Quantaloid-enriched categories: an introduction (and some results)

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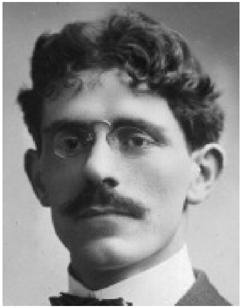
Université du Littoral, France

Fès, February 2026

Fréchet: metric spaces (1906)



Maurice Fréchet
(1878 – 1973)

SUR QUELQUES POINTS DU CALCUL FONCTIONNEL;

Par M. **Maurice Fréchet** (Paris) *).

*) Thèse présentée à la Faculté des Sciences de Paris pour obtenir le grade de Docteur ès Sciences.

Adunanza del 22 aprile 1906.

49. *Introduction de l'écart.* — Lorsque nous appliquerons les résultats généraux de la Première Partie à des exemples concrets, nous reconnaîtrons d'abord que, dans chaque cas, on peut faire correspondre à tout couple d'éléments A, B un nombre $(A, B) \geq 0$, que nous appellerons *l'écart des deux éléments* et qui jouit des deux propriétés suivantes: *a*) L'écart (A, B) n'est nul que si A et B sont identiques. *b*) Si A, B, C, sont trois éléments quelconques, on a toujours $(A, B) \leq (A, C) + (C, B)$.

[...]

discerner si deux d'entre eux sont ou non identiques et tels, de plus, qu'à deux quelconques d'entre eux A, B, on puisse faire correspondre un nombre $(A, B) = (B, A) \geq 0$

Il fallait d'abord voir comment transformer les énoncés des théorèmes pour qu'ils conservent un sens dans le cas général. Il fallait ensuite, soit transcrire les démonstrations dans un langage plus général, soit, lorsque cela n'était pas possible, donner des démonstrations nouvelles et plus générales. Il s'est trouvé que les démonstrations que nous avons ainsi obtenues sont souvent aussi simples, et quelquefois même plus simples, que les démonstrations particulières qu'elles remplaçaient. Cela tient sans doute à ce que la position de la question obligeait à ne faire usage que de ses particularités vraiment essentielles.

Lawvere: metric spaces are categories (1973)

Bill Lawvere
(1937-2023)

METRIC SPACES, GENERALIZED LOGIC,
AND CLOSED CATEGORIES

(*Conferenza tenuta il 30 marzo 1973*) *

By taking account of a certain natural generalization of category theory within itself, namely the consideration of strong categories whose hom-functors take their values in a given « closed category » $\mathcal{V}$ (not necessarily in the category $\mathcal{S}$ of abstract sets), we will show below that it is possible to regard a metric space as a (strong) category and that moreover by specializing the constructions and theorems of general category theory we can deduce a large part of *general* metric space theory.

Quantale-enriched categories

A partially ordered set $(X, \leq)$ is

a binary relation "$\leq$" on X

such that, for all $x, y, z \in X$,

if $x \leq y$ and $y \leq z$ then $x \leq z$,

$x \leq x$,

if $x \leq y$ and $y \leq x$ then $x = y$.

A metric space (X, d) is

a function $d \colon X \times X \to \mathbb{R}$

such that, for all $x, y, z \in X$,

$d(x, y) \geq 0$,

$d(x, y) + d(y, z) \geq d(x, z)$,

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Both these ordered monoids are residuated complete lattices:

$$\frac{u \wedge v \quad \leq \quad w}{u \quad \leq \quad (v \Rightarrow w)}$$

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That is to say, $(\{0,1\}, \leq, \wedge, 1)$ and $([0,\infty], \geq, +, 0)$ are examples of **quantales**; and (partially) ordered sets, resp. (generalized) metric spaces, are **quantale-enriched categories**.

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$$a \circ (\bigvee_i b_i) = \bigvee_i (a \circ b_i) \quad \text{and} \quad (\bigvee_i a_i) \circ b = \bigvee_i (a_i \circ b)$$

for all $a, b, (a_i)_i, (b_i)_i$ in Q. (So Q is a monoid in Sup, and a *very particular* monoidal category.)

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A Q-**category** $\mathbb{A}$ is

a function $\mathbb{A} \colon \mathbb{A}_0 \times \mathbb{A}_0 \to Q \colon (x, y) \mapsto \mathbb{A}(x, y)$

assigning a "hom" to each pair of "objects", such that

$\mathbb{A}(x, y) \circ \mathbb{A}(y, z) \leq \mathbb{A}(x, z)$ for any x, y, z,

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(Small) Q-categories and Q-functors form a (large) category $\mathsf{Cat}(Q)$ in the obvious way.

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Examples of $\mathsf{Cat}(Q)$:

Quantale-enriched categories

Examples of Cat(Q):

Cat$(\{0,1\},\vee,\wedge,1)$: ordered sets and monotone maps.

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The sup-lattice $([0,1],\bigvee)$ carries several (non-isomorphic) closed monoid structures, e.g.

$$x * y = x \cdot y \quad \text{or} \quad x * y = \min\{x,y\} \quad \text{or} \quad x * y = \max\{x+y-1,0\}.$$

A **left-continuous t-norm** is exactly a commutative, integral quantale $([0,1],\bigvee,*,1)$; these are used in many-valued logic.

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$\mathbb{A}(x,y)(t)$ is the probability that the distance from x to y is less than t.

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There are many more examples, but for some of these we must generalize our setup...

