans and cospans. Let $\underline{C}$ and $\overline{\underline{C}}$ be any categories with pushouts and $F\colon \underline{C} \longrightarrow \overline{\underline{C}}$ be a functor. Choose pushouts in $\underline{C}$ and $\overline{\underline{C}}$, and define:

(i) A map $F\colon \mathrm{Ob}(\mathrm{Cosp}\,\underline{C}) = \mathrm{Ob}(\underline{C}) \longrightarrow \mathrm{Ob}(\mathrm{Cosp}\,\overline{\underline{C}}) = \mathrm{Ob}(\overline{\underline{C}})$, $X \rightsquigarrow FX$.

(ii) Functors $F(A,B)\colon \mathrm{Cosp}\,\underline{C}(A,B) \longrightarrow \mathrm{Cosp}\,\overline{\underline{C}}(FA,FB)$ by:

$$(S\colon A \xrightarrow{\ \alpha\ } X \xleftarrow{\ \beta\ } B) \rightsquigarrow (FS\colon FA \xrightarrow{\ F\alpha\ } FX \xleftarrow{\ F\beta\ } FB).$$

(iii) For each $A \in \mathrm{Ob}\,\underline{C}$ an arrow $\varphi_A = \mathrm{Id}\colon \overline{I}_{FA} \longrightarrow F_{IA}$ of $\mathrm{Cosp}\,\underline{C}(FA,FA)$

(iv) If $S\colon A \longrightarrow X \longleftarrow B$ and $T\colon B \longrightarrow Y \longleftarrow C$, a map

$\varphi(A,B,C)(S,T)\colon FS \circ FT \longrightarrow F(S \circ T)$ in $\mathrm{Cosp}\,\overline{\underline{C}}(FA,FC)$ to be the diagram:

$$\begin{tikzcd}
& FX \coprod_{FB} FY \arrow[dd, "?"] & \\
FX \arrow[ur, "j(FX)"] \arrow[dr, "F(j(X))"'] & & FY \arrow[ul, "j(FY)"'] \arrow[dl, "F(j(Y))"] \\
& F(X \coprod_{B} Y) &
\end{tikzcd}$$

where $j(FX), j(FY), j(X), j(Y)$ are the canonical maps in the pushouts and ? is the unique map making the diagram commutative (there is always one such).

<u>(5.8.1) Proposition.</u> With the previous notations:

(i) (F, φ) determine a strictly unitary morphism $\mathrm{Cosp}F: \mathrm{Cosp}\underline{C} \longrightarrow \mathrm{Cosp}\overline{\underline{C}}$.

(ii) $\mathrm{Cosp}F$ is a homomorphism iff F commutes with pushouts.

(iii) $\mathrm{Cosp}F$ is a strict homomorphism iff F commutes with the chosen pushouts.

(iv) If $\overline{F}: \overline{\underline{C}} \longrightarrow \overline{\overline{\underline{C}}}$ is another functor, choosing pushouts in $\overline{\overline{\underline{C}}}$ we get $\mathrm{Cosp}(\overline{F}F) = \mathrm{Cosp}(\overline{F})\, \mathrm{Cosp}(F)$.

Proof long but straightforward.

Note that if $\underline{C}$ and $\overline{\underline{C}}$ have pullbacks and $F: \underline{C} \longrightarrow \overline{\underline{C}}$ is a functor, $\mathrm{Sp}F$ defined dually is a comorphism from $\mathrm{Sp}\underline{C}$ to $\mathrm{Sp}\overline{\underline{C}}$ (i.e., a morphism of the conjugates). In particular, if $\underline{C}$ and $\overline{\underline{C}}$ have final objects 1 and $\overline{1}$ any functor defines a comorphism of the multiplicative categories $\underline{C}$ and $\overline{\underline{C}}$ (with $\prod$ as multiplication), since $\underline{C}$ and $\overline{\underline{C}}$ are equivalent to the c.m.'s $\mathrm{Sp}\underline{C}(1,1)$ and $\mathrm{Sp}\overline{\underline{C}}(\overline{1},\overline{1})$ by restrictions of $\mathrm{Sp}F$. It will be a homomorphism (resp. strict) of c.m. iff F commutes with products (resp. with chosen products).

§6. Some Corollaries of Theorem (4.3.1):

The interest of defining mathematical objects as morphisms of bi-categories, is the possibility to compose them with other morphisms to get new objects as direct or inverse images. We give a few instances, many others can be obtained by choosing a pair of composable morphisms in the list of §5.

(6.1) Proposition: Let $\underline{M}$ and $\overline{\underline{M}}$ be multiplicative categories, $\Phi = (F, \varphi, \lambda): \underline{M} \longrightarrow \overline{\underline{M}}$ a morphism, and $M = (T; \mu; \eta)$ a monoid of $\underline{M}$; then $(FT; F\mu \circ \varphi(T,T); F\eta \circ \lambda)$ is a monoid of $\overline{\underline{M}}$ called image of M by Φ, and denoted $\Phi(M)$.

Identify M with a morphism $\overline{\Phi} : \mathbb{1} \longrightarrow \underline{M}$, then $\Phi(M)$ is identified with $\Phi\,\overline{\Phi}$.

As an example, take (i) $\underline{M} = (\underline{A}, \otimes, \ldots)$, $\overline{\underline{M}}$ = the category of endomorphisms of $\underline{A}$ and Φ the left representation [B.1]; $A \rightsquigarrow A \otimes -$. To each monoid in $\underline{M}$ corresponds a monad over $\underline{A}$.
(ii) $\underline{M}$ and $\overline{\underline{M}}$ to be the endomorphisms of two categories $\underline{K}$ and $\underline{L}$. $S: \underline{K} \longrightarrow \underline{L}$ and $T: \underline{L} \longrightarrow \underline{K}$ a pair of adjoint functors, and $\Phi: \underline{M} \longrightarrow \overline{\underline{M}}$ the morphism determined by the adjunction (cf [B.1]), to each monad on $\underline{K}$ corresponds a monad on $\underline{L}$.

(6.2). If $\underline{C}$ is a category with pushouts, $C = (X_0, X_1, \partial_0, \partial_1, \sigma, \gamma)$ a cocategory inside $\underline{C}$, cf (5.4.3) and $F: \underline{C}^* \longrightarrow \overline{\underline{C}}$ a functor which commutes with pullbacks, then $(FX_0, FX_1, \ldots)$ define a category FC

inside $\overline{\underline{C}}$. Identify C with a morphism $\mathbb{1} \longrightarrow \mathrm{Sp}\,\underline{C}^*$ and note that F determines a morphism $\mathrm{Sp}\,\underline{C}^* \longrightarrow \mathrm{Sp}\,\overline{\underline{C}}$.

In particular, for each object X of $\underline{C}$, Hom(C, X) is a category, and $X \rightsquigarrow \mathrm{Hom}(C,X)$ a functor $\mathrm{Hom}(C,-)\colon \underline{C} \longrightarrow \mathrm{Cat}^{[1]}$. Taking $\underline{C} = \mathrm{Cat}^{[1]}$ and $C = \mathbb{2}$, we find that for each category $\underline{X}$, $\mathrm{Cat}^{[1]}(\mathbb{2}, \underline{X})$ is a category. The structure of Cat as a 2-category comes from this remark which will be generalized to get the 2 and 3 dimensional parts of Bicat.

(6.3). Let $\underline{M}$ be a c.m., $\underline{C}$ be an $\underline{M}$ category (cf (5.5.2)), $\underline{C}'_o$ a set and $f\colon \underline{C}'_o \longrightarrow \mathrm{Ob}(\underline{C})$ a map. For all X', Y', Z' in $\underline{C}'_o$ define

$$\underline{C}'(X',Y') = \underline{C}(fX',fY') \ , \quad i_{\underline{C}'}(X') = i_{\underline{C}}(fX') \ , \quad c'(X',Y',Z') = c(fX',fY',fZ').$$

<u>(6.3.1) Proposition.</u> With these notations, $(\underline{C}'_o, \underline{C}'(X,Y), i_{\underline{C}'}, c')$ is an $\underline{M}$ category $f^*(\underline{C})$ called <u>inverse image</u> of $\underline{C}$ by f, and the inverse images are transitive (i.e., $g^*f^*(\underline{C}) = (fg)^*(\underline{C})$; $\mathrm{id}^*(\underline{C}) = \underline{C}$).