Bénabou: Bicategories (1967)

Jean Bénabou
(1932-2022)

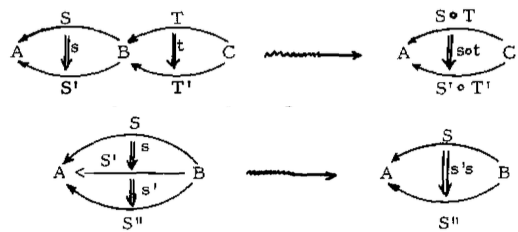
Reports of the
Midwest Category Seminar

1967

INTRODUCTION TO BICATEGORIES

Jean Bénabou

(1.1) Local definition. A bicategory S is determined by the following



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(5.5.1) Definition. Let $\overline{S}$ be a bicategory. A polyad in $\overline{S}$ (or $\overline{S}$-polyad) is a morphism of bicategories $\Phi = (F, \varphi)$: $S \longrightarrow \overline{S}$ where S is locally punctual. The set Ob S is called set of objects or indices of the polyad. (The monads are obtained when Ob S = $\mathbb{1}$, hence the name of polyad.)

If in the data of S all the S(A, B) are partially ordered sets, identified to categories, the coherence conditions are automatically satisfied,

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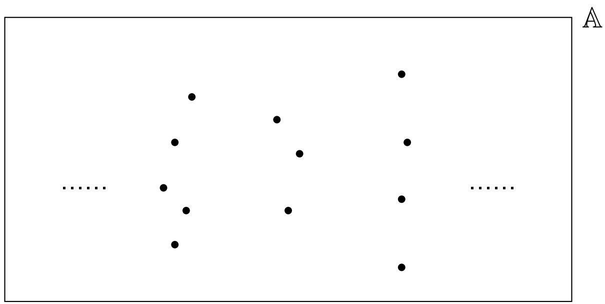

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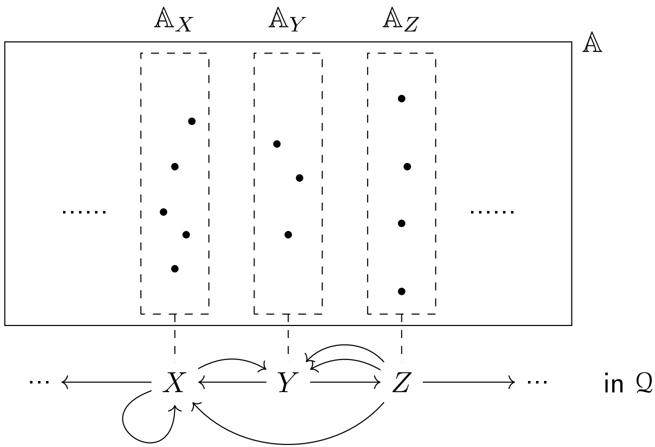

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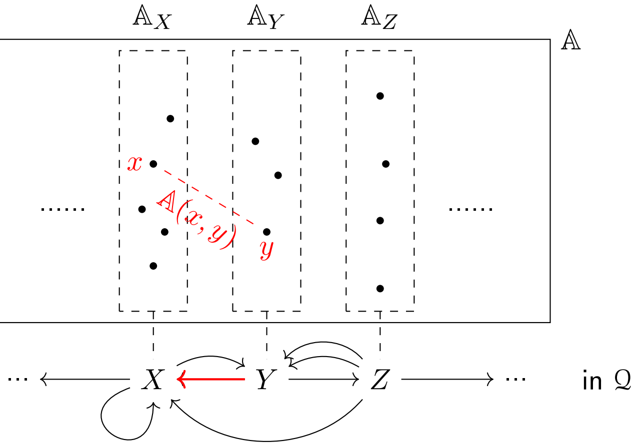

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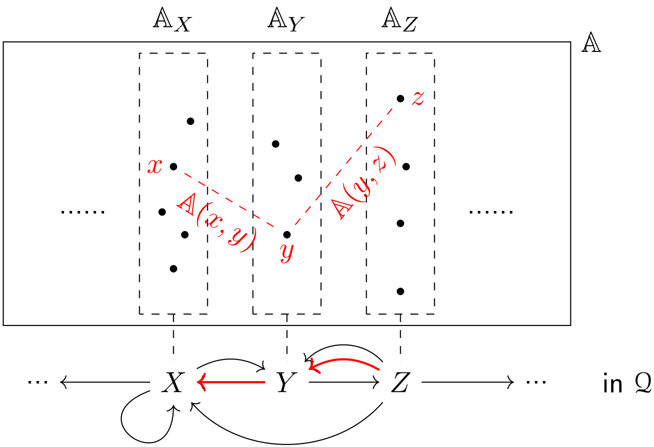

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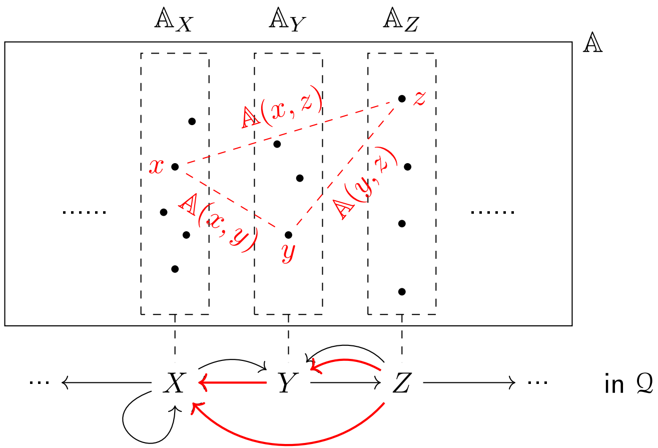

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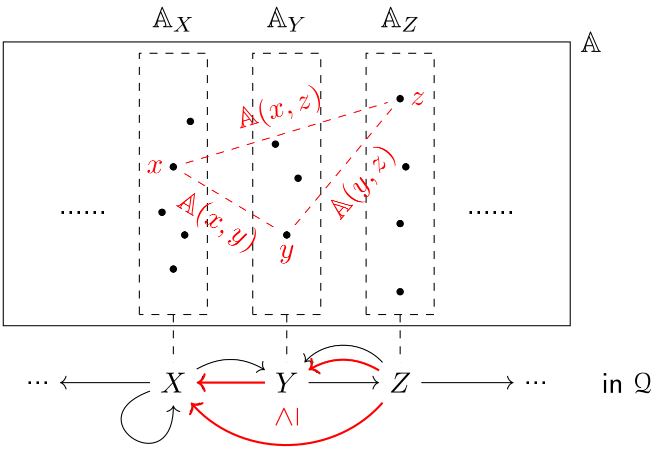

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Universal constructions produce quantaloids, which underly new examples!

Intermezzo: diagonals

In any quantaloid $\mathcal{Q}$, say that d is a **diagonal** from f to g

$$\begin{tikzcd}
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$$\mathcal{Q} \to \mathcal{D}(\mathcal{Q})\colon \left(A \xrightarrow{f} B \right) \mapsto \left(\begin{tikzcd} A \arrow[d, "1_A"'] \arrow[dr, "f"] & B \arrow[d, "1_B"] \\ A & B \end{tikzcd} \right)$$

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Even for a quantale Q, the diagonal construction produces a genuine quantaloid $\mathcal{D}(Q)$!

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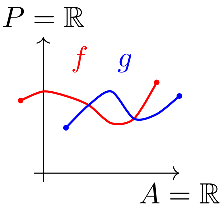

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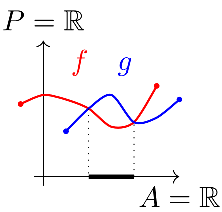

It is natural to compute the "extent to which f is smaller than g":

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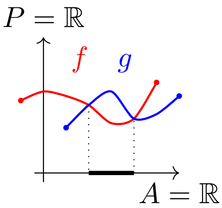

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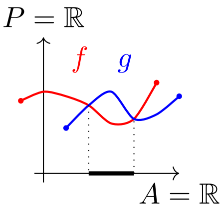

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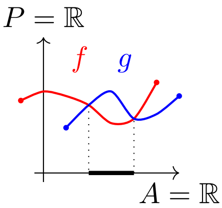

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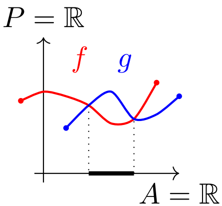

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$\mathsf{Cat}(\mathcal{D}(L, \bigvee, \wedge, \top))$: orders of "local" elements over a locale L,
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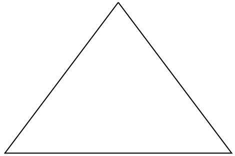
$\mathsf{Cat}(\mathcal{D}(\Delta, \bigwedge, +, 0))$: partial probabilistic metric spaces.

It takes three to tango

To "do category theory", we need:

categories



functors ... and a third ingredient!

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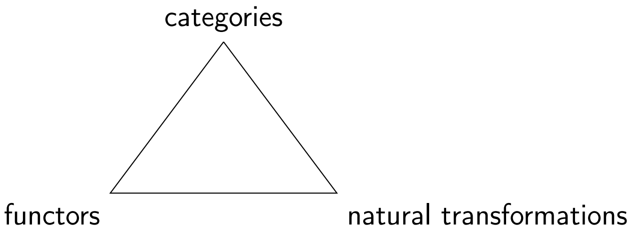

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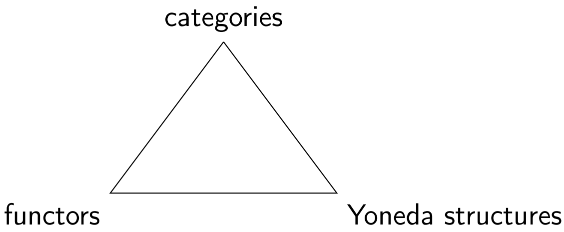

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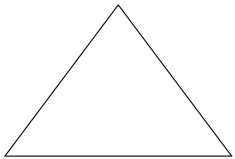

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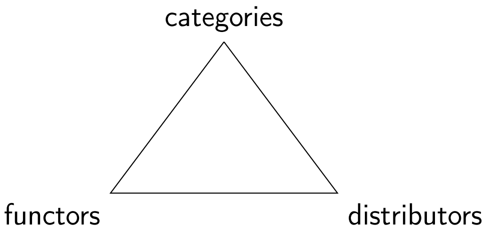

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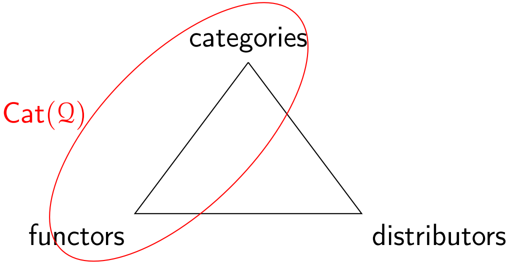