Let $\underline{L}$ and $\underline{L}'$ be the locally punctual bicategories having $\mathrm{Ob}\,\underline{C}$ and $\underline{C}'_o$ as set of objects, $\Phi'\colon \underline{L}' \longrightarrow \underline{L}$ the morphism obviously determined by f and $\Psi\colon \underline{L} \longrightarrow \mathrm{I}\underline{M}$ the morphism identified with $\underline{C}$ (cf (5.5.3)). Then $f^*(\underline{C})$ is the $\underline{M}$-category identified with the morphism $\Psi\Phi'\colon \underline{L}' \longrightarrow \mathrm{I}\underline{M}$.

Let $\overline{\underline{M}}$ be another c.m. and $\Phi = (F, \varphi, \lambda)\colon \underline{M} \longrightarrow \overline{\underline{M}}$ a morphism. For all X, Y, Z in $\mathrm{Ob}(\underline{C})$ define: $\overline{\underline{C}}(X,Y) = F(\underline{C}(X,Y))$, $\overline{c}(X,Y,Z)$ to be the composite morphism:

$$F(\underline{C}(X,Y))\overline{\otimes}F(\underline{C}(Y,Z)) \xrightarrow{\varphi} F(\underline{C}(X,Y)\otimes\underline{C}(Y,Z)) \xrightarrow{F(c)} F(\underline{C}(X,Z))$$

and $i_{\overline{\underline{C}}}(X)$ to be the composite:

$$\overline{\Lambda} \xrightarrow{\lambda} F(\Lambda) \xrightarrow{F(i_{\underline{C}}(X))} F(\underline{C}(X,X)) = \overline{\underline{C}}(X,X).$$

<u>(6.3.2) Proposition:</u> With these notations $(\mathrm{Ob}(\underline{C}), \overline{\underline{C}}(X,Y), i_{\overline{\underline{C}}}, \tau)$ is an $\overline{\underline{M}}$-category, $\Phi_*(\underline{C})$ called direct image of $\underline{C}$ by Φ, and direct images are transitive.

If $\underline{C}$ is identified with $\Psi\colon \underline{L} \longrightarrow I\underline{M}$, then $\Phi_*(\underline{C})$ is identified with the composite $I\Phi\circ\Psi\colon \underline{L} \longrightarrow I\underline{M} \longrightarrow I\overline{\underline{M}}$.

Note that, moreover, from the associativity of composition of morphisms, it follows that direct and inverse image commute with each other, that is: $\Phi_*(f^*(\underline{C})) = f^*(\Phi_*(\underline{C}))$.

(6.4). Let P be a pseudo-functor from $\underline{E}^*$ to Cat (cf. (5.6)), and $g\colon \overline{\underline{E}} \longrightarrow \underline{E}$ be a functor. P can be identified with a morphism $\Phi\colon D(\underline{E}^*) \longrightarrow \mathrm{Cat}$, g determines a morphism $D(g^*)\colon D(\overline{\underline{E}}^*) \longrightarrow D(\underline{E}^*)$ and the composite $\Phi\circ D(g^*)\colon D(\overline{\underline{E}}^*) \longrightarrow \mathrm{Cat}$ defines a pseudo-functor $g^*(P)$ called <u>inverse image</u> of P by g ; again transitive. If $p\colon \underline{F} \longrightarrow \underline{E}$ is the cleaved category over $\underline{E}$ associated with P, then the cleaved category associated with $g^*(P)$ is the inverse image $g^*(p)$ (in the sense of cleaved categories i.e., the pullback $\overline{\underline{E}} \times \underline{F}$ with the cleavage pulled back from $\underline{F}$).

§7. Some Basic Constructions.

(7.1) Poincaré category. Let $\underline{S}$ be a bicategory. For each pair (A, B) of objects of $\underline{S}$, let $\Pi\underline{S}(A, B)$ be the set of connected components of the category $\underline{S}(A, B)$. If S is an object of $\underline{S}(A, B)$ we write $[S]_\Pi$ for its equivalence class. We define composition:

$$\Pi\underline{S}(A, B) \times \Pi\underline{S}(B, C) \longrightarrow \Pi\underline{S}(A, C) \text{ by } [S]_\Pi \circ [T]_\Pi = [S \circ T]_\Pi .$$

It is well-defined, associative and the $[I_A]_\Pi$ are identities, giving rise to a category ΠS having the same objects as $\underline{S}$, called the Poincaré category of $\underline{S}$.

For example, if $\underline{A}$ is an abelian category, and $\underline{\underline{Ext}}_{\underline{A}}$ is the bicategory of extensions in $\underline{A}$ (cf. (2.8)) then $\Pi\,\underline{\underline{Ext}}_{\underline{A}}$, written $\underline{Ext}_{\underline{A}}$, is the category having $\underline{A}$ as set of objects, with maps the equivalence classes of extensions under the usual equivalence relation.

If $\Phi = (F, \varphi): \underline{S} \longrightarrow \overline{\underline{S}}$ is a morphism of bicategories, we define a functor $\Pi\Phi: \Pi\underline{S} \longrightarrow \Pi\overline{\underline{S}}$ by $\Pi\Phi A = FA$ and $\Pi\Phi[S]_\Pi = [FS]_\Pi$. Thus as a map of diagrams $\Pi\Phi$ depends only on the F part, but it is a functor thanks to the φ part which connects $FS \circ FT$ and $F(S \circ T)$ and also FI_A and I_{FA}. We clearly define thus the Poincaré functor

$$\Pi: \mathrm{Bicat}^{[1]} \longrightarrow \mathrm{Cat}^{[1]} .$$

(7.1.1) Proposition.

(i) The Poincaré functor is left adjoint to the degeneracy functor

$$D: \mathrm{Cat}^{[1]} \longrightarrow \mathrm{Bicat}^{[1]} \text{ of } (5.1).$$

(ii) The composite $\Pi \circ D$ is isomorphic to the identity functor of $\mathrm{Cat}^{[1]}$.

The proof, straightforward, is omitted.

(7.2) Classifying category. In many cases the equivalence relation defining $\Pi\underline{S}$ is too coarse. Thus if all the $\underline{S}(A, B)$ are connected and non-empty (e.g. for $\underline{S} = \mathrm{Bim}$) the category $\Pi\underline{S}$ is equivalent to one point. A more precise category is defined as follows: Let $\underline{S}$ be a bicategory. For each object S of $\underline{S}(A, B)$ let $[S]$ be the set of all objects of $\underline{S}(A, B)$ isomorphic to S. Define $C\underline{S}(A, B)$ to be the set of all such isomorphism classes. We have a composition:

$$C\underline{S}(A, B) \times C\underline{S}(B, C) \longrightarrow C\underline{S}(A, C) \qquad ([S], [T]) \rightsquigarrow [S \circ T]$$

giving rise to a category $C\underline{S}$ having same objects as $\underline{S}$ called the classifying category of $\underline{S}$.

If $\underline{S} = \mathrm{Cat}$, $C\underline{S}$ is the category with objects the categories, and maps isomorphism classes of functors; if $\underline{S}$ is the c.m. of modules over a commutative ring Λ, $C\underline{S}$ is the monoid with elements classes of isomorphic modules and composition induced by $\otimes$, etc....

The category $C\underline{S}$ is less functorial than the Poincare category: If $\Phi = (F, \varphi): \underline{S} \longrightarrow \underline{\bar{S}}$ is a morphism, the correspondence $A \rightsquigarrow FA$ $[S] \rightsquigarrow [FS]$ defines a map of the underlying graphs of $C\underline{S}$ and $C\underline{\bar{S}}$, however $[FS] \circ [FT] \neq [F(S \circ T)]$. However, if Φ is a homomorphism this map is a functor $C\Phi: C\underline{S} \longrightarrow C\underline{\bar{S}}$. Thus we obtain a classifying functor

$$C: \text{H-Bicat}^{[1]} \longrightarrow \mathrm{Cat}^{[1]}.$$

Clearly, we have a natural surjection $C\underline{S} \longrightarrow \Pi\underline{S}$ which is an isomorphism when $\underline{S}$ is locally a groupoid (i.e., all the $\underline{S}(A, B)$'s are groupoids).