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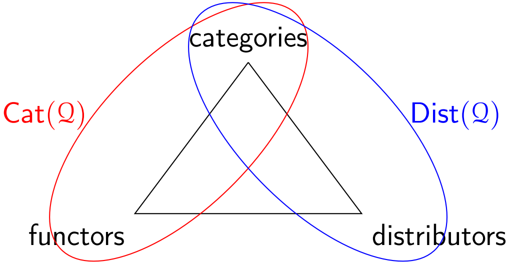

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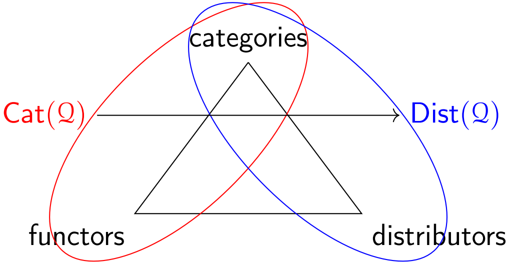

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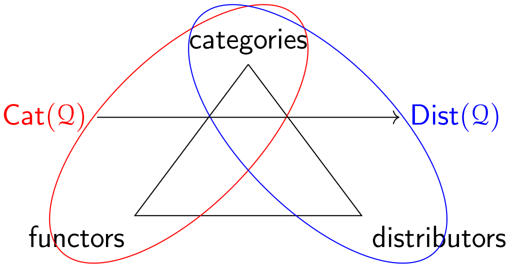

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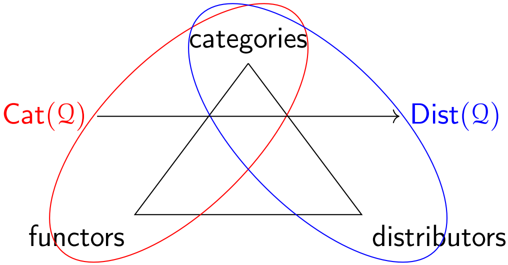

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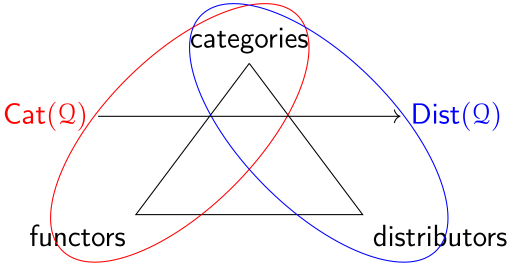

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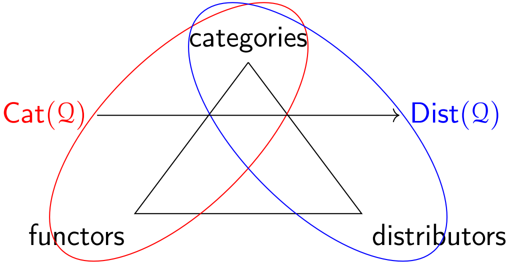

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For rings, distributors are (bi)modules.

Advantage: distributors are “actions”, and this works perfectly in “non-commutative” contexts too.

Bénabou: Distributors (1973)

Jean Bénabou
(1932-2022)

LES DISTRIBUTEURS
d'après le cours de "Questions spéciales de mathématique"

par
J. BENABOU *rédigé par Jean-Roger ROISIN*

Rapport n^o 33, janvier 1973
Séminaires de Mathématique Pure

Bâtiment Sc. I, Avenue du Cyclotron, 2 1348 Louvain-La-Neuve

Nous supposerons maintenant que $\mathcal{V}$ est un cosmos c'est-à-dire une catégorie multiplicative symétrique fermée complète à gauche et à droite.

Une flèche de $\mathcal{A}$ vers $\mathcal{B}$, appelée un distributeur, est un $\mathcal{V}$-bifoncteur vers $\mathcal{V}$, contravariant en $\mathcal{B}$ et covariant en $\mathcal{A}$.

4.3. *Proposition.*

Dist($\mathcal{V}$) est une bicatégorie fermée.

Distributors

A $\mathcal{Q}$-**distributor** $\Phi\colon \mathbb{A} \begin{array}{c}\circ\\[-1.5ex]\longrightarrow\end{array} \mathbb{B}$ between two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ is

a matrix $\Phi\colon \mathbb{B}_0 \times \mathbb{A}_0 \to \mathcal{Q}_1\colon (b,a) \mapsto \Big(\Phi(b,a)\colon ta \to tb\Big)$ of $\mathcal{Q}$-arrows

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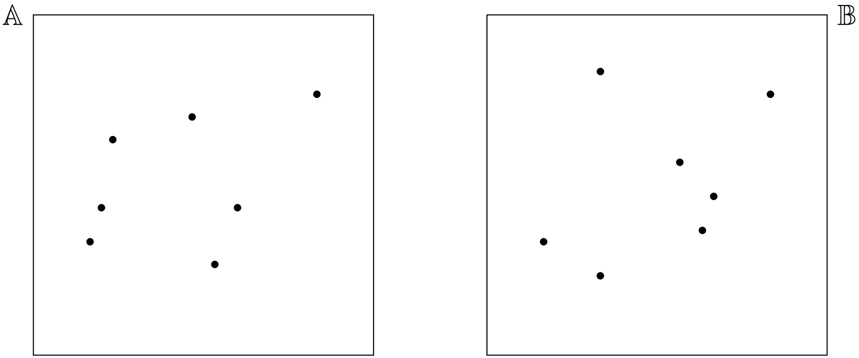

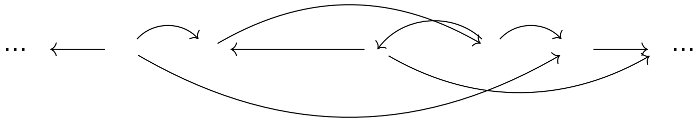

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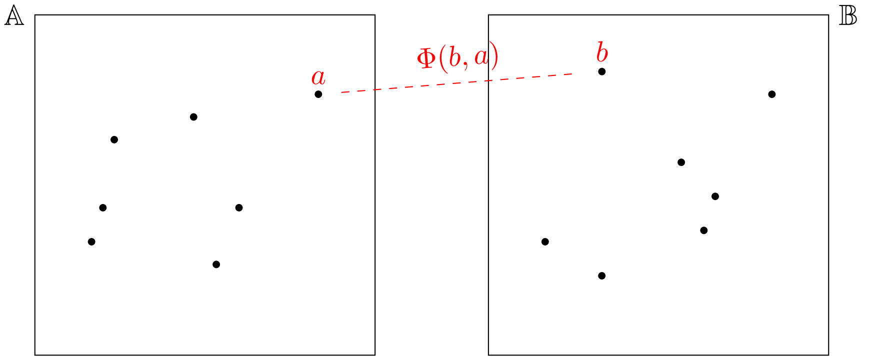

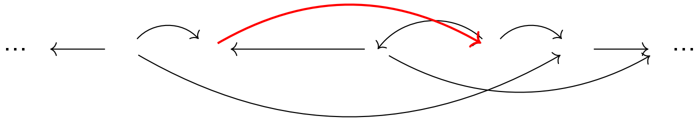

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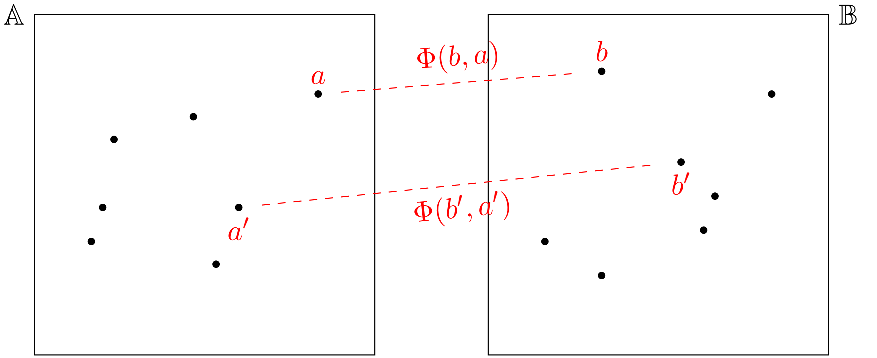

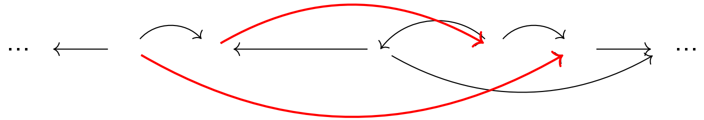

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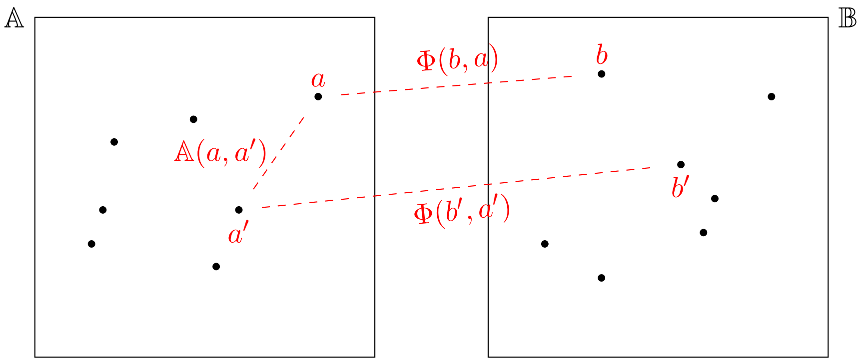

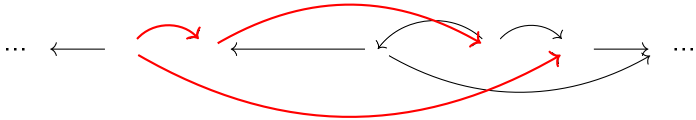

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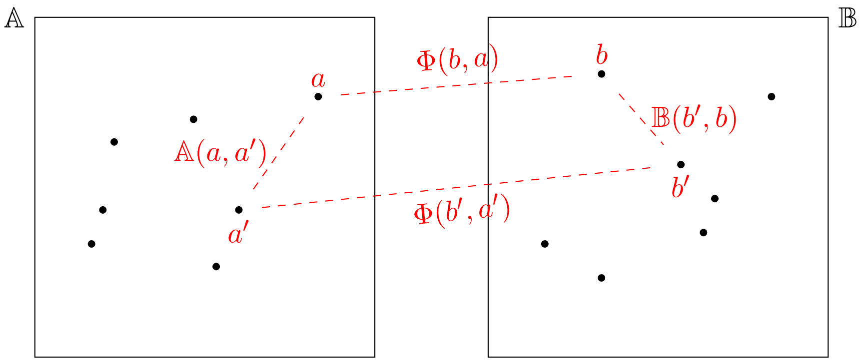