(7.3) Picard groupoid. If $\underline{S}$ is a bicategory, the invertible maps of the classifying category $C\underline{S}$ form a groupoid $\mathrm{Pic}\,\underline{S}$, called the Picard groupoid of $\underline{S}$. Clearly we obtain the Picard functor

$$\mathrm{Pic}\colon \text{H-Bicat}^{[1]} \longrightarrow \text{Groupoid}^{[1]}.$$

The definition is motivated by:

(7.3.1) Theorem: Let R be a commutative ring with identity, Mod(R) the c.m. of R-modules (with $\otimes_R$ as multiplication). Then Pic Mod(R) is canonically isomorphic to the Picard group of R, Pic R.

All there is to show is that, if M is an R-module such that there exists an R-module N with $M \otimes N \simeq R$ and $N \otimes M \simeq R$, then M is finitely generated projective. The proof is left to the reader since it will result from general considerations of Part II.

(7.4) Inverse limits. The general notion of limits of bicategories shall be examined in Part II, in connection with bi-adjoints. We will need immediately the following:

(7.4.1) Proposition: (i) The category $S\circ\text{H-Bicat}^{[1]}$ has inverse limits (and even a canonical choice of limits). (ii) The inclusion functors of $S\circ\text{H-Bicat}^{[1]}$ in $\text{H-Bicat}^{[1]}$, $S\circ\text{U-Bicat}^{[1]}$, $\text{U-Bicat}^{[1]}$ and $\text{Bicat}^{[1]}$ commute with the inverse limits.

Proof. (i) follows from the fact that bicategories are algebraic structures (cf. (1.4)(iii)) and that their morphisms as algebraic structures are the strict homomorphisms (cf. (4.2)). If $\underline{T}$ is an indexing category and $\underline{S}_i$ a family of bicategories indexed by $\underline{T}$, the transition maps being strict homomorphisms, $\varprojlim \underline{S}_i = \underline{S}$ is constructed pointwise, i.e., $\mathrm{Ob}\,\underline{S} = \varprojlim \mathrm{Ob}\,\underline{S}_i$; $\underline{S}(A, B) = \varprojlim \underline{S}_i(A_i, B_i)$ for $A = (A_i)$, $B = (B_i)$, etc..., the maps $\underline{S} \longrightarrow \underline{S}_i$ are the obvious projections. The proof of (ii) is straightforward and is omitted.

8. Transformations between Morphisms

(8.1) Introduction. Starting with categories, which are one-dimensional graphs with one operation, we get for the system of "all possible maps" (functors and natural transformations) a bicategory Cat which is a 2-dimensional complex with two operations. Similarly, "all the maps" between bicategories should constitute a 3-dimensional complex with three (partially defined) operations. Apart from internal coherence the examples given in §5 already oblige us to construct completely this 3-dimensional structure: We have shown that many notions usually thought of as objects -- e.g., algebras, categories, monads, ... -- could be identified with morphisms of bicategories $\Phi: \underline{S} \longrightarrow \underline{S}'$ for suitable $\underline{S}$ and $\underline{S}'$. However, if Φ and Ψ are two such objects, there are usually maps between them which sould correspond to transformations between morphisms of bicategories, i.e., 2-cells. Moreover, if Φ and Ψ were categories, the functors $\Phi \longrightarrow \Psi$ would give 2-cells, but we would, and will indeed, interpret natural transformations as 3-cells of Bicat.

To construct the 2-dimensional skeleton $\mathrm{Bicat}^{[2]}$ of Bicat we use the following idea of category theory: If f_o and f_1 are functors $\underline{X} \longrightarrow \underline{Y}$ a natural transformation can be defined in either of these two ways $^{(*)}$:

(i) A functor $h: \mathbb{2} \times \underline{X} \longrightarrow \underline{Y}$ such that $h \circ (\partial_i \times \mathrm{Id}) = f_i$ $(i = 0, 1)$ where $\partial_i : \mathbb{1} \longrightarrow \mathbb{2}$ are the obvious functors.

$^{(*)}$ Compare with the definition of a homotopy by $I \times X \longrightarrow Y$ or $X \longrightarrow Y^I$.

(ii) A functor $k\colon \underline{X} \longrightarrow \underline{Y}^{\mathbf{2}}$ such that $d_i k = f_i$ where

$$d_i\colon \underline{Y}^{\mathbf{2}} \xrightarrow{\underline{Y}^{\partial_i}} \underline{Y}^{\mathbf{1}} \longrightarrow \underline{Y} .$$

However none of these definitions suffices to define the composition of natural transformations $f_o \longrightarrow f_1 \longrightarrow f_2$. It is obtained by means of

(i) A functor $\gamma\colon \mathbf{2} \longrightarrow \mathbf{2} \coprod_{(\partial_o, \partial_1)} \mathbf{2}$ or

(ii) A functor $c\colon \underline{Y}^{\mathbf{2}} \prod_{(d_o, d_1)} \underline{Y}^{\mathbf{2}} \longrightarrow \underline{Y}^{\mathbf{2}}$

such that $\mathbf{2}$ is a cocategory inside $\mathrm{Cat}^{[1]}$ (resp. $\underline{Y}^{\mathbf{2}}$ is a category inside $\mathrm{Cat}^{[1]}$). In $\mathrm{Cat}^{[1]}$ the passage from (i) to (ii) is trivial, but it is far from being so in $\mathrm{Bicat}^{[1]}$, and the analogue of (ii), being less complicated, will be used. Thus, the aim of the section is to assign to each bicategory $\underline{S}$ a bicategory called $\mathrm{Cyl}\,\underline{S}$, equipped with strict homomorphisms $d_o, d_1\colon \mathrm{Cyl}\,\underline{S} \longrightarrow \underline{S}$ and $c\colon \mathrm{Cyl}\,\underline{S} \prod_{(d_o, d_1)} \mathrm{Cyl}\,\underline{S} \longrightarrow \mathrm{Cyl}\,\underline{S}$ (the pullback exists because of (7.4.1)). $\mathrm{Cyl}\,\underline{S}$ plays the same universal role in this context as the space of paths in topology.

<u>(8.2) Squares and cylinders.</u> Let $\underline{S}$ be a bicategory, $U\colon A \longrightarrow \overline{A}$ and $V\colon B \longrightarrow \overline{B}$ be arrows of $\underline{S}$.

(8.2.1) A <u>square from V to U</u> $Q = (\overline{S}, u, S)\colon V \longrightarrow U$ is defined (*) by:

(i) two arrows $S\colon B \longrightarrow A$ and $\overline{S}\colon \overline{B} \longrightarrow \overline{A}$.

(ii) a 2-cell $u\colon \overline{S} \circ V \Longrightarrow U \circ S$ (i.e., an arrow of $\underline{S}(\overline{A}, B)$).

(*) See (8.5) below for a geometric interpretation.

The square Q is said <u>commutative</u> if $U\circ S = \overline{S}\circ V$ and u is the identity, commutative up to isomorphism, or <u>iso-commutative</u> if u is invertible (in $\underline{S}(\overline{A}, B)$).

Let $Q_1 = (\overline{S}_1, u_1, S_1)\colon V \longrightarrow U$ and $Q_2 = (\overline{S}_2, u_2, S_2)\colon V \longrightarrow U$ be two squares with the same domain V and codomain U.