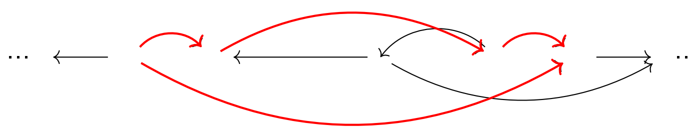

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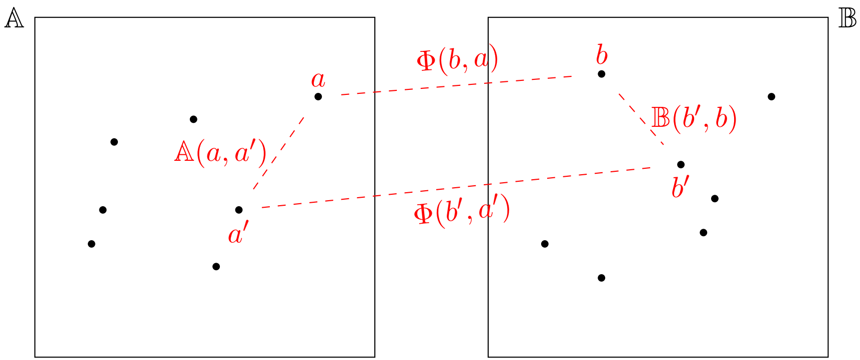

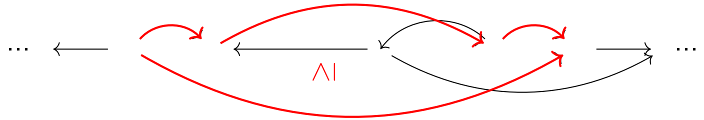

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Each object X of $\mathcal{Q}$ determines a trivial one-object $\mathcal{Q}$-category,

$$\mathbb{1}_X \text{ is defined by } (\mathbb{1}_X)_0 = \{*\},\ t* = X,\ \mathbb{1}_X(*,*) = 1_X,$$

and each morphism $f\colon X \to Y$ in $\mathcal{Q}$ determines a one-element distributor,

$$(f)\colon \mathbb{1}_X \xrightarrow{\circ} \mathbb{1}_Y \text{ is defined by } (f)(*,*) = f.$$

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Two $\mathcal{Q}$-distributors $\Phi\colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ and $\Psi\colon \mathbb{B} \xrightarrow{\circ} \mathbb{C}$ compose by the "matrix" formula

$$(\Psi \circ \Phi)(c,a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c,b) \circ \Phi(b,a)$$

and (with elementwise order) this makes for a **quantaloid** $\mathsf{Dist}(\mathcal{Q})$.

Thus we have access to **all 2-categorical notions in** $\mathsf{Dist}(\mathcal{Q})$ (such as adjoints, ...).

There is a full embedding of quantaloids

$$\mathcal{Q} \to \mathsf{Dist}(\mathcal{Q})\colon \Big(f\colon X \to Y\Big) \mapsto \Big((f)\colon \mathbb{1}_X \xrightarrow{\circ} \mathbb{1}_Y\Big)$$

which displays $\mathsf{Dist}(\mathcal{Q})$ as the universal lax (co)completion of $\mathcal{Q}$.

Distributors

A $\mathcal{Q}$-**distributor** $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$ between two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ is

a matrix $\Phi\colon \mathbb{B}_0 \times \mathbb{A}_0 \to \mathcal{Q}_1\colon (b,a) \mapsto \Big(\Phi(b,a)\colon ta \to tb\Big)$ of $\mathcal{Q}$-arrows

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Even for a quantale Q, $\mathsf{Dist}(Q)$ is a genuine quantaloid.

Representability

Whereas $\mathsf{Cat}(\mathcal{Q})$ is (a priori) "just" a category, the quantaloid $\mathsf{Dist}(\mathcal{Q})$ has a much richer structure.

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Yet, every $\mathcal{Q}$-functor $F\colon \mathbb{A} \to \mathbb{B}$ **represents** an **adjoint pair of distributors**

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "F^*"] \end{tikzcd} \quad \text{defined by} \begin{cases} F_*(b,a) = \mathbb{B}(b,Fa) \\ F^*(a,b) = \mathbb{B}(Fa,b) \end{cases}$$

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We make $\mathsf{Cat}(\mathcal{Q})$ a **2-category** by putting, for $F, G\colon \mathbb{A} \to \mathbb{B}$,

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so that, from now on, we can use **all 2-categorical notions in** $\mathsf{Cat}(\mathcal{Q})$ too.

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But when a distributor is the (co)graph of a functor, then it is so for an **essentially unique** functor: the functor **(co)represents** the distributor.

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Representability is at the heart of $\mathcal{Q}$-enriched category theory!

Illustration: weighted (co)limits in a $\mathcal{Q}$-category

$$\begin{tikzcd} \mathbb{B} \arrow[r, "F"] & \mathbb{C} \\ \mathbb{A} \arrow[u, "\Phi"] & \end{tikzcd}$$

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Thus we define $K = \mathrm{colim}(\Phi, F)$ (whenever it exists) to mean that $K^* = \Phi \searrow F^*$.