(8.2.2) A <u>cylinder</u> from Q_2 to Q_1, $q = (\overline{s}, s)\colon Q_2 \longrightarrow Q_1$ is defined (*) by a pair of 2-cells:

$$S_1 \xleftarrow{\ s\ } S_2 \text{ in } \underline{S}(A,B) \quad\text{and}\quad \overline{S}_1 \xleftarrow{\ \overline{s}\ } \overline{S}_2 \text{ in } \underline{S}(\overline{A},\overline{B})$$

making the following diagram of $\underline{S}(\overline{A}, B)$ commutative:

(8.2.3)
$$\begin{tikzcd}
\overline{S}_1\circ V \arrow[d, "u_1"'] & \overline{S}_2\circ V \arrow[l, "\overline{s}\circ V"'] \arrow[d, "u_2"] \\
U\circ S_1 & U\circ S_2 \arrow[l, "U\circ s"]
\end{tikzcd}$$

that is, satisfying the equation:

(Cyl): $$(U\circ s)u_2 = u_1(\overline{s}\circ V).$$

<u>(8.3) The categories</u> $\mathrm{Cyl}\,\underline{S}(U,V)$: Let $Q^i = (\overline{S}^i, u^i, S^i)\colon V \longrightarrow U$ $(i = 1,2,3)$ be three squares with same domain and codomain, and $q^j = (\overline{s}^j, s^j)\colon Q^{j+1} \longrightarrow Q^j$ $(j = 1,2)$ be two cylinders, then the composite $\overline{s}^1\overline{s}^2$ and $s^1 s^2$ are defined, and we have:

(*) See (8.5) below for a geometric interpretation.

<u>(8.3.1) Lemma.</u> (i) The pair $(\bar{s}^1\bar{s}^2, s^1s^2)$ defines a cylinder from Q^3 to Q^1 written q^1q^2.

(ii) With the composition $(q^1, q^2) \rightsquigarrow q^1q^2$ we obtain a category, denoted $\mathrm{Cyl}\,\underline{S}(U, V)$, having as objects the squares from V to U, and as maps the cylinders between these squares. If $Q = (\bar{S}, u, S): V \longrightarrow U$ is an object of $\mathrm{Cyl}\,\underline{S}(U, V)$, its identity is the cylinder i_Q defined by $(i_{\bar{S}}, i_S)$.

<u>Proof.</u> The equation (Cyl) for $(\bar{s}^1\bar{s}^2, s^1s^2)$ is equivalent to the commutativity of the outside of the following diagram in $\underline{S}(\bar{A}, B)$

$$
\begin{tikzcd}
\bar{S}^1 \circ V \arrow[dd, "u^1"'] \arrow[dr, phantom, "\mathrm{II}"] & & \bar{S}^3 \circ V \arrow[ll, "(\bar{s}^1\bar{s}^2)\circ V"'] \arrow[dl, "\bar{s}^2\circ V"] \arrow[dd, "u^3"] \arrow[ll, phantom, shift right=4, "\mathrm{I}"] \\
& \bar{S}^2 \circ V \arrow[ul, "\bar{s}^1\circ V"] \arrow[d, "u^2"] \arrow[ur, phantom, "\mathrm{II'}", shift left=6] & \\
U \circ S^1 & U \circ S^2 \arrow[l, "U\circ s^1"'] & U \circ S^3 \arrow[l, "U\circ s^2"] \arrow[ll, bend left, "U\circ(s^1s^2)", phantom, "\mathrm{I'}"'] \arrow[ll, bend left=40, "U\circ(s^1s^2)"]
\end{tikzcd}
$$

But the regions numbered I and I' commute because $\circ$ are bifunctors, and II and II' because q^1 and q^2 are cylinders. This proves (i), then (ii) follows trivially from the fact that $\underline{S}(A, B)$ and $\underline{S}(\bar{A}, \bar{B})$ are categories.

<u>(8.4) The functors $c(U, V, W)$.</u> Let $U: A \longrightarrow \bar{A}$, $V: B \longrightarrow \bar{B}$ and $W: C \longrightarrow \bar{C}$ be arrows of a bicategory $\underline{S}$.

(8.4.1). If $Q = (\overline{S}, u, S): V \longrightarrow U$ and $R = (\overline{T}, v, T): W \longrightarrow V$ are two squares, we define a square $Q \circ R$ from W to U by

$$Q \circ R = (\overline{S} \circ \overline{T}, u/v, S \circ T): W \longrightarrow U$$

where u/v is the composite map in $\underline{S}(\overline{A}, C)$:

$$U \circ (S \circ T) \xleftarrow{a} (U \circ S) \circ T \xleftarrow{u \circ T} (\overline{S} \circ V) \circ T \xleftarrow{a^{-1}} \overline{S} \circ (V \circ T) \xleftarrow{S \circ v} \overline{S} \circ (\overline{T} \circ W) \xleftarrow{a} (\overline{S} \circ T) \circ W.$$

(8.4.2). Suppose we are given furthermore two squares:

$$Q_1 = (\overline{S}_1, u_1, S_1): V \longrightarrow U \quad \text{and} \quad R_1 = (\overline{T}_1, v_1, T_1): W \longrightarrow V$$

and two cylinders:

$$q = (\overline{s}, s): Q_1 \longrightarrow Q \text{ and } \quad r = (\overline{t}, t): R_1 \longrightarrow R$$

then the composites $s \circ t$ and $\overline{s} \circ \overline{t}$ are defined and we have:

<u>(8.4.3) Lemma</u>: (i) The pair $(\overline{s} \circ \overline{t}, s \circ t)$ determines a cylinder from $Q_1 \circ R_1$ to $Q \circ R$, written $q \circ r$.

(ii) The composition $(Q, R) \rightsquigarrow Q \circ R$, $(q, r) \rightsquigarrow q \circ r$ is a bifunctor:

$$c(U,V,W): \mathrm{Cyl}\,\underline{S}(U, V) \times \mathrm{Cyl}\,\underline{S}(V, W) \longrightarrow \mathrm{Cyl}\,\underline{S}(U, W).$$

<u>Proof.</u> To show (i) we must prove that the equation (Cyl) holds for $(\overline{s} \circ \overline{t}, s \circ t)$ and the squares $Q_1 \circ R_1$ and $Q \circ R$, which means, according to the definition of $Q \circ R$, that the exterior of the following diagram commutes:

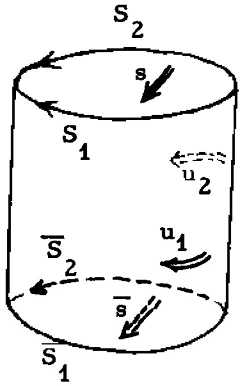
$$
\begin{tikzcd}[column sep=large]
(\overline{S}\circ\overline{T})\circ W \arrow[d, "a"'] \arrow[drr, phantom, "\mathrm{I}"] & & (\overline{S}_1\circ\overline{T}_1)\circ W \arrow[ll, "(\overline{s}\circ\overline{t})\circ W"'] \arrow[d, "a"] \\
\overline{S}\circ(\overline{T}\circ W) \arrow[ddd, "\overline{S}\circ v"'] \arrow[dr, phantom, "\mathrm{II}"] \arrow[dddr, phantom, "\mathrm{III}"] & & \overline{S}_1\circ(\overline{T}_1\circ W) \arrow[ll, "\overline{s}\circ(\overline{t}\circ W)"'] \arrow[dl, "\overline{s}\circ(\overline{T}_1\circ W)"] \arrow[ddd, "\overline{S}_1\circ v_1"] \arrow[dddl, phantom, "\mathrm{IV}"] \\
& \overline{S}\circ(\overline{T}_1\circ W) \arrow[ul, "\overline{S}\circ(\overline{t}\circ W)"'] \arrow[d, "\overline{S}\circ v_1"] & \\
& \overline{S}\circ(V\circ T_1) \arrow[dl, "\overline{S}\circ(V\circ t)"'] \arrow[d, phantom, "\mathrm{II}"] & \\
\overline{S}\circ(V\circ T) \arrow[d, "a^{-1}"'] \arrow[drr, phantom, "\mathrm{I}"] & & \overline{S}_1\circ(V\circ T_1) \arrow[ll, "\overline{s}\circ(V\circ t)"] \arrow[ul, "\overline{s}\circ(V\circ T_1)"'] \arrow[d, "a^{-1}"] \\
(\overline{S}\circ V)\circ T \arrow[ddd, "u\circ T"'] \arrow[dr, phantom, "\mathrm{II}"] \arrow[dddr, phantom, "\mathrm{IV}"] & & (\overline{S}_1\circ V)\circ T_1 \arrow[ll, "(\overline{s}\circ V)\circ t"'] \arrow[dl, "(\overline{s}\circ V)\circ T_1"] \arrow[ddd, "u_1\circ T_1"] \arrow[dddl, phantom, "\mathrm{III'}"] \\
& (\overline{S}\circ V)\circ T_1 \arrow[ul, "(\overline{S}\circ V)\circ t"'] \arrow[d, "u\circ T_1"] & \\
& (U\circ S)\circ T_1 \arrow[dl, "(U\circ S)\circ t"'] \arrow[d, phantom, "\mathrm{II}"] & \\
(U\circ S)\circ T \arrow[d, "a"'] \arrow[drr, phantom, "\mathrm{I}"] & & (U\circ S_1)\circ T_1 \arrow[ll, "(U\circ s)\circ t"] \arrow[ul, "(U\circ s)\circ T_1"'] \arrow[d, "a"] \\
U\circ(S\circ T) & & U\circ(S_1\circ T_1) \arrow[ll, "U\circ(s\circ t)"]
\end{tikzcd}
$$

<u>(8.5) Geometric representation.</u> The definitions and results of (8.1) to (8.4) admit the following geometric interpretation

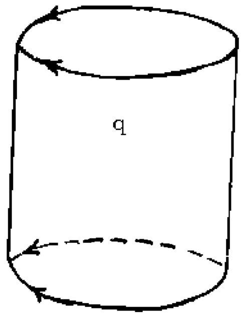
(8.5.1) A square $Q = (\overline{S}, u, S)\colon V \longrightarrow U$ can be represented by:

$$
\begin{tikzcd}
A \arrow[d, "U"'] & B \arrow[l, "S"'] \arrow[d, "V"] \arrow[dl, phantom, "\Leftarrow u"] \\
\overline{A} & \overline{B} \arrow[l, "\overline{S}"]
\end{tikzcd}
\qquad \text{abbreviated in} \qquad
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l] \arrow[d] \arrow[dl, phantom, "\Leftarrow Q"] \\
\cdot & \cdot \arrow[l]
\end{tikzcd}
$$

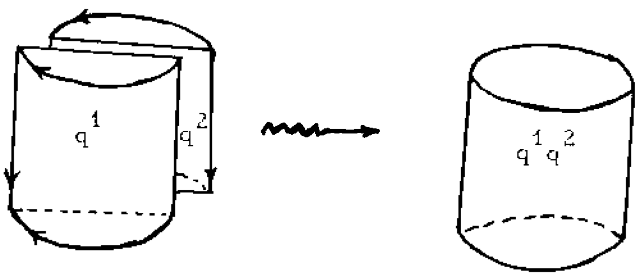
(8.5.2) A cylinder q: $(\overline{s}, s)$: $Q_2 \Longrightarrow Q_1$, by :



abbreviated in



(8.5.3) The part (i) of the Lemma (8.3.1) means that cylinders can be pasted according to the picture:

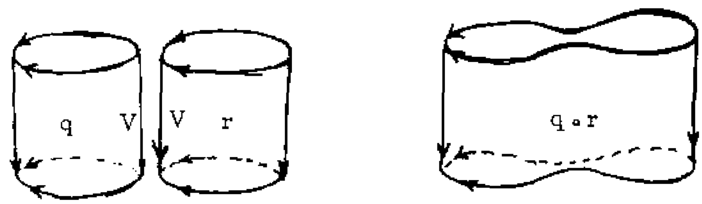


and the part (ii) essentially means that this pasting is associative.

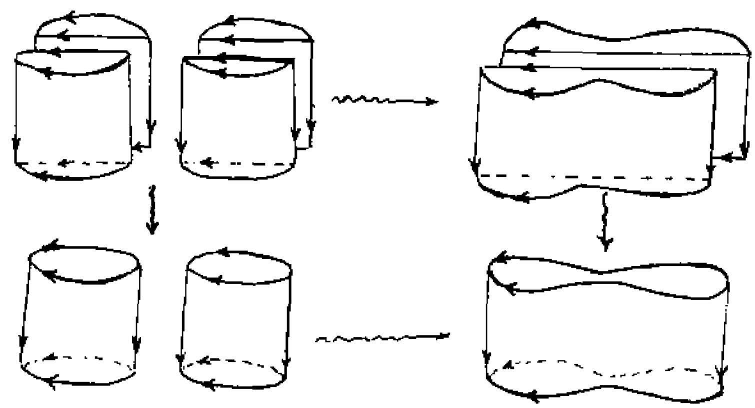
(8.5.4) The composition $(Q, R) \rightsquigarrow Q \circ R$ of (8.4.1) corresponds to:

$$\begin{tikzcd} \cdot \arrow[d, "U"'] & \cdot \arrow[l, "S"'] \arrow[d, "V"] \arrow[dl, phantom, "\overset{u}{\Leftarrow}"] \\ \cdot & \cdot \arrow[l, "\overline{S}"] \end{tikzcd} \quad \begin{tikzcd} \cdot \arrow[d, "V"'] & \cdot \arrow[l, "T"'] \arrow[d, "W"] \arrow[dl, phantom, "\overset{v}{\Leftarrow}"] \\ \cdot & \cdot \arrow[l, "\overline{T}"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \cdot \arrow[d, "U"'] & & \cdot \arrow[ll, "S \circ T"'] \arrow[d, "W"] \arrow[dll, phantom, "\overset{w}{\Leftarrow}"] \\ \cdot & & \cdot \arrow[ll, "\overline{S} \circ \overline{T}"] \end{tikzcd}$$

(8.5.5) The part (i) of Lemma (8.4.3) means that cylinders q and r can be pasted along V to get a new cylinder:



And part (ii) means essentially that the following "diagram" is commutative



In what follows we shall use frequently this geometrical representation which motivates and makes comprehensible definitions such as (8.4.1) or (8.4.2), and makes plausible results such as (8.3.1) or (8.4.3). However, the suggestions of geometry cannot replace proofs, and should be taken with a "grain of salt" because of the lack of associativity of $\circ$; thus the pairing $(q, r) \longrightarrow q \circ r$ of cylinders is not associative, and neither is their superposition (8.8). Nevertheless, to avoid diagrams such as (8.6.2) we shall replace many proofs by their geometrical analogs.

<u>(8.6) The associativity and identity isomorphisms.</u> Let $\underline{S}$ be a bicategory and $Q_i = (\overline{S}_i, u_i, S_i)\colon U_i \longrightarrow U_{i-1}$ be three squares satisfying the incidence relations depicted by:

$$
\begin{tikzcd}
A_0 \arrow[d, "U_0"'] & A_1 \arrow[l, "S_1"'] \arrow[d, "U_1"'] \arrow[dl, phantom, "\overset{u_1}{\Longleftarrow}"] & A_2 \arrow[l, "S_2"'] \arrow[d, "U_2"'] \arrow[dl, phantom, "\overset{u_2}{\Longleftarrow}"] & A_3 \arrow[l, "S_3"'] \arrow[d, "U_3"'] \arrow[dl, phantom, "\overset{u_3}{\Longleftarrow}"] \\
\overline{A}_0 & \overline{A}_1 \arrow[l, "\overline{S}_1"] & \overline{A}_2 \arrow[l, "\overline{S}_2"] & \overline{A}_3 \arrow[l, "\overline{S}_3"]
\end{tikzcd}
$$

They determine:

(i) Two, in general distinct, squares from U_3 to U_0 :

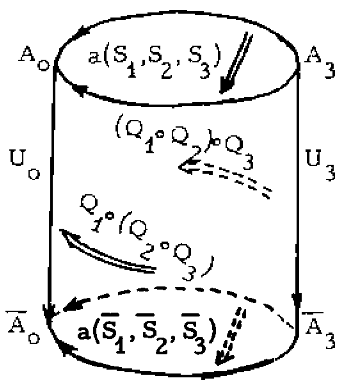
$$(Q_1 \circ Q_2) \circ Q_3 \qquad \text{and} \qquad Q_1 \circ (Q_2 \circ Q_3)$$