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In a particular case involving ordered sets,

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\begin{tikzcd}
(B, \leq) \arrow[r, "F"] & (C, \leq) \\
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(C, \leq) \arrow[r, "\mathsf{id}"] & (C, \leq) \\
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$K = \mathrm{colim}(\Phi, \mathrm{id})$ precisely when $K(*) \in C$ is the **supremum** of the down-closed set $\Phi \subseteq C$.

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\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{B} & & \mathbb{C} \arrow[ll, "F^*"', "\circ" marking] \arrow[ddll, dashed, "\circ" marking, "\Phi \searrow F^* = K^*"] \\
& \leq & \\
\mathbb{A} \arrow[uu, "\Phi", "\circ" marking] & &
\end{tikzcd}$$

Thus we define $K = \mathrm{colim}(\Phi, F)$ (whenever it exists) to mean that $K^* = \Phi \searrow F^*$.

Dually, we define $L = \lim(\Psi, F)$ (whenever it exists) to mean that $L_* = F_* \swarrow \Psi$

$$\begin{tikzcd}
\mathbb{B} \arrow[rr, "F"] \arrow[dd, "\Psi"', "\circ" marking] & & \mathbb{C} \\
& & \\
\mathbb{A} \arrow[uurr, dashed, "L"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{B} \arrow[rr, "F_*", "\circ" marking] \arrow[dd, "\Psi"', "\circ" marking] & & \mathbb{C} \\
& \leq & \\
\mathbb{A} \arrow[uurr, dashed, "\circ" marking, "F_* \swarrow \Psi = L_*"'] & &
\end{tikzcd}$$

(and for ordered sets this reduces to the computation of infima).

Illustration: (co)completion doctrines

Let $\mathcal{W}$ be a class of distributors ("weights"), then $\mathbb{C}$ is a $\mathcal{W}$-**cocomplete** $\mathcal{Q}$-category if, for each $\Phi \in \mathcal{W}$, all Φ-weighted colimits exist in $\mathbb{C}$.

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\mathbb{B} \arrow[r, "F"] & \mathbb{C} \\
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Under mild conditions on $\mathcal{W}$, there is a monad $T_{\mathcal{W}} : \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ whose algebras are precisely the $\mathcal{W}$-cocomplete $\mathcal{Q}$-categories; the unit of this monad at a $\mathcal{Q}$-category $\mathbb{C}$ is the $\mathcal{W}$-**cocompletion** of $\mathbb{C}$.

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Under mild conditions on $\mathcal{W}$, there is a monad $T_{\mathcal{W}}$: Cat$(\mathcal{Q}) \to$ Cat$(\mathcal{Q})$ whose algebras are precisely the $\mathcal{W}$-cocomplete $\mathcal{Q}$-categories; the unit of this monad at a $\mathcal{Q}$-category $\mathbb{C}$ is the $\mathcal{W}$-**cocompletion** of $\mathbb{C}$.

Conversely, all "**cocompletion doctrines**" are determined by such a class $\mathcal{W}$ (thus we can classify cocompletion doctrines).

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Conversely, all "**cocompletion doctrines**" are determined by such a class $\mathcal{W}$ (thus we can classify cocompletion doctrines).

Some examples:

$\mathcal{W} = \{$all distributors$\}$: $\mathsf{Cat}(\mathcal{Q})^{T_{\mathcal{W}}} =$ $\mathcal{Q}$-modules,

$\mathcal{W} = \{$left adjoint distributors$\}$: $\mathsf{Cat}([0,\infty])^{T_{\mathcal{W}}} =$ Cauchy complete metric spaces,

$\mathcal{W} = \{$"conical" distributors$\}$: $\mathsf{Cat}([0,\infty])^{T_{\mathcal{W}}} =$ Hausdorff metrics.

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Let $\mathcal{W} = \{$all distributors$\}$, then we have both

free cocompletion doctrine $T_{\mathcal{W}} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$,

free completion doctrine $S_{\mathcal{W}} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$.

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Moreover there is always a distributive law of the free cocompletion over the free completion:

$\lambda\colon T_{\mathcal{W}} \circ S_{\mathcal{W}} \Longrightarrow S_{\mathcal{W}} \circ T_{\mathcal{W}}$.

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The algebras for the composite monad $S_{\mathcal{W}} \circ T_{\mathcal{W}} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ are those $\mathcal{Q}$-categories which are **(co)complete and completely codistributive**:

$$\begin{tikzcd}[column sep=huge]
\mathbb{C} \arrow[r, bend left=40, "\eta_{\mathbb{C}}"] \arrow[r, bend left=20, phantom, "\perp"] \arrow[r, bend right=20, phantom, "\perp"] \arrow[r, bend right=40] & S_{\mathcal{W}}\mathbb{C} \arrow[l]
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$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left=50, "\eta_{\mathbb{C}}"] \arrow[r, bend right=50] \arrow[r, phantom, bend left=25, "\perp"] \arrow[r, phantom, bend right=25, "\perp"] & S_{\mathcal{W}}\mathbb{C} \arrow[l]
\end{tikzcd}$$

For an ordered set $(A, \leq)$, this means precisely that it is a complete lattice in which each $a \in A$ is the infimum of the elements **totally above** a.

Illustration: quantitative domains

Replacing colimits by limits, one finds **completion doctrines**.