(ii) Two isomorphisms, in the categories $\underline{S}(A_0, A_3)$ and $\underline{S}(\overline{A}_0, \overline{A}_3)$:

$$a(A_0, A_1, A_2, A_3)(S_1, S_2, S_3)\colon (S_1 \circ S_2) \circ S_3 \xrightarrow{\sim} S_1 \circ (S_2 \circ S_3)$$

$$a(\overline{A}_0, \overline{A}_1, \overline{A}_2, \overline{A}_3)(\overline{S}_1, \overline{S}_2, \overline{S}_3)\colon (\overline{S}_1 \circ \overline{S}_2) \circ \overline{S}_3 \xrightarrow{\sim} \overline{S}_1 \circ (\overline{S}_2 \circ \overline{S}_3)$$

satisfying the incidence relations depicted by:



<u>(8.6.1) Lemma.</u> With the previous notations:

(i) The pair $(a(A_o, A_1, A_2, A_3)(S_1, S_2, S_3)\ ,\ a(\overline{A}_o, \overline{A}_1, \overline{A}_2, \overline{A}_3)(\overline{S}_1, \overline{S}_2, \overline{S}_3))$ defines a cylinder:

$$a(U_o, U_1, U_2, U_3)(Q_1, Q_2, Q_3) : (Q_1 \circ Q_2) \circ Q_3 \longrightarrow Q_1 \circ (Q_2 \circ Q_3).$$

(ii) For U_i fixed, the family $a(U_o, U_1, U_2, U_3)(Q_1, Q_2, Q_3)$ is functorial in the Q_j's, that is, determines a natural transformation:

$$a(U_o, U_1, U_2, U_3) : c(U_o, U_2, U_3)\,(c(U_o, U_1, U_2) \times \mathrm{Id}) \longrightarrow c(U_o, U_1, U_3)\,(\mathrm{Id} \times c(U_1, U_2, U_3))$$

between the composite functors bounding the diagram:

$$\begin{tikzcd}
\mathrm{Cyl}\,\underline{S}(U_o, U_1) \times \mathrm{Cyl}\,\underline{S}(U_1, U_2) \times \mathrm{Cyl}\,\underline{S}(U_2, U_3) \arrow[r, "\mathrm{Id} \times c(U_1, U_2, U_3)"] \arrow[d, "c(U_o, U_1, U_2) \times \mathrm{Id}"'] & \mathrm{Cyl}\,\underline{S}(U_o, U_1) \times \mathrm{Cyl}\,\underline{S}(U_1, U_3) \arrow[d, "c(U_o, U_1, U_3)"] \\
\mathrm{Cyl}\,\underline{S}(U_o, U_2) \times \mathrm{Cyl}\,\underline{S}(U_2, U_3) \arrow[r, "c(U_o, U_2, U_3)"'] & \mathrm{Cyl}\,\underline{S}(U_o, U_3)
\end{tikzcd}$$

(iii) The natural transformation $a(U_o, U_1, U_2, U_3)$ are isomorphisms.

(iv) The $a(U_o, U_1, U_2, U_3)$, for the U_i's ranging in the arrows of $\underline{S}$, satisfy the associativity coherence (A.C).

Proof. Remembering the definition of u/v and $Q \circ R$ in (8.4.1) the equation (Cyl) is here equivalent to the commutativity of the exterior of diagram (8.6.2) below.

But the regions called I commute because the natural transformations a's of $\underline{S}$ satisfy (A.C) of §1; the II's by naturality of the a's ; the III's by definition of u/v; IV is the image of the diagram defining u_2/u_3 by the functor $S_1 \circ (\ )$ and similarly IV' "is" $(u_1/u_2) \circ S_3$. This proves (i). Then (ii), (iv) follow immediately from the similar statements which hold in $\underline{S}$ for $a(A_o, A_1, A_2, A_3)$ and $a(\overline{A}_o, \overline{A}_1, \overline{A}_2, \overline{A}_3)$. Finally (iii) will hold if we know that the pair:

$([a(A_o, A_1, A_2, A_3)(S_1, S_2, S_3)]^{-1}, [a(\overline{A}_o, \overline{A}_1, \overline{A}_2, \overline{A}_3)(\overline{S}_1, \overline{S}_2, \overline{S}_3)]^{-1})$ defines a cylinder from $Q_1 \circ (Q_2 \circ Q_3)$ to $(Q_1 \circ Q_2) \circ Q_3$, but this follows from (i) by conjugation.

$$
\begin{tikzcd}[column sep=huge]
(\bar S_1\circ(\bar S_2\circ \bar S_3))\circ U_3 \arrow[d, "a"] \arrow[ddddddddd, bend right=60, "u_1|(u_2|u_3)"'] \arrow[dr, phantom, "\mathrm{I}"] & ((\bar S_1\circ \bar S_2)\circ \bar S_3)\circ U_3 \arrow[l, "a\circ U_3"'] \arrow[d, "a"] \arrow[ddddddddd, bend left=60, "(u_1|u_2)|u_3"] \\
\bar S_1\circ((\bar S_2\circ \bar S_3)\circ U_3) \arrow[d, "\bar S_1\circ a"] \arrow[ddddd, bend right=50, "\bar S_1\circ(u_2|u_3)"'] \arrow[dr, phantom, "\mathrm{II}"] & (\bar S_1\circ \bar S_2)\circ(\bar S_3\circ U_3) \arrow[dl, "a"'] \arrow[d, "(\bar S_1\circ\bar S_2)\circ u_3"] \\
\bar S_1\circ(\bar S_2\circ(\bar S_3\circ U_3)) \arrow[d, "\bar S_1\circ(\bar S_2\circ u_3)"] \arrow[dr, phantom, "\mathrm{I}"] & (\bar S_1\circ \bar S_2)\circ(U_2\circ S_3) \arrow[dl, "a"'] \arrow[d, "a^{-1}"] \\
\bar S_1\circ(\bar S_2\circ(U_2\circ S_3)) \arrow[d, "\bar S_1\circ a^{-1}"] \arrow[dr, phantom, "\mathrm{II}"] & ((\bar S_1\circ \bar S_2)\circ U_2)\circ S_3 \arrow[d, "a\circ S_3"] \arrow[ddddd, bend left=50, "(u_1|u_2)\circ S_3"] \\
\bar S_1\circ((\bar S_2\circ U_2)\circ S_3) \arrow[d, "\bar S_1\circ(u_2\circ S_3)"] \arrow[dr, phantom, "\mathrm{I}"] & (\bar S_1\circ(\bar S_2\circ U_2))\circ S_3 \arrow[l, "a"'] \arrow[d, "(\bar S_1\circ u_2)\circ S_3"] \\
\bar S_1\circ((U_1\circ S_2)\circ S_3) \arrow[d, "\bar S_1\circ a"] \arrow[dr, phantom, "\mathrm{II}"] & (\bar S_1\circ(U_1\circ S_2))\circ S_3 \arrow[l, "a"] \arrow[d, "a^{-1}\circ S_3"] \\
\bar S_1\circ(U_1\circ(S_2\circ S_3)) \arrow[d, "a^{-1}"] \arrow[dr, phantom, "\mathrm{I}"] & ((\bar S_1\circ U_1)\circ S_2)\circ S_3 \arrow[dl, "a"'] \arrow[d, "(u_1\circ S_2)\circ S_3"] \\
(\bar S_1\circ U_1)\circ(S_2\circ S_3) \arrow[d, "u_1\circ(S_2\circ S_3)"] \arrow[dr, phantom, "\mathrm{II}"] & ((U_0\circ S_1)\circ S_2)\circ S_3 \arrow[dl, "a"'] \arrow[d, "a\circ S_3"] \\
(U_0\circ S_1)\circ(S_2\circ S_3) \arrow[d, "a"] \arrow[dr, phantom, "\mathrm{I}"] & (U_0\circ(S_1\circ S_2))\circ S_3 \arrow[d, "a"] \\
U_0\circ(S_1\circ(S_2\circ S_3)) & U_0\circ((S_1\circ S_2)\circ S_3) \arrow[l, "U_0\circ a"]
\end{tikzcd}
$$