Example:

Let $\mathcal{W} = \{$all distributors$\}$, then we have both

free cocompletion doctrine $T_{\mathcal{W}} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$,

free completion doctrine $S_{\mathcal{W}} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$.

Moreover there is always a distributive law of the free cocompletion over the free completion:

$\lambda \colon T_{\mathcal{W}} \circ S_{\mathcal{W}} \Longrightarrow S_{\mathcal{W}} \circ T_{\mathcal{W}}$.

The algebras for the composite monad $S_{\mathcal{W}} \circ T_{\mathcal{W}} \colon \mathsf{Cat}(\mathcal{Q}) \to \mathsf{Cat}(\mathcal{Q})$ are those $\mathcal{Q}$-categories which are **(co)complete and completely codistributive**:

$$
\begin{tikzcd}
\mathbb{C} \arrow[rr, bend left, "\eta_{\mathbb{C}}"] \arrow[rr, bend right] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] & & S_{\mathcal{W}}\mathbb{C} \arrow[ll]
\end{tikzcd}
$$

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For general $\mathcal{Q}$, this is the starting point for **quantitative domain theory** (metric domains, fuzzy domains).

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Recall, for a locale $(L, \bigvee, \wedge, \top)$:

$\mathsf{Cat}(\mathcal{D}(L)) =$ orders of "local" elements over L (without "global" reflexivity)

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locally localic,
map-discrete,
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modular quantal frame with
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(A long but fascinating story, with many aspects open for further inverstigation!)

And I didn't even mention...

... exponentiability in, and cartesian closedness of, $\mathsf{Cat}(\mathcal{Q})$,

... fixpoint theorem for $\mathcal{Q}$-enriched categories,

... $\mathcal{W}$-continuous $\mathcal{Q}$-categories,

... divisible quantales and quantaloids,

... extending monads on $\mathsf{Cat}(\mathcal{Q})$ to monads on $\mathsf{Dist}(\mathcal{Q})$,

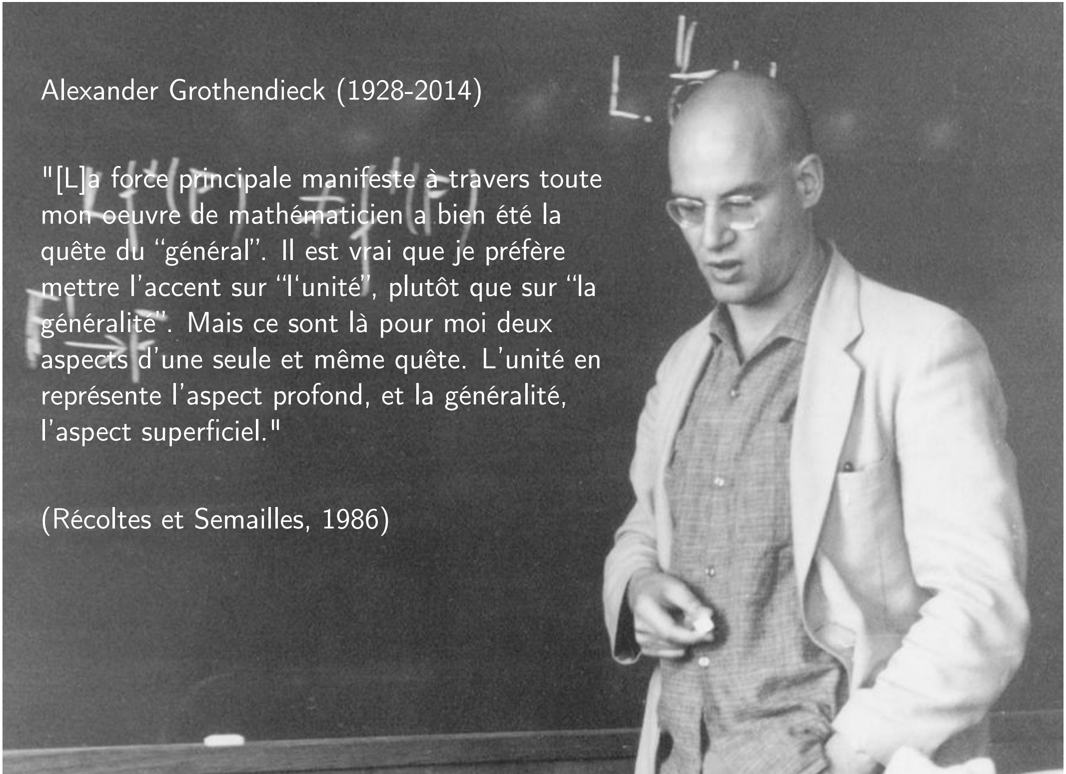
... Hilbert $\mathcal{Q}$-modules for toposes,

... bilateral (co)completion doctrines on $\mathsf{Cat}(\mathcal{Q})$,

... and many other interesting subjects!

(I will share a reference list for this talk as soon as possible.)

Mot de la fin



Alexander Grothendieck (1928-2014)

"[L]a force principale manifeste à travers toute mon oeuvre de mathématicien a bien été la quête du “général”. Il est vrai que je préfère mettre l’accent sur “l‘unité”, plutôt que sur “la généralité”. Mais ce sont là pour moi deux aspects d’une seule et même quête. L’unité en représente l’aspect profond, et la généralité, l’aspect superficiel."

(Récoltes et Semailles, 1986)