Diagram (8. 6. 2)

(8.6.3). Let $U: A \longrightarrow \bar{A}$ be an arrow of $\underset{\sim}{S}$. We define a square I_U from U to U by $I_U = (I_{\bar{A}}, k_U, I_A): U \longrightarrow U$, where k_U is the unique arrow of the category $\underset{\sim}{S}(\bar{A}, A)$ making commutative the diagram

$$\begin{tikzcd}
U \circ I_A \arrow[dr, "\sim", "r_U"'] & & I_{\bar{A}} \circ U \arrow[ll, "k_U"'] \arrow[dl, "\sim"', "\ell_U"] \\
& U &
\end{tikzcd}$$

(8.6.4). Let $Q = (\bar{S}, u, S): V \longrightarrow U$ be a square. According to the picture

$$\begin{tikzcd}
A \arrow[d, dash, "U"'] & A \arrow[l, "I_A"'] \arrow[d, dash, "U"] & B \arrow[l, "S"'] \arrow[d, dash, "V"] & B \arrow[l, "I_B"'] \arrow[d, dash, "V"] \\
\bar{A} & \bar{A} \arrow[l, "I_{\bar{A}}"] \arrow[ul, phantom, "I_U \; k_U \Leftarrow"] & \bar{B} \arrow[l, "\bar{S}"] \arrow[ul, phantom, "Q \; \overset{u}{\Leftarrow}"] & \bar{B} \arrow[l, "I_{\bar{B}}"] \arrow[ul, phantom, "I_V \; \overset{k_V}{\Leftarrow}"]
\end{tikzcd}$$

the square Q determines:

(i) Two squares $I_U \circ Q$ and $Q \circ I_V$ from V to U.

(ii) Four isomorphisms

$\ell(A, B)(S): I_A \circ S \xrightarrow{\sim} S$ and $r(A, B)(S): S \circ I_B \xrightarrow{\sim} S$ in $\underset{\sim}{S}(A, B)$

$\ell(\bar{A}, \bar{B})(\bar{S}): I_{\bar{A}} \circ \bar{S} \xrightarrow{\sim} \bar{S}$ and $r(\bar{A}, \bar{B})(\bar{S}): \bar{S} \circ I_{\bar{B}} \xrightarrow{\sim} \bar{S}$ in $\underset{\sim}{S}(\bar{A}, \bar{B})$.

<u>(8.6.5) Lemma.</u> With the previous notations:

(i) The pairs $(\ell(A, B)(S), \ell(\bar{A}, \bar{B})(\bar{S}))$ and $(r(A, B)(S), r(\bar{A}, \bar{B})(\bar{S}))$ define two cylinders:

$\ell(U, V)(Q): I_U \circ Q \longrightarrow Q$ and $r(U, V)(Q): Q \circ I_V \longrightarrow Q$.

(ii) For U and V fixed, the families $\ell(U,V)(Q)$ and $r(U,V)(Q)$ are functorial in Q, that is, determine natural transformations $\ell(U,V)$ and $r(U,V)$ as shown in the diagram:

$$\begin{tikzcd}
\mathbf{1} \times \mathrm{Cyl}\,\underline{S}(U,V) \arrow[rr, "I_U \times \mathrm{Id}"] \arrow[dr, "\sim"'] \arrow[rr, phantom, shift right=4ex, "\swarrow \ell(U,V)"] & & \mathrm{Cyl}\,\underline{S}(U,U) \times \mathrm{Cyl}\,\underline{S}(U,V) \arrow[dl, "c(U,U,V)"] \\
& \mathrm{Cyl}\,\underline{S}(U,V) &
\end{tikzcd}$$

and the obvious analogue for $r(U,V)$.

(iii) The $r(U,V)$ and $\ell(U,V)$ are isomorphisms.

(iv) The system of natural transformations $a(U_o, U_1, U_2, U_3)$, $\ell(U,V)$ and $r(U,V)$ satisfy the coherence axiom (I. C) of 1.

The proof, completely similar to that of (8. 6. 1) except for smaller diagrams, is left to the reader.

(8. 6. 6). If $\underline{S}$ is a bicategory we define $\mathrm{Ob}(\mathrm{Cyl}\,\underline{S})$ to be the set of arrows of $\underline{S}$.

Putting together the Lemmas (8. 3. 1), (8. 4. 3), (8. 6. 1), and (8. 6. 5), we obtain:

<u>(8. 6. 7) Theorem:</u> The data: $\mathrm{Ob}(\mathrm{Cyl}\,\underline{S})$, $\mathrm{Cyl}\,\underline{S}(U,V)$, $c(U,V,W)$, I_U, $a(U_o, U_1, U_2, U_3)$, $\ell(U,V)$, $r(U,V)$ determine a bicategory $\mathrm{Cyl}\,\underline{S}$ called <u>bicategory of cylinders</u> of $\underline{S}$

(8.6.8) Remark: In all the steps of the passage from $\underline{S}$ to Cyl $\underline{S}$ there appears a shift of dimension in the definitions as well as the coherence properties involved: Thus to define the binary operations $(Q, R) \rightsquigarrow Q \circ R$ in Cyl $\underline{S}$, we need the existence in $\underline{S}$ of the ternary associativity isomorphisms $a(S, T, U)$. To define a in Cyl $\underline{S}$, we need to know that a is coherent up to the order 4 (see proof of (8.6.1)(i)). This suggests the conditions to require in higher dimensional cases (*).

(8.7) The top and bottom homomorphisms Let $\underline{S}$ be a bicategory, we define

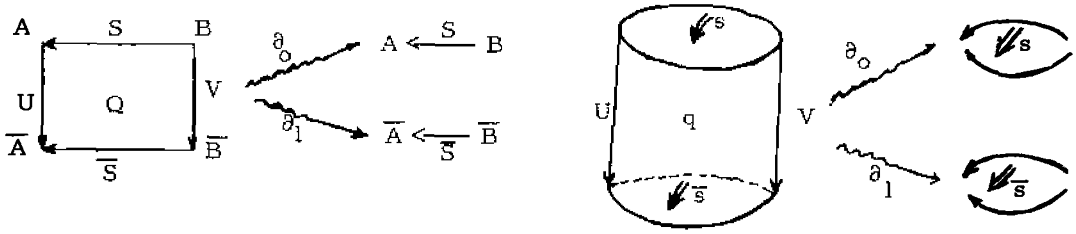
(i) Two maps ∂_i: Ob Cyl $\underline{S} \longrightarrow$ Ob $\underline{S}$, $i = 0, 1$, by

$$\partial_o U = A \ , \ \partial_1 U = \overline{A} \quad \text{for each arrow } U: A \longrightarrow \overline{A} \text{ of } \underline{S}$$

(ii) Two families of functors, indexed by pairs (U, V) of arrows of $\underline{S}$

$$\partial_i(U, V): \mathrm{Cyl}\,\underline{S}(U, V) \longrightarrow \underline{S}(\partial_i U, \partial_i V) \ , \quad Q \rightsquigarrow \partial_i Q \ , \quad q \rightsquigarrow \partial_i q \quad \text{by:}$$

$$\partial_o Q = S \ , \ \partial_1 Q = \overline{S} \ , \quad \partial_o q = s \ , \ \partial_1 q = \overline{s} \quad \text{for } Q \text{ and } q \text{ as below:}$$



It follows from the definitions that the ∂_i commute strictly with the $\circ$'s and I's and define thus strict homomorphisms called respectively top and bottom

$$\partial_o \text{ and } \partial_1 : \ \mathrm{Cyl}\,\underline{S} \longrightarrow \underline{S} \ .$$

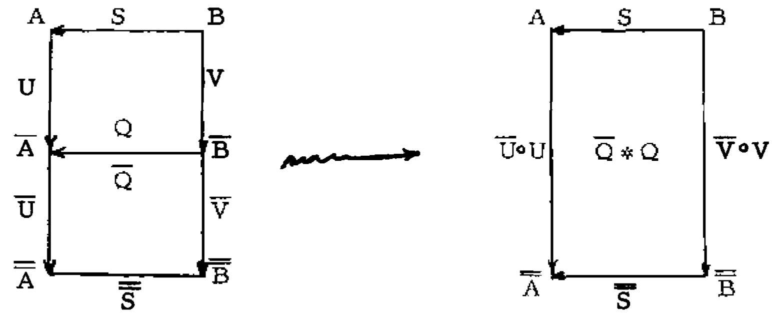
(*) Compare with $\Pi_i(\Omega X) \cong \Pi_{i+1}(X)$ in homotopy theory (see footnote p. 57).

<u>(8. 8) Superposition morphism.</u> Let $Q = (\bar{S}, u, S): V \longrightarrow U$ and $\bar{Q} = (\bar{\bar{S}}, \bar{u}, \bar{S}): \bar{V} \longrightarrow \bar{U}$ be two squares such that $\partial_o \bar{Q} = \bar{S} = \partial_1 Q$, (see picture (8.8.1) below). In the category $\underline{S}(\bar{\bar{A}}, B)$ we have the composite map, written $\bar{u}{*}u$:

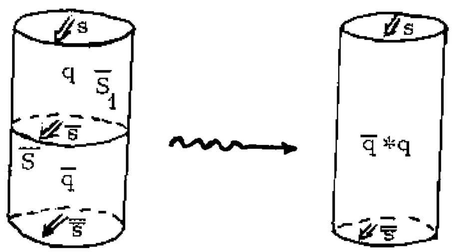
$$(\bar{u}\circ u)\circ S \xleftarrow{a^{-1}} \bar{U}\circ(U\circ S) \xleftarrow{\bar{U}\circ u} \bar{U}\circ(\bar{S}\circ V) \xleftarrow{a} (\bar{U}\circ\bar{S})\circ V \xleftarrow{\bar{u}\circ V} (\bar{\bar{S}}\circ\bar{V})\circ V \xleftarrow{a^{-1}} \bar{\bar{S}}\circ(\bar{V}\circ V)$$

which determines a square $\bar{Q}*Q = (\bar{\bar{S}}, \bar{u}{*}u, S): \bar{V}\circ V \longrightarrow \bar{U}\circ U$, called superposition of Q on $\bar{Q}$ (along $\bar{S}$).

(8.8.1)



Similarly, let $q = (\bar{s}, s): Q_1 \longrightarrow Q$ and $\bar{q} = (\bar{\bar{s}}, s): \bar{Q}_1 \longrightarrow \bar{Q}$ be two cylinders such that $\partial_o \bar{q} = \bar{s} = \partial_1 q$, as depicted below:



Using a diagram similar to (8. 6. 2) one can prove:

<u>(8. 8. 2) Lemma.</u> The pair $(\bar{\bar{s}}, s)$ defines a cylinder from $\bar{Q}_1 * Q_1$ to $\bar{Q} * Q$ written $\bar{q}\,{}_{\bar{s}}\,q$ and called superposition of q on $\bar{q}$ (along $\bar{s}$).

(8.8.3) Proposition. The superposition of cylinders and squares define a strict homomorphism

$$* : \ \mathrm{Cyl}\,\underline{S} \prod_{(\partial_o, \partial_1)} \mathrm{Cyl}\,\underline{S} \longrightarrow \mathrm{Cyl}\,\underline{S}\,.$$

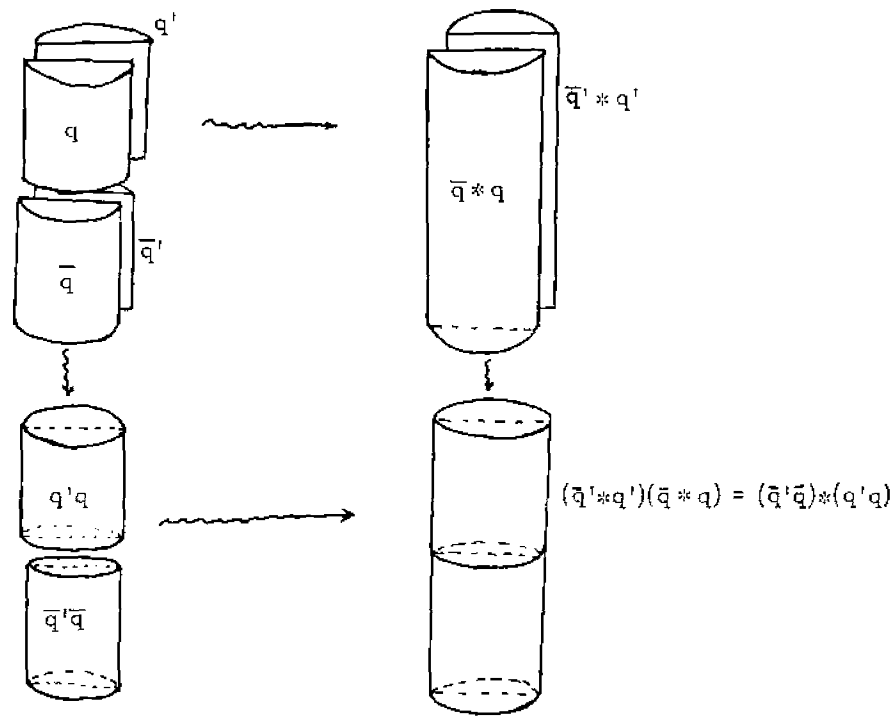
We have to show that the equations (i) and (ii) below, which state that * commutes with the compositions, hold (and similar equations for the identities, left to the reader.)

(i) $(\bar{q}'*q')(\bar{q}*q) = (\bar{q}'\bar{q})*(q'q)$

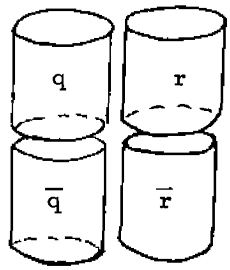
(ii) $(\bar{q}*q)\circ(\bar{r}*r) = (\bar{q}\circ\bar{r})*(q\circ r)$

whenever both sides are defined.

The equation (i) corresponds to the commutativity of:



Similarly (ii) means that the pasting of the four cylinders



does not depend on the order. Proofs are omitted.

(B.1) Bénabou, J., Catégories avec multiplication, C.R. Acad. Sci., Paris 256 (1963), 1887-1890.

(B.2) __________, Algèbre élémentaire dans les catégories avec multiplication, C.R. Acad. Sci., Paris 258 (1964), 771-774.

(B.3) __________, Catégories relatives, C.R. Acad. Sci., Paris 260 (1965), 3824-3827.

(B.4) __________, Structures algébriques dans les catégories, to appear.

(E) Epstein, D., Steenrod operations in homological algebra, Invent. Math. 1 (1966), 152-208.

(EL) Ehresmann, C., Catégories structurées, Ann. Sci. Ecole Norm. Sup. 80 (1963), 349-425.

(EH) Eckmann-Hilton, Group-like structures in categories, Math. Ann. 145 (1962), 227-255.

(E.K) Eilenberg-Kelly, Closed categories, Proceedings of the La Jolla Conference on Categories, Springer 1967, 424-562.

(E.M) Eilenberg-Moore, Adjoint functors and triples, Illinois Journal of Mathematics, 9 (1965), 381-398.

(G) Godement, R., Théorie des faisceaux, Hermann Paris 1957.

(Gr) Grothendieck, A., Catégories fibrées et descente, Séminaire de Géometrie Algébrique 1961. Exposé VI.

(M) MacLane, S., Natural associativity and commutativity, Rice Univ. Studies 49 (1963), 28-46.

(Y) Yoneda, N., On Ext and exact sequences, J. Fac. Sci. Univ. Tokyo, Sec. I, 7 (1954), 193-227